Математички Билтен
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QUADRATIC LIPSCHITZIAN INEQUALITIES FOR THE $\mathcal{AT}$-INTEGRAL TRANSFORM OF SELFADJOINT OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. For μ a positive measure on $[0,1]$ and $p:[0,1]\to[0,\infty)$ a continuous function, we consider the following *$\mathcal{AT}$-integral transform*

$$\mathcal{AT}(p,\mu)(T):=\int_0^1 \frac{p(\lambda)}{\lambda^2}\left[1-\left(\lambda^2T^2+1\right)^{-1}\right]d\mu(\lambda),$$

where the integral is assumed to exist for T a selfadjoint operator on a complex Hilbert space H.

In this paper, we show among others that, if A, B are selfadjoint operators with $A^2\geq m_1^2$, $B^2\geq m_2^2$ and $m_1, m_2>0$, then

$$\|\mathcal{AT}(p,\mu)(B)-\mathcal{AT}(p,\mu)(A)\|$$
$$\leq \left\|B^2-A^2\right\|\begin{cases}\frac{\mathcal{AT}(p,\mu)(m_2)-\mathcal{AT}(p,\mu)(m_1)}{m_2^2-m_1^2} \text{ if } m_2^2\neq m_1^2,\\ \frac{1}{2m}\mathcal{AT}'(p,\mu)(m) \text{ if } m_2^2=m_1^2=m^2,\end{cases}$$

where $\mathcal{AT}'(p,\mu)(t)$ is the derivative of $\mathcal{AT}(p,\mu)$ as a function of t. Some examples for integral transforms $\mathcal{AT}(p,\mu)$ related to the arctan function are also provided.

1. Introduction

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A|:=(A^*A)^{1/2}$.

It is known that [3] in the infinite-dimensional case the map $f(A):=|A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no constant $L>0$ such that

$$\||A|-|B|\|\leq L\|A-B\|$$

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for any $A, B \in \mathcal{B}(H)$.

However, as shown by Farforovskaya in [8], [9] and Kato in [11], the following inequality holds

$$\||A| - |B|\| \leq \frac{2}{\pi} \|A - B\| \left(2 + \log\left(\frac{\|A\| + \|B\|}{\|A - B\|}\right)\right) \tag{1.1}$$

for any $A, B \in \mathcal{B}(H)$ with $A \neq B$.

If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS} := (\operatorname{tr} C^* C)^{\frac{1}{2}}$ of an operator C, then the following inequality is true [1]

$$\||A| - |B|\|_{HS} \leq \sqrt{2} \|A - B\|_{HS}$$

for any $A, B \in \mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

It has been shown in [3] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\||A| - |B|\| \leq a_1 \|A - B\| + a_2 \|A - B\|^2 + O\left(\|A - B\|^3\right)$$

where

$$a_1 = \left\|A^{-1}\right\| \|A\| \text{ and } a_2 = \left\|A^{-1}\right\| + \left\|A^{-1}\right\|^3 \|A\|^2.$$

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

In [2] the author also obtained the following *Lipschitz type inequality*

$$\|f(A) - f(B)\| \leq f'(a) \|A - B\|$$

where f is an *operator monotone function* on $(0, \infty)$ and $A, B \geq a > 0$.

One of the problems in perturbation theory is to find bounds for $\|f(A) - f(B)\|$ in terms of $\|A - B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [4], [10] and the references therein.

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [5, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.2}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln\left(\frac{u + t}{u + 1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.3}$$

for all $t > 0$.

Motivated by these representations, we introduced in [6], for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\mu(\lambda),\ t > 0, \tag{1.4}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.4) exists for all $t > 0$.

For μ the Lebesgue measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\lambda,\ t > 0. \tag{1.5}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t),\ t > 0. \tag{1.6}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda+1)^{-1}$, $t > 0$, then we have the representation

$$\ln t = (t-1)\mathcal{D}(w_{\ln})(t),\ t > 0. \tag{1.7}$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator [6]

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\mu(\lambda), \tag{1.8}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\lambda, \tag{1.9}$$

for $T > 0$.

We also defined the *logarithmic transform* for a continuous and positive function $w(\lambda)$, $\lambda > 0$ by [7]

$$\mathcal{L}og(w,\mu)(t) := \int_0^\infty w(\lambda)\ln(\lambda+t)\, d\mu(\lambda), \tag{1.10}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.4) exists for all $t > 0$.Also, when μ is the usual Lebesgue measure, then

$$\mathcal{L}og(w)(t) := \int_0^\infty w(\lambda)\ln(\lambda+t)\, d\lambda. \tag{1.11}$$

Now, for a continuous and positive function $p(\lambda)$, $\lambda \in [0,1]$ and a positive measure μ on $[0,1]$, we define the integral transform

$$\mathcal{AT}(p,\mu)(t) := \begin{cases} \int_0^1 \frac{p(\lambda)}{\lambda^2+t^{-2}} d\mu(\lambda), \ t \neq 0, \\ 0, \ t = 0. \end{cases} \tag{1.12}$$

If μ is the usual Lebesgue measure, then we put

$$\mathcal{AT}(p)(t) := \begin{cases} \int_0^1 \frac{p(\lambda)}{\lambda^2+t^{-2}} d\lambda, \ t \neq 0, \\ 0, \ t = 0. \end{cases} \tag{1.13}$$

For $p(\lambda) = 1$, $\lambda \in [0,1]$ and μ is the usual Lebesgue measure, we have

$$\mathcal{AT}(t) := \begin{cases} \int_0^1 \frac{1}{\lambda^2+t^{-2}} d\lambda, \ t \neq 0, \\ 0, \ t = 0. \end{cases} = t \arctan t, \ t \in \mathbb{R}. \tag{1.14}$$

In the case when $p(\lambda) = \ell^2(\lambda) = \lambda^2$, we derive

$$\mathcal{AT}\left(\ell^2\right)(t) = \begin{cases} \int_0^1 \frac{\lambda^2}{\lambda^2+t^{-2}} d\lambda, \ t \neq 0, \\ 0, \ t = 0. \end{cases} = \begin{cases} 1 - \frac{\arctan t}{t}, \ t \neq 0, \\ 0, \ t = 0. \end{cases}$$

We observe that (1.12) can be written in equivalent form as

$$\begin{aligned} \mathcal{AT}(p,\mu)(t) &:= \int_0^1 \frac{t^2 p(\lambda)}{t^2\lambda^2+1} d\mu(\lambda) \\ &= \int_0^1 \left(1 - \frac{1}{t^2\lambda^2+1}\right) \frac{p(\lambda)}{\lambda^2} d\mu(\lambda), \ t \in \mathbb{R}. \end{aligned} \tag{1.15}$$

For a selfadjoint operator T we have that $\lambda^2 T^2 + 1$ is invertible for all $\lambda \in [0,1]$ and by the continuous functional calculus for selfadjoint operators we can define the *$\mathcal{AT}$-operator transform* by

$$\mathcal{AT}(p,\mu)(T) := \int_0^1 \frac{p(\lambda)}{\lambda^2} \left[1 - \left(\lambda^2 T^2 + 1\right)^{-1}\right] d\mu(\lambda), \tag{1.16}$$

and, in particular, for the Lebesgue measure

$$\mathcal{AT}(p)(T) := \int_0^1 \frac{p(\lambda)}{\lambda^2} \left[1 - \left(\lambda^2 T^2 + 1\right)^{-1}\right] d\lambda. \tag{1.17}$$

We observe that

$$\mathcal{AT}(T) = \int_0^1 \frac{1}{\lambda^2} \left[1 - \left(\lambda^2 T^2 + 1\right)^{-1}\right] d\lambda = T \arctan T,$$

for any selfadjoint operator T.

If T is invertible, then

$$\mathcal{AT}\left(\ell^2\right)(T) = \int_0^1 \left[1 - \left(\lambda^2 T^2 + 1\right)^{-1}\right] d\lambda = 1 - T^{-1} \arctan T.$$

In this paper, we show among others that, if A, B are selfadjoint operators with $A^2 \geq m_1^2$, $B^2 \geq m_2^2$ and $m_1, m_2 > 0$, then

$$\begin{aligned}
&\left\| \mathcal{AT}\left(p,\mu\right)\left(B\right) - \mathcal{AT}\left(p,\mu\right)\left(A\right) \right\| \\
&\leq \left\| B^2 - A^2 \right\| \begin{cases} \frac{\mathcal{AT}(p,\mu)(m_2) - \mathcal{AT}(p,\mu)(m_1)}{m_2^2 - m_1^2} \text{ if } m_2^2 \neq m_1^2, \\ \\ \frac{1}{2m}\mathcal{AT}'\left(p,\mu\right)\left(m\right) \text{ if } m_2^2 = m_1^2 = m^2, \end{cases}
\end{aligned}$$

where $\mathcal{AT}'\left(p,\mu\right)\left(t\right)$ is the derivative of $\mathcal{AT}\left(p,\mu\right)$ as a function of t. Some examples for integral transforms $\mathcal{AT}\left(p,\mu\right)$ related to the arctan function are also provided.

2. Main Results

We have the following identity of interest for the difference $\mathcal{AT}\left(p,\mu\right)\left(B\right) - \mathcal{AT}\left(p,\mu\right)\left(A\right)$.

Lemma 1. *Let A, B be selfadjoint operators, then*

$$\begin{aligned}
&\mathcal{AT}\left(p,\mu\right)\left(B\right) - \mathcal{AT}\left(p,\mu\right)\left(A\right) \\
&= \int_0^1 p\left(\lambda\right)\left[\int_0^1 \left(1 + \lambda^2\left[\left(1-t\right)A^2 + tB^2\right]\right)^{-1}\left(B^2 - A^2\right)\right. \\
&\left.\times \left(1 + \lambda^2\left[\left(1-t\right)A^2 + tB^2\right]\right)^{-1} dt\right] d\mu\left(\lambda\right).
\end{aligned} \tag{2.1}$$

If $B^2 \geq A^2$, then

$$\mathcal{AT}\left(p,\mu\right)\left(B\right) \geq \mathcal{AT}\left(p,\mu\right)\left(A\right). \tag{2.2}$$

Proof. Let A, B selfadjoint operators. By employing (1.16) we have

$$\begin{aligned}
&\mathcal{AT}\left(p,\mu\right)\left(B\right) - \mathcal{AT}\left(p,\mu\right)\left(A\right) \\
&= \int_0^1 \frac{p\left(\lambda\right)}{\lambda^2}\left[1 - \left(\lambda^2 B^2 + 1\right)^{-1}\right] d\mu\left(\lambda\right) \\
&- \int_0^1 \frac{p\left(\lambda\right)}{\lambda^2}\left[1 - \left(\lambda^2 A^2 + 1\right)^{-1}\right] d\mu\left(\lambda\right) \\
&= \int_0^1 \frac{p\left(\lambda\right)}{\lambda^2}\left[\left(\lambda^2 A^2 + 1\right)^{-1} - \left(\lambda^2 B^2 + 1\right)^{-1}\right] d\mu\left(\lambda\right).
\end{aligned} \tag{2.3}$$

The function $g\left(t\right) = -t^{-1}$ is operator monotone on $\left(0,\infty\right)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla g_T\left(S\right) := \lim_{t\to 0}\left[\frac{g\left(T+tS\right) - g\left(T\right)}{t}\right] = T^{-1}ST^{-1} \tag{2.4}$$

for $T, S > 0$.

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C, D]$:

$\{(1-t)C+tD,\ t\in[0,1]\}$ for C, D selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$g_{C,D}(t) := f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$g(D)-g(C)=\int_0^1 \frac{d}{dt}\left(g_{C,D}(t)\right)dt=\int_0^1 \nabla g_{(1-t)C+tD}(D-C)\,dt. \tag{2.5}$$

If we write this equality for the function $g(t)=-t^{-1}$ and C, $D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1 ((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.6}$$

If in (2.6) we take $C=\lambda^2A^2+1$ and $D=\lambda^2B^2+1$, then we get

$$\begin{aligned}
&\left(\lambda^2A^2+1\right)^{-1}-\left(\lambda^2B^2+1\right)^{-1} \qquad (2.7)\\
&=\int_0^1\left((1-t)\left(\lambda^2A^2+1\right)+t\left(\lambda^2B^2+1\right)\right)^{-1}\left(\lambda^2B^2-\lambda^2A^2\right)\\
&\times\left((1-t)\left(\lambda^2A^2+1\right)+t\left(\lambda^2B^2+1\right)\right)^{-1}dt\\
&=\lambda^2\int_0^1\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right)\\
&\times\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}dt
\end{aligned}$$

for all t, $\lambda\in[0,1]$.

Therefore

$$\begin{aligned}
&\int_0^1\frac{p(\lambda)}{\lambda^2}\left[\left(\lambda^2A^2+1\right)^{-1}-\left(\lambda^2B^2+1\right)^{-1}\right]d\mu(\lambda)\\
&=\int_0^1\frac{p(\lambda)}{\lambda^2}\left[\left(\lambda^2A^2+1\right)^{-1}-\left(\lambda^2B^2+1\right)^{-1}\right]d\mu(\lambda)\\
&=\int_0^1 p(\lambda)\left[\int_0^1\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right)\right.\\
&\left.\times\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}dt\right]d\mu(\lambda)
\end{aligned}$$

and by (2.3) we get the representation (2.1).

If $B^2-A^2\geq 0$, then by multiplying both sides by $\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}$ we get

$$\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right)\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\geq 0$$

for all t, $\lambda\in[0,1]$.

If we integrate over $t \in [0,1]$, then multiply by $p(\lambda) \geq 0$ and integrate over the positive measure $d\mu(\lambda)$ on $[0,1]$, we have

$$\int_0^1 p(\lambda) \left[\int_0^1 \left(1+\lambda^2 \left[(1-t)A^2+tB^2\right]\right)^{-1} \left(B^2-A^2\right) \right.$$
$$\left. \times \left(1+\lambda^2 \left[(1-t)A^2+tB^2\right]\right)^{-1} dt \right] d\mu(\lambda) \geq 0$$

and the operator quadratic monotonicity of the transform $\mathcal{AT}(p,\mu)$ is proved. □

We have the following main result:

Theorem 1. *For A, B selfadjoint operators with $A^2 \geq m_1^2$, $B^2 \geq m_2^2$ and $m_1, m_2 > 0$, we have*

$$\|\mathcal{AT}(p,\mu)(B) - \mathcal{AT}(p,\mu)(A)\| \tag{2.8}$$
$$\leq \left\|B^2-A^2\right\| \begin{cases} \frac{\mathcal{AT}(p,\mu)(m_2)-\mathcal{AT}(p,\mu)(m_1)}{m_2^2-m_1^2} \text{ if } m_2^2 \neq m_1^2, \\ \\ \frac{1}{2m}\mathcal{AT}'(p,\mu)(m) \text{ if } m_2^2 = m_1^2 = m^2, \end{cases}$$

where $\mathcal{AT}'(p,\mu)(t)$ is the derivative of $\mathcal{AT}(p,\mu)$ as a function of t.

Proof. From the identity (2.1), we get, by taking the norm, that

$$\|\mathcal{AT}(p,\mu)(B) - \mathcal{AT}(p,\mu)(A)\| \tag{2.9}$$
$$\leq \int_0^1 p(\lambda) \left[\int_0^1 \left\| \left(1+\lambda^2 \left[(1-t)A^2+tB^2\right]\right)^{-1} \left(B^2-A^2\right) \right.\right.$$
$$\left.\left. \times \left(1+\lambda^2 \left[(1-t)A^2+tB^2\right]\right)^{-1} \right\| dt \right] d\mu(\lambda)$$
$$\leq \left\|B^2-A^2\right\|$$
$$\times \int_0^1 p(\lambda) \left(\int_0^1 \left\| \left(1+\lambda^2 \left[(1-t)A^2+tB^2\right]\right)^{-1} \right\|^2 dt \right) d\mu(\lambda)$$

for A, B selfadjoint operators.

Assume that $m_1^2 < m_2^2$, then

$$(1-t)m_1^2 + tm_2^2 \leq (1-t)A^2 + tB^2$$

for $t \in [0,1]$, which implies that

$$1+\lambda^2 \left[(1-t)m_1^2+tm_2^2\right] \leq 1+\lambda^2 \left[(1-t)A^2+tB^2\right],$$

namely

$$\left(1+\lambda^2 \left[(1-t)A^2+tB^2\right]\right)^{-1} \leq \left(1+\lambda^2 \left[(1-t)m_1^2+tm_2^2\right]\right)^{-1}$$

for $t, \lambda \in [0,1]$.

This implies that

$$\left\|\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\right\| \leq \left(1+\lambda^2\left[(1-t)m_1^2+tm_2^2\right]\right)^{-1},$$

which implies that

$$\left\|\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\right\|^2 \leq \left(1+\lambda^2\left[(1-t)m_1^2+tm_2^2\right]\right)^{-2}, \quad (2.10)$$

for $t, \lambda \in [0,1]$.

If we integrate (2.10) over $t \in [0,1]$, then multiply by $p(\lambda) \geq 0$ and integrate over the positive measure $d\mu(\lambda)$ on $[0,1]$, we have by (2.1) that,

$$\begin{aligned}&\int_0^1 p(\lambda)\left(\int_0^1\left\|\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\right\|^2 dt\right)d\mu(\lambda) \quad (2.11)\\ &\leq \int_0^1 p(\lambda)\left(\int_0^1\left(1+\lambda^2\left[(1-t)m_1^2+tm_2^2\right]\right)^{-2}dt\right)d\mu(\lambda).\end{aligned}$$

If we use the identity (2.1) for $A = m_1$ and $B = m_2$, then we get

$$\begin{aligned}&\mathcal{AT}(p,\mu)(m_2)-\mathcal{AT}(p,\mu)(m_1)\\ &=\int_0^1 p(\lambda)\left[\int_0^1\left(1+\lambda^2\left[(1-t)m_1^2+tm_2^2\right]\right)^{-1}\left(m_2^2-m_1^2\right)\right.\\ &\left.\times\left(1+\lambda^2\left[(1-t)m_1^2+tm_2^2\right]\right)^{-1}dt\right]d\mu(\lambda)\\ &=\left(m_2^2-m_1^2\right)\int_0^1 p(\lambda)\left(\int_0^1\left(1+\lambda^2\left[(1-t)m_1^2+tm_2^2\right]\right)^{-2}dt\right)d\mu(\lambda),\end{aligned}$$

which gives that

$$\begin{aligned}&\int_0^1 p(\lambda)\left(\int_0^1\left(1+\lambda^2\left[(1-t)m_1^2+tm_2^2\right]\right)^{-2}dt\right)d\mu(\lambda)\\ &=\frac{\mathcal{AT}(p,\mu)(m_2)-\mathcal{AT}(p,\mu)(m_1)}{m_2^2-m_1^2}\end{aligned}$$

and by (2.9) and (2.11), we get the first inequality in (2.8).

If $m_1^2 > m_2^2$, then we can prove (2.8) in a similar way.

Assume that $m_1^2 = m_2^2 = m^2$ with $m > 0$. Let $\varepsilon > 0$ such that $m^2 - \varepsilon > 0$. Then $B^2 \geq m^2$ and $A^2 \geq m^2 > m^2 - \varepsilon > 0$. If we take $m_2^2 := m^2$, $m_1^2 = m^2 - \varepsilon > 0$, then $m_2^2 > m_1^2$ and by the first inequality in (2.8,)

$$\begin{aligned}&\|\mathcal{AT}(p,\mu)(B)-\mathcal{AT}(p,\mu)(A)\|\\ &\leq\left\|B^2-A^2\right\|\frac{\mathcal{AT}(p,\mu)(m)-\mathcal{AT}(p,\mu)\left(\sqrt{m^2-\varepsilon}\right)}{\varepsilon}\end{aligned}$$

and by taking the limit over $\varepsilon \to 0+$, then we get

$$\|\mathcal{AT}(p,\mu)(B)-\mathcal{AT}(p,\mu)(A)\| \quad (2.12)$$

$$\leq \left\|B^2-A^2\right\| \lim_{\varepsilon\to 0+} \frac{\mathcal{AT}\left(p,\mu\right)\left(m\right)-\mathcal{AT}\left(p,\mu\right)\left(\sqrt{m^2-\varepsilon}\right)}{\varepsilon}$$

$$= \left\|B^2-A^2\right\| \lim_{\varepsilon\to 0+} \frac{\mathcal{AT}\left(p,\mu\right)\mathcal{AT}\left(p,\mu\right)\left(m\right)-\mathcal{AT}\left(p,\mu\right)\left(\sqrt{m^2-\varepsilon}\right)}{m-\sqrt{m^2-\varepsilon}}$$

$$\times \lim_{\varepsilon\to 0+} \frac{m-\sqrt{m^2-\varepsilon}}{\varepsilon}.$$

Since

$$\lim_{\varepsilon\to 0+} \frac{\mathcal{AT}\left(p,\mu\right)\left(m\right)-\mathcal{AT}\left(p,\mu\right)\left(\sqrt{m^2-\varepsilon}-m+m\right)}{m-\sqrt{m^2-\varepsilon}}$$

$$= \lim_{h\to 0+} \frac{\mathcal{AT}\left(p,\mu\right)\left(m\right)-\mathcal{AT}\left(p,\mu\right)\left(m-h\right)}{h} = \mathcal{AT}'\left(p,\mu\right)\left(m\right)$$

and

$$\lim_{\varepsilon\to 0+} \frac{m-\sqrt{m^2-\varepsilon}}{\varepsilon} = \lim_{\varepsilon\to 0+} \frac{1}{m+\sqrt{m^2-\varepsilon}} = \frac{1}{2m},$$

hence by (2.12) we obtain the second part of (2.8). □

Remark 2.1: For A, B selfadjoint operators with $A^2 \geq m_1^2$, $B^2 \geq m_2^2$ and m_1, $m_2 > 0$, we have

$$\left\|B\arctan\left(B\right)-A\arctan\left(A\right)\right\| \tag{2.13}$$

$$\leq \left\|B^2-A^2\right\| \begin{cases} \frac{m_2\arctan(m_2)-m_1\arctan(m_1)}{m_2^2-m_1^2} \text{ if } m_2^2 \neq m_1^2, \\ \\ \frac{1}{2m}\left(\arctan m+\frac{m}{m^2+1}\right) \text{ if } m_2^2=m_1^2=m^2. \end{cases}$$

For any A, B selfadjoint and invertible operators with $A^2 \geq m_1^2$, $B^2 \geq m_2^2$ and m_1, $m_2 > 0$, we have

$$\left\|A^{-1}\arctan A-B^{-1}\arctan B\right\| \tag{2.14}$$

$$\leq \left\|B^2-A^2\right\| \begin{cases} \frac{m_1^{-1}\arctan(m_1)-m_2^{-1}\arctan(m_2)}{m_2^2-m_1^2} \text{ if } m_2^2 \neq m_1^2, \\ \\ \frac{1}{2m^3}\left(\arctan m-\frac{m}{m^2+1}\right) \text{ if } m_2^2=m_1^2=m^2. \end{cases}$$

3. A Hermite-Hadamard Type Inequality

We have:

Proposition 3.1. *For A, B selfadjoint operators, assume that $\left((1-s)A+sB\right)^2 \geq m^2$ for $s\in[0,1]$ with $m>0$, then*

$$\left\|\int_0^1 \mathcal{AT}\left(p,\mu\right)\left((1-s)A+sB\right)ds-\mathcal{AT}\left(p,\mu\right)\left(\frac{A+B}{2}\right)\right\| \tag{3.1}$$

$$\leq \frac{1}{2m}\mathcal{AT}'(p,\mu)(m)\int_0^1 \left\|((1-s)A+sB)^2-\left(\frac{A+B}{2}\right)^2\right\| ds$$
$$\leq \frac{1}{12m}\left(\|A\|^2+\frac{1}{2}\|AB+BA\|+\|B\|^2\right)\mathcal{AT}'(p,\mu)(m).$$

Proof. Since

$$\left(\frac{A+B}{2}\right)^2 \geq m^2,$$

then by (2.8) we have

$$\left\|\mathcal{AT}(p,\mu)((1-s)A+sB)-\mathcal{AT}(p,\mu)\left(\frac{A+B}{2}\right)\right\|$$
$$\leq \frac{1}{2m}\left\|((1-s)A+sB)^2-\left(\frac{A+B}{2}\right)^2\right\|\mathcal{AT}'(p,\mu)(m),$$

for $s\in[0,1]$.

If we take the integral and use the norm properties, then we get

$$\left\|\int_0^1 \mathcal{AT}(p,\mu)((1-s)A+sB)\,ds-\mathcal{AT}(p,\mu)\left(\frac{A+B}{2}\right)\right\| \tag{3.2}$$
$$\leq \int_0^1 \left\|\mathcal{AT}(p,\mu)((1-s)A+sB)-\mathcal{AT}(p,\mu)\left(\frac{A+B}{2}\right)\right\| ds$$
$$\leq \frac{1}{2m}\mathcal{AT}'(p,\mu)(m)\int_0^1 \left\|((1-s)A+sB)^2-\left(\frac{A+B}{2}\right)^2\right\| ds.$$

We have

$$((1-s)A+sB)^2-\left(\frac{A+B}{2}\right)^2$$
$$=(1-s)^2A^2+s(1-s)(AB+BA)+s^2B^2$$
$$-\frac{1}{4}\left(A^2+AB+BA+B^2\right)$$
$$=\left[(1-s)^2-\frac{1}{4}\right]A^2+\left[s(1-s)-\frac{1}{4}\right](AB+BA)+\left(s^2-\frac{1}{4}\right)B^2$$

for $s\in[0,1]$.

If we take the norm, then we get

$$\left\|((1-s)A+sB)^2-\left(\frac{A+B}{2}\right)^2\right\|$$
$$\leq \left|(1-s)^2-\frac{1}{4}\right|\|A\|^2+\left|s(1-s)-\frac{1}{4}\right|\|AB+BA\|$$
$$+\left|s^2-\frac{1}{4}\right|\|B\|^2$$

and, by integration, we derive

$$\begin{aligned}
&\int_0^1 \left\| \left((1-s)\,A+sB\right)^2 - \left(\frac{A+B}{2}\right)^2 \right\| ds \qquad (3.3)\\
&\leq \int_0^1 \left|(1-s)^2 - \frac{1}{4}\right| ds \,\|A\|^2 + \int_0^1 \left|s\,(1-s) - \frac{1}{4}\right| ds \,\|AB+BA\| \\
&+ \int_0^1 \left|s^2 - \frac{1}{4}\right| ds \,\|B\|^2 \\
&= \frac{1}{6}\,\|A\|^2 + \frac{1}{12}\,\|AB+BA\| + \frac{1}{6}\,\|B\|^2 .
\end{aligned}$$

By making use of the inequalities (3.2) and (3.3), we derive the desired result (3.1). □

Remark 3.1: We observe that, if $A,\ B \geq m > 0$, then $(1-s)\,A+sB \geq m > 0$, which implies that $\left((1-s)\,A+sB\right)^2 \geq m^2$ for $s \in [0,1]$.

With the assumptions in Proposition 3.1, we have

$$\begin{aligned}
&\left\| \int_0^1 \left((1-s)\,A+sB\right) \arctan\left((1-s)\,A+sB\right) ds \right. \qquad (3.4)\\
&\left. - \left(\frac{A+B}{2}\right) \arctan\left(\frac{A+B}{2}\right) \right\| \\
&\leq \frac{1}{2m}\left(\arctan m + \frac{m}{m^2+1}\right) \int_0^1 \left\| \left((1-s)\,A+sB\right)^2 - \left(\frac{A+B}{2}\right)^2 \right\| ds \\
&\leq \frac{1}{12m}\left(\arctan m + \frac{m}{m^2+1}\right) \left(\|A\|^2 + \frac{1}{2}\,\|AB+BA\| + \|B\|^2\right).
\end{aligned}$$

REFERENCES

[1] H. Araki and S. Yamagami, An inequality for Hilbert-Schmidt norm, *Commun. Math. Phys.* **81** (1981), 89-96.

[2] R. Bhatia, First and second order perturbation bounds for the operator absolute value, *Linear Algebra Appl.* **208/209** (1994), 367-376.

[3] R. Bhatia, Perturbation bounds for the operator absolute value. *Linear Algebra Appl.* **226/228** (1995), 639–645.

[4] R. Bhatia, D. Singh and K. B. Sinha, Differentiation of operator functions and perturbation bounds. *Comm. Math. Phys.* **191** (1998), no. 3, 603–611.

[5] R. Bhatia, *Matrix Analysis*. Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp. ISBN: 0-387-94846-5.

[6] S. S. Dragomir, Operator monotonicity of an integral transform of positive operators in Hilbert spaces with applications, Preprint *RGMIA Res. Rep. Coll.* **23** (2020), Art. 65, 15 pp. [Online https://rgmia.org/papers/v23/v23a65.pdf].

[7] S. S. Dragomir, Operator monotonicity of the logarithmic integral transform for positive operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.* **23** (2020), Art. 114, 13 pp. [Online https://rgmia.org/papers/v23/v23a114.pdf].

[8] Yu. B. Farforovskaya, Estimates of the closeness of spectral decompositions of self-adjoint operators in the Kantorovich-Rubinshtein metric (in Russian), *Vesln. Leningrad. Gos. Univ. Ser. Mat. Mekh. Astronom.* **4** (1967), 155-156.

[9] Yu. B. Farforovskaya, An estimate of the norm $\|f(B) - f(A)\|$ for self-adjoint operators A and B (in Russian) *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst.* **56** (1976), 143-162 .
[10] Yu. B. Farforovskaya and L. Nikolskaya, Modulus of continuity of operator functions. *Algebra i Analiz* **20** (2008), no. 3, 224–242; translation in St. Petersburg Math. J. **20** (2009), no. 3, 493–506.
[11] T. Kato, Continuity of the map $S \to |S|$ for linear operators, *Proc. Japan Acad.* **49** (1973), 143-162.

[1] APPLIED MATHEMATICS RESEARCH GROUP, ISILC,
VICTORIA UNIVERSITY, PO BOX 14428
MELBOURNE CITY, MC 8001, AUSTRALIA.

[2] SCHOOL OF COMPUTER SCIENCE
& APPLIED MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND,
JOHANNESBURG, SOUTH AFRICA.
Email address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

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