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TOPOLOGICALLY MIXING SOLUTIONS OF ABSTRACT PARTIAL FRACTIONAL CAUCHY INCLUSIONS

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Abstract. In this paper, we investigate topologically mixing properties of solutions for some classes of abstract partial fractional Cauchy inclusions. We introduce the notion of subspace (strong) topologically mixing property in the multi-dimensional setting, providing also several new structural characterizations, illustrative examples and useful observations.

1. Introduction and Preliminaries

In this paper, E denotes a separable Fréchet space over the field of complex numbers. A linear continuous mapping $T: E \to E$ is said to be hypercyclic if there exists an element $x \in E$ whose orbit $\mathrm{Orb}(x, T) := \{T^k x : k \in \mathbb{N}_0\}$ is dense in E, while T is said to be topologically transitive if for any pair of open non-empty subsets U, V of E there exists $k \in \mathbb{N}$ such that $T^k(U) \cap V \neq \emptyset$. Here, $\mathbb{N}_0 = \{0, 1, 2, ...\}$; if $m \in \mathbb{N}$, then $\mathbb{N}_m = \{1, ..., m\}$ henceforth. Using the Baire theorem, we can simply prove that T is hypercyclic if and only if T is topologically transitive. The most commonly used criterion for proving hypercyclicity of single operators, the Hypercyclicity Criterion, was discovered independently by C. Kitai [13] (1982) and R. M. Gethner, J. Shapiro [8] (1987). If T is hypercyclic, then E must be infinite-dimensional.

From the period 1987 onwards, the study of hypercyclicity of single-valued linear operators and various generalizations of this concept have experienced a great development; see the research monographs [1] by F. Bayart, E. Matheron, [6] by K.-G. Grosse-Erdmann, A. Peris and [16] by M. Kostić. For our study, it should be worthwhile to recall that the class of topologically mixing operators

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was introduced by G. Costakis and M. Sambarino in [3] (2003): a linear continuous mapping $T: E \to E$ is said to be topologically mixing if for any pair of open non-empty subsets U, V of E there exists $k_0 \in \mathbb{N}$ such that $T^k(U) \cap V \neq \emptyset$ for all $k_0 \in \mathbb{N}$ with $k \geq k_0$.

The hypercyclic tuples of operators were considered for the first time by L. Kérchy in [22] (2005) and N. Feldman in [7] (2008): A tuple $(T_1, ..., T_n)$ of commuting linear continuous operators on E, where $n \in \mathbb{N} \setminus \{1\}$, is said to be hypercyclic if there exists $x \in E$ such that the orbit $\{T_1^{k_1} \cdots T_n^{k_n} x : k_1 \in \mathbb{N}_0, ..., k_n \in \mathbb{N}_0\}$ is dense in E. A tuple $(T_1, ..., T_n)$ of commuting linear continuous operators on E is hypercyclic if and only if it is topologically transitive, which means that for any pair of open non-empty subsets U, V of E there exist integers $k_1 \in \mathbb{N}_0, ..., k_n \in \mathbb{N}_0$ such that $T_1^{k_1} \cdots T_n^{k_n}(U) \cap V \neq \emptyset$. It is clear that the hypercyclicity of some operator T_i, where $1 \leq i \leq n$, implies the hypercyclicity of tuple $(T_1, ..., T_n)$; the converse statement is not true in general. In contrast with the usually examined hypercyclic operators, the hypercyclic tuples of operators exist on finite-dimensional spaces (see [7, Example 2.7] for the first example in this direction). Many interesting features of hypercyclic tuples of operators on finite-dimensional spaces have been established by S. Shkarin in [26] and G. Costakis, D. Hadjiloukas, A. Manoussos in [2].

The topologically mixing tuples of operators were first considered by M. Habibi and B. Yousefi in [9] (2010); cf. also [10, 23, 28]. Following [9, Definition 1.1], we say that a tuple $(T_1, ..., T_n)$ of commuting linear continuous operators on E is topologically mixing if for any pair of open non-empty subsets U, V of E there exists an integer $k_0 \in \mathbb{N}$ such that, for every integers $k_1 \geq k_0, ..., k_n \geq k_0$ we have $T_1^{k_1} \cdots T_n^{k_n}(U) \cap V \neq \emptyset$. In contrast with the hypercyclic tuples of operators, the topologically mixing tuples of operators do not exist on finite-dimensional spaces. The reason is quite simple: if a tuple $(T_1, ..., T_n)$ of commuting linear continuous operators on E is topologically mixing, then the operator $T_1 \cdots T_n$ is topologically mixing itself and therefore hypercyclic, which immediately implies that the pivot space E must be infinite-dimensional. Some extensions of topologically mixing tuples of operators were considered by W. Wang, Y. Shu and X. Wang in [27] (2016), who investigated the class of weakly mixing tuples of operators and the class of S-mixing tuples of operators in the setting od complex Banach spaces.

Besides the above mentioned classes, we can also introduce and analyze the class of strong topologically mixing tuples of operators, which makes a very special subclass of topologically mixing tuples of operators: a tuple $(T_1, ..., T_n)$ of commuting linear continuous operators on E is said to be strongly topologically mixing if for any pair of open non-empty subsets U, V of E there exists an integer $k_0 \in \mathbb{N}$ such that, for every integers $k_1 \in \mathbb{N}_0, ..., k_n \in \mathbb{N}_0$ with $k_1 + ... + k_n \geq k_0$, we have $T_1^{k_1} \cdots T_n^{k_n}(U) \cap V \neq \emptyset$. If this is the case, then any single operator T_i of tuple $(T_1, ..., T_n)$ is topologically mixing ($1 \leq i \leq n$), which is not true for topologically mixing tuples of operators; for example, if the operator T is topologically mixing on E and I is the identity operator on E, then the tuple (T, I) is topologically mixing.

On the other hand, the hypercyclic one-parameter strongly continuous semigroups of linear operators were systematically analyzed by W. Desch, W. Schappacher and G. F. Webb in [4] (1997), while the hypercyclic multiparameter strongly continuous semigroups of linear operators were introduced and analyzed by M. Janfada and A. N. Baghan in [12] (2019); cf. [17, pp. 363–365, pp. 593–595] and [12, Section 2, Section 3] for the notion and more details about multiparameter strongly continuous semigroups of operators. The study of hypercyclicity of abstract fractional differential equations goes back to the paper [18] (2012); cf. also the research monographs [14, 15, 16] for comprehensive surveys of results.

The main aim of this research article is to initiate the study of subspace (strong) topologically mixing solutions to the abstract n-parameter fractional Sobolev problem (FP)= $(FP)_R$ or (FP)= $(FP)_L$, where

$$(\mathrm{FP})_R : \mathbf{D}_{t_i}^{\alpha_i} B_i u(t) = A_i u(t), \quad t \in [0,+\infty)^n,\ 1 \leq i \leq n,$$
$$B_i u(0) = B_i x;\ \Big[\frac{\partial^j}{\partial t_i^j} B_i u(t_1,...,t_{i-1},t_i,t_{i+1},...,t_n)\Big]_{t_i=0} = 0$$
$$\text{for all } 1 \leq i \leq n,\ 1 \leq j \leq \lceil \alpha_i \rceil - 1 \text{ and } (t_1,...,t_{i-1},t_{i+1},...,t_n) \in [0,+\infty)^{n-1},$$

and

$$(\mathrm{FP})_L : B_i \mathbf{D}_{t_i}^{\alpha_i} u(t) = A_i u(t), \quad t \in [0,+\infty)^n,\ 1 \leq i \leq n,$$
$$u(0) = x;\ \Big[\frac{\partial^j}{\partial t_i^j} u(t_1,...,t_{i-1},t_i,t_{i+1},...,t_n)\Big]_{t_i=0} = 0$$
$$\text{for all } 1 \leq i \leq n,\ 1 \leq j \leq \lceil \alpha_i \rceil - 1 \text{ and } (t_1,...,t_{i-1},t_{i+1},...,t_n) \in [0,+\infty)^{n-1};$$

here, $n \in \mathbb{N}$, A_i, B_i are closed linear operators on E, $x \in E$, $\alpha_i \in (0,2)$ for $1 \leq i \leq n$ and $D_{t_i}^{\alpha_i} u(t)$ is the partial Caputo fractional derivative of order α_i of function $u(\cdot)$ with respect to the variable t_i, which will be defined a little bit later ($1 \leq i \leq n$).

The organization and main ideas of this work can be briefly described as follows. First of all, we fix the notation and terminology used throughout the paper. Section 2 is devoted to the study of subspace (strong) topologically mixing solutions of abstract n-parameter fractional Sobolev problem (FP). In Definition 2.1, we introduce the notion of a subspace (strong) topologically mixing solution of problem (FP). The main result of Section 2 is Theorem 1, where we extend the statement of Theorem 2.10.3 of [15] to the multidimensional setting.

At this place, we would like to emphasize that it would be very difficult to prove the subspace *strong* topologically mixing property for solutions of problem (FP) in the pure fractional case since the Mittag-Leffler function $E_\alpha(\cdot)$ has infinitely many zeroes for $\alpha \in (0,2) \setminus \{1\}$ (see [24]-[25], and especially, [5, Theorem 3.1] for more details about this issue). To the best of our knowledge, we can do that only for the problem (FP) in which $\alpha_i = 1$ for all $i \in \mathbb{N}_n$ and, in particular, for the n-parameter abstract Cauchy problem

$$(ACP) : \begin{cases} u \in C([0,+\infty)^n : E) \cap C^1((0,+\infty)^n : E), \\ u_{t_i}(t) = A_i u(t),\ t \in (0,+\infty)^n,\ 1 \leq i \leq n, \\ u(0) = x, \quad x \in \bigcap_{i \in \mathbb{N}_n} D(A_i). \end{cases}$$

The well-posedness of (ACP) has been analyzed by many authors; for example, M. Janfada and A. Niknam have proved in [11, Theorem 2.1], that if the tuple $(A_1, A_2, \cdots, A_n)$ is the infinitesimal generator of multiparameter strongly continuous semigroup $(T(t))_{t\in[0,+\infty)^n}$ and $x \in \bigcap_{i\in\mathbb{N}_n} D(A_i)$, then $u(t) := T(t)x$, $t \in [0,+\infty)^n$ is a unique strong solution of the above problem. We illustrate Theorem 1 by some interesting examples in Example 1; cf. also Remark 2.1–Remark 2.2.

In the first part of Section 2, we do not assume the existence of any fractional solution operator family for (FP). Concerning this issue, we would like to emphasize that the author has recently explored the multiparameter solution operator families of form

$$R(t_1, ..., t_n) = R_1(t_1) \cdot ... \cdot R_n(t_n), \quad t = (t_1, ..., t_n) \in [0,+\infty)^n, \tag{1.1}$$

where $(R_i(t_i))_{t_i\geq 0}$ is a global (a,k)-regularized C_i-resolvent family with a subgenerator $\mathcal{A}_i$ and $R_i(t_i)R_j(t_j) = R_j(t_j)R_i(t_i)$ for all $1 \leq i, j \leq n$, $t_i \geq 0$ and $t_j \geq 0$. Here, $\mathcal{A}_i$ is a multivalued linear operator (MLO) on E for $1 \leq i \leq n$; cf. [19, 20, 21] for the basic source of information concerning abstract Volterra integro-differential inclusions with multiple variables.

In Subsection 2.1, we investigate the existence of subspace (strong) topologically mixing solutions to the following abstract n-parameter fractional differential Cauchy inclusion (P):

$$(P): \begin{cases} u \in \mathcal{C}([0,+\infty)^n : E), \\ \mathbf{D}^{\alpha_i}_{t_i} u(t) \in \mathcal{A}_i u(t), \ t \in [0,+\infty)^n, \ 1 \leq i \leq n, \\ u(0) = x, \quad x \in \bigcap_{i\in\mathbb{N}_n} D(\mathcal{A}_i), \\ \left[\frac{\partial^j}{\partial t_i^j} u(t_1, ..., t_{i-1}, t_i, t_{i+1}, ..., t_n)\right]_{t_i=0} = 0 \text{ for all } 1 \leq i \leq n, \\ 1 \leq j \leq \lceil \alpha_i \rceil - 1 \text{ and } (t_1, ..., t_{i-1}, t_{i+1}, ..., t_n) \in [0,+\infty)^{n-1}, \end{cases}$$

where $\alpha_i > 0$, $\mathcal{A}_i$ are closed multivalued linear operators on E $(1 \leq i \leq n)$, $\mathcal{C}([0,+\infty)^n : E)$ denotes the vector space consisting of all continuous functions $u : [0,+\infty)^n \to E$ such that the partial Caputo fractional derivative $\mathbf{D}^{\alpha_i}_{t_i} u(\cdot)$ is well-defined for $1 \leq i \leq n$, and some extra assumptions are satisfied. In order to do that, we can use the following recent extension of Theorem 2.1 from [11]:

Lemma 1. *(cf.* [21, Theorem 2.1]*) Suppose that* $x \in C_1 \cdot ... \cdot C_n(\bigcap_{i\in\mathbb{N}_n} D(\mathcal{A}_i))$ *and condition* (C1) *holds, where:*

(C1) $\alpha_i > 0$, $\mathcal{A}_i$ *is a closed subgenerator of a global* (g_{α_i}, C_i)*-regularized resolvent family* $(R_i(t_i))_{t_i\geq 0}$ $(i \in \mathbb{N}_n)$, $R_i(t_i)R_j(t_j) = R_j(t_j)R_i(t_i)$ *and* $C_iR_j(t_j) = R_j(t_j)C_i$ *for all* $1 \leq i, j \leq n$ *such that* $i \neq j$, $t_i \geq 0$ *and* $t_j \geq 0$.

Then there exists a unique strong solution of problem (P) and we have

$$u(t_1, ..., t_n) = R_1(t_1) \cdot ... \cdot R_n(t_n)[C_1 \cdot ... \cdot C_n]^{-1}x, \quad t \in [0,+\infty)^n.$$

Subspace (strong) topologically mixing property for operator family $(R(t))_{t\in[0,+\infty)^n}$ of form (1.1) is introduced in Definition 2.2. The main result of Subsection 2.1 is Theorem 2; cf. also Example 2 and some final observations given in

Section 3. We will thoroughly analyze the subspace (strong) topologically mixing tuples of operators somewhere else; the interested reader may also try to transfer assertions of [23, Theorem 2.3], [28, Theorem 2.2] and multidimensional Godefroy-Shapiro criteria [27, Theorem 3, Theorem 6] to continuous setting.

Before explaining the notation used throughout the paper, we feel it is our duty to say that the fractional differential Cauchy inclusion (P) seems to be an artificial system since, in the present situation, we really do not know to tell whether it can be applied in modeling of some real world phenomena; also, the examples presented in this paper are only illustrative and they are not motivated by some real-life problems (here we can only comment that the chaotic operators and chaotic strongly continuous semigroups are extremely special and highly unlikely). We want also to emphasize that this is a pure theoretical study and that we will not provide here any figures, numerical simulations or tables.

Notation and preliminaries. The Gamma function will be denoted by $\Gamma(\cdot)$ and the principal branch will be always used to take the powers. We define $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$ and $0^\zeta := 0$ $(\zeta > 0, t > 0)$. If $s \in \mathbb{R}$, then we set $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$. We will use the fractional derivatives of Caputo type, only. Suppose that $\alpha > 0$, $m = \lceil \alpha \rceil$ and $D = [0, T)$, $D = [0, T]$ or $D = [0, +\infty)$ for some real number $T > 0$. Then the Caputo fractional derivative $\mathbf{D}_t^\alpha u(\cdot)$ is defined for those functions $u \in C^{m-1}(D : E)$ for which $g_{m-\alpha} * (u - \sum_{k=0}^{m-1} u_k g_{k+1}) \in C^m(D : E)$, by

$$\mathbf{D}_t^\alpha u(\cdot) := \frac{d^m}{dt^m}\left[g_{m-\alpha} * \left(u - \sum_{k=0}^{m-1} u_k g_{k+1}\right)\right].$$

We similarly define the notion of Caputo partial fractional derivative $\mathbf{D}_{t_i}^{\alpha_i} u$ for a continuous function $u \in C([0, +\infty)^n : E)$, where $\alpha_i > 0$ for $1 \leq i \leq n$. Finally, if $0 < \theta \leq \pi$, then we define $\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \theta\}$.

For more details about the multidimensional fractional calculus and its applications, we refer the reader to [19] and references quoted therein. For basic information concerning multivalued linear operators in sequentially complete locally convex spaces and (a, k)-regularized C-resolvent solution operator families subgenerated by them, we refer the reader to [15].

2. Topologically Mixing Solutions of Abstract n-Parameter Fractional Differential Cauchy Inclusions (FP) and (P)

Let $n \in \mathbb{N}$, let A_i and B_i be closed linear operators on E, and let $\alpha_i > 0$ $(1 \leq i \leq n)$. In this section, we analyze the existence of (strong) topologically mixing solutions to the n-parameter abstract fractional Sobolev problems $(\mathrm{FP})_R$, $(\mathrm{FP})_L$ and (P). By (FP) we denote either $(\mathrm{FP})_R$ or $(\mathrm{FP})_L$.

By a strong solution of $(\mathrm{FP})_L$ we mean any function $u \in \mathcal{C}([0, +\infty)^n : E)$ such that $B_i \mathbf{D}_{t_i}^{\alpha_i} u(t) = A_i u(t)$, $t \in [0, +\infty)^n$, $1 \leq i \leq n$, the initial values in $(\mathrm{FP})_L$ hold and the functions $t \mapsto A_i u(t)$, $t \in [0, +\infty)^n$ are continuous for $1 \leq i \leq n$. By Z_L we denote the collection of all initial values $x \in \bigcap_{i \in \mathbb{N}_n} D(A_i)$ for which there exists a strong solution of $(\mathrm{FP})_L$; we similarly define the notion of a strong solution of

$(FP)_R$ and the space Z_R; the abbreviation Z stands for Z_L or Z_R (without going into further details, we will only point out here that our main results can be also formulated for the n-parameter abstract fractional Sobolev problems in which the equations $\mathbf{D}_{t_i}^{\alpha_i} B_i u(t) = A_i u(t)$, $t \in [0,+\infty)^n$ and $B_i \mathbf{D}_{t_i}^{\alpha_i} u(t) = A_i u(t)$, $t \in [0,+\infty)^n$ are mixed).

Now we would like to introduce the following notion:

Definition 2.1. *Suppose that $\tilde{E}$ is a closed linear subspace of E. Then we say that the problem (FP) is $\tilde{E}$-strongly topologically mixing if for any pair of open non-empty subsets U, V of $\tilde{E}$ there exists a positive real number $M > 0$ such that, for every point $t = (t_1, ..., t_n) \in [0,+\infty)^n$ such that $t_1 + ... + t_n \geq M$, there exists an element $x_t \in U \cap Z$ such that the mapping $s \mapsto u(s; x_t)$, $s = (s_1, ..., s_n) \in [0,+\infty)^n$ is continuous and $u(t; x_t) \in V \cap Z$. Here, $u(\cdot; x_t)$ denotes a strong solution of problem (FP) with the initial value $x = x_t$.*

We introduce the notion of $\tilde{E}$-topologically mixing property of problem (FP) in the same way as above, by replacing the term "$t_1 + ... + t_n \geq M$" with the term "$t_1 \geq M, ..., t_n \geq M$". We omit the term "$\tilde{E}$-" from the notation if $\tilde{E} = E$.

From the practical point of view (cf. also [15, p. 225]), it is much better to use the space Z_L in Definition 2.1 than the space $Z_{L:uniq}$ consisting of all initial values $x \in \bigcap_{i \in \mathbb{N}_n} D(A_i)$ for which there exists a unique strong solution of $(FP)_L$; a similar comment can be given for the problem Z_R.

Now we are ready to state the following result:

Theorem 1. *Suppose that $\alpha_i \in (0,2)$ for $1 \leq i \leq n$, Ω is an open connected subset of $\mathbb{C}$, $f: \Omega \to E \setminus \{0\}$ is an analytic mapping, $g_i : R(f) \to \mathbb{C} \setminus \{0\}$ and $A_i f(\lambda) = g_i(f(\lambda)) B_i f(\lambda)$ for all $\lambda \in \Omega$ and $1 \leq i \leq n$. Set $\tilde{E} := \overline{span\{f(\lambda) : \lambda \in \Omega\}}$. If there exist two non-empty subsets Ω_∞ and Ω_0 of Ω, both of which admit a cluster point in Ω, such that*

$$g_i(f(\lambda)) \in \Sigma_{\alpha_i \pi/2}, \quad \lambda \in \Omega_\infty, \ 1 \leq i \leq n \tag{2.1}$$

and

$$-g_i(f(\lambda)) \in \Sigma_{\pi-(\alpha_i \pi/2)}, \quad \lambda \in \Omega_0, \ 1 \leq i \leq n, \tag{2.2}$$

then the problem (FP) is $\tilde{E}$-topologically mixing; furthermore, if $A_i f(\lambda) \in \tilde{E}$ for all $\lambda \in \Omega$ and $1 \leq i \leq n$, then the problem $(FP)_{\tilde{E}}$, obtained by replacing the operators A_i and B_i in (FP) with the operators $A_{i|\tilde{E}}$ and $B_{i|\tilde{E}}$, respectively, is topologically mixing in the Fréchet space $\tilde{E}$.

Assuming additionally that $\alpha_i = 1$ for all $i \in \mathbb{N}_n$ as well as that there exists $\epsilon > 0$ such that $\Re(g_i(f(\lambda))) \leq -\epsilon$ for all $\lambda \in \Omega_0$ and $1 \leq i \leq n$ as well as $\Re(g_i(f(\lambda))) \geq \epsilon$ for all $\lambda \in \Omega_\infty$ and $1 \leq i \leq n$, then the problem (FP) is $\tilde{E}$-strongly topologically mixing; furthermore, if $A_i f(\lambda) \in \tilde{E}$ for all $\lambda \in \Omega$ and $1 \leq i \leq n$, then the problem $(FP)_{\tilde{E}}$, obtained by replacing the operators A_i and B_i in (FP) with the operators $A_{i|\tilde{E}}$ and $B_{i|\tilde{E}}$, respectively, is strongly topologically mixing in the Fréchet space $\tilde{E}$.

Proof. If Ω_0 is an arbitrary non-empty subset of Ω which admits a cluster point in Ω, then the (weak) analyticity of mapping $\lambda \mapsto f(\lambda) \in \tilde{E}$, $\lambda \in \Omega$ implies that

$\Psi(\Omega_0) := span\{f(\lambda) : \lambda \in \Omega_0\}$ is dense in $\tilde{E}$. Due to the inclusions (2.1)-(2.2) and the asymptotic expansion formulae for Mittag-Leffler functions (see, e.g., the equations [15, (40)-(42)]), we have:

$$E_{\alpha_i}(g_i(f(\lambda))s^{\alpha_i}) \to \infty,\ s \to +\infty,\ \lambda \in \Omega_\infty,\ 1 \leq i \leq n,\ \text{and}$$
$$E_{\alpha_i}(g_i(f(\lambda))s^{\alpha_i}) \to 0,\ s \to +\infty,\ \lambda \in \Omega_0,\ 1 \leq i \leq n. \tag{2.3}$$

Suppose that $l,\ m \in \mathbb{N}$, $y \in \Psi(\Omega_0)$, $z \in \Psi(\Omega_\infty)$, $y = \sum_{i=1}^{l} \beta_i f(\lambda_i)$, $z = \sum_{j=1}^{m} \gamma_j f(\tilde{\lambda}_j)$, $\beta_i, \gamma_j \in \mathbb{C}$, $\lambda_i \in \Omega_0$ and $\tilde{\lambda}_j \in \Omega_\infty$ for $1 \leq i \leq l$ and $1 \leq j \leq m$. Using (2.3), we get that there exists a real number $s_0 > 0$ such that $\prod_{i=1}^{n} E_{\alpha_i}(g_i(f(\tilde{\lambda}_j))s^{\alpha_i}) \neq 0$, $s \geq s_0$, $1 \leq j \leq m$. If $\min(t_1, ..., t_n) \geq s_0$, then we set

$$z_t := \sum_{j=1}^{m} \frac{\gamma_j}{E_{\alpha_i}(g_i(f(\tilde{\lambda}_j))t_i^{\alpha_i})} f(\tilde{\lambda}_j)$$

and $x_t := y + z_t$. Using [15, Lemma 2.10.2], it follows that $f(\lambda) \in Z$, $\lambda \in \Omega$ and

$$u(t; f(\lambda)) = \left[\prod_{i=1}^{l} E_{\alpha_i}(g_i(f(\lambda))t_i^{\alpha_i})\right] \cdot f(\lambda),\ t = (t_1, ..., t_n) \in [0, +\infty)^n,\ \lambda \in \Omega; \tag{2.4}$$

hence, $\{x_t,\ y,\ z_t\} \subseteq Z \cap \tilde{E}$ provided that $\min(t_1, .., t_n) \geq s_0$ (here, $u(\cdot; x)$ denotes a strong solution of (FP) with the initial value $u(0) = x$). Keeping in mind (2.3), we have $\lim_{\min(t_1,...,t_n)\to+\infty} u(t; y) = 0$, $u(t; z_t) = z$ and $\lim_{\min(t_1,..,t_n)\to+\infty} z_t = 0$; this implies that, for any pair of open non-empty subsets U, V of $\tilde{E}$, there exists a real number $M > s_0$ such that the assumption $\min(t_1, .., t_n) \geq M$ implies $x_t \in U \cap Z$, the mapping $s \mapsto u(s; x_t)$, $s = (s_1, ..., s_n) \in [0, +\infty)^n$ is continuous and $u(t; x_t) \in V \cap Z$. Hence, the problem (FP) is $\tilde{E}$-topologically mixing; if, additionally, $A_i f(\lambda) \in \tilde{E}$ for all $\lambda \in \Omega$ and $1 \leq i \leq n$, then we can repeat verbatim the above arguments in order to see that the problem $(FP)_{\tilde{E}}$, obtained by replacing the operators A_i and B_i in (FP) with the operators $A_{i|\tilde{E}}$ and $B_{i|\tilde{E}}$, respectively, is topologically mixing in the Fréchet space $\tilde{E}$.

Suppose now that $\alpha_i = 1$ for all $i \in \mathbb{N}_n$ and there exists $\epsilon > 0$ such that $\Re(g_i(f(\lambda))) \leq -\epsilon$ for all $\lambda \in \Omega_0$ and $1 \leq i \leq n$ as well as $\Re(g_i(f(\lambda))) \geq \epsilon$ for all $\lambda \in \Omega_0$ and $1 \leq i \leq n$. Then for each $\lambda \in \Omega_0$ we have

$$\left|\prod_{i=1}^{n} E_{\alpha_i}(g_i(f(\lambda))t_i^{\alpha_i})\right| = \left|e^{g_1(f(\lambda))t_1+...+g_n(f(\lambda))t_n}\right| \leq e^{-\epsilon(t_1+...+t_n)},$$

for any $t = (t_1, ..., t_n) \in [0, +\infty)^n$. Furthermore, if $\lambda \in \Omega_\infty$, then we have

$$\left|\prod_{i=1}^{n} E_{\alpha_i}(g_i(f(\lambda))t_i^{\alpha_i})\right| = \left|e^{g_1(f(\lambda))t_1+...+g_n(f(\tilde{\lambda}_j))t_n}\right| \geq e^{\epsilon(t_1+...+t_n)},$$

for any $t = (t_1, ..., t_n) \in [0, +\infty)^n$, which implies by the foregoing arguments that the problem (FP) is $\tilde{E}$-strongly topologically mixing. The remainder of proof is simple and therefore omitted. □

Now we would like to present the following illustrative applications of Theorem 1:

Example 1. *(i) In this part, we consider the abstract n-parameter Cauchy problem (ACP) with $A_1 := A$ being the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on E. Suppose that $\xi \in \mathbb{R}$, $\delta > 0$, $\Omega = B(i\xi, \delta) \subseteq \sigma_p(A)$, $Af(\lambda) = \lambda f(\lambda)$, $\lambda \in \Omega$ and $f : \Omega \to E \setminus \{0\}$ is analytic. Let $\epsilon > 0$ and $\epsilon_j > 0$ $(2 \leq j \leq n)$ satisfy $\epsilon + \max(\epsilon_2, ..., \epsilon_n) < \delta$. Set now $A_j := A + \epsilon_j$ for $2 \leq j \leq n$. Then $g_1(f(\lambda)) = \lambda$, $\lambda \in \Omega$ and $g_j(f(\lambda)) = \lambda + \epsilon_j$, $\lambda \in \Omega$, $2 \leq j \leq n$, so that the abstract n-parameter Cauchy problem (ACP) is $\tilde{E}$-strongly topologically mixing by Theorem 1. For some concrete examples of differential operators which can be used here, we refer the reader to* [14, Example 3.2.37(i), (iii); Example 3.3.12(iii)] *and* [15, pp. 257–258].

(ii) (cf. also [15, Example 2.10.6]*) Suppose that A is a closed linear operator on E such that there exist an open connected subset $\emptyset \neq \Omega' \subseteq \mathbb{C}$ and an analytic mapping $g : \Omega' \to E \setminus \{0\}$ such that $Af(\lambda) = \lambda f(\lambda)$, $\lambda \in \Omega'$. Let $P_i(z)$ and $Q_i(z)$ be non-zero complex polynomials and $0 < \alpha_i < 2$ for $1 \leq i \leq n$. Define $S := \{z \in \mathbb{C} : P_i(z) = 0 \text{ for some } 1 \leq i \leq n\}$, $\Omega := \Omega' \setminus S$, $A_i := Q_i(A)$ and $B_i := P_i(A)$ for $1 \leq i \leq n$. Then Theorem 1 can be successfully applied provided that the requirements of this result hold for functions*

$$g_i(f(\lambda)) := \frac{Q_i(\lambda)}{P_i(\lambda)}, \quad \lambda \in \Omega,\ 1 \leq i \leq n.$$

In particular, we can apply Theorem 1 in the analysis of existence of (subspace) topologically mixing solutions to linearized Benney–Luke system

$$(z_{0,i} - \Delta)\mathbf{D}_{t_i}^{\alpha_i} u(t,x) = \alpha_{0,i}\Delta u(t,x) - \beta_{0,i}\Delta^2 u(t,x), \quad t \in [0,+\infty)^n,\ 1 \leq i \leq n,$$

where $0 < \alpha_i < 2$, $\alpha_{0,i} > 0$, $\beta_{0,i} > 0$, $z_{0,i} \in \mathbb{C}$ for $1 \leq i \leq n$, and the space variable x belongs to some symmetric space of non-compact type or Heckman–Opdam root space. More concretely, we can apply Theorem 1 in the case that $\alpha_{0,i} \sim z_{0,i}$, $\beta_{0,i} \sim 1$ and $\Omega \cap \{\exp(t\alpha_i\pi/2) : t \geq 0\} \neq \emptyset$ for $1 \leq i \leq n$, when we have $g_i(f(\lambda)) \sim \lambda$, $\lambda \in \Omega$ $(1 \leq i \leq n)$.

Before we move ourselves to Subsection 2.1, we will provide two useful observations regarding Theorem 1:

Remark 2.1: It is worth observing that Theorem 1 cannot be applied provided that $\alpha_i = 2$ for some index $1 \leq i \leq n$. Furthermore, if $\alpha_i = 2$ for all indexes $1 \leq i \leq n$, then we cannot so easily transform the problem (FP) into an equivalent first-order problem on the product space $E \times E$, as in the case that $n = 1$. It would be very tempting to find some applications of Theorem 1 with A_i and B_i being matricial operators on product space $E \times E \times ... \times E$ $(1 \leq i \leq n)$; cf. [15, Remark 2.10.4, Theorem 2.10.24] for more details in this direction.

Remark 2.2: It is worth noting that for each $\lambda \in \Omega$ the function $u(\cdot; f(\lambda))$, given by (2.4), is also a solution of the following partial fractional differential equation with Caputo derivatives:

$$\begin{aligned} &B_1B_2 \cdot ... \cdot B_n \mathbf{D}_C^{\alpha_1}\mathbf{D}_C^{\alpha_2} \cdot ... \cdot \mathbf{D}_C^{\alpha_n} u(t_1, t_2, ..., t_n) \\ &= A_1A_2 \cdot ... \cdot A_n u(t_1, t_2, ..., t_n),\ t = (t_1, t_2, ..., t_n) \in [0,+\infty)^n. \end{aligned} \tag{2.5}$$

For simplicity, we will consider the problem (2.5) without any initial conditions (cf. [19, Subsection 6.1] and the first application made in [20, Subsection 2.2] for more details about this issue). If we introduce the notion of $\tilde{E}$-(strong) topologically mixing property of problem (2.5) in the same way as in Definition 2.1, then Theorem 1 continues to hold provided that condition $A_i f(\lambda) = g_i(f(\lambda))B_i f(\lambda)$ for all $\lambda \in \Omega$ and $1 \leq i \leq n$ is replaced with condition $B_1 B_2 \cdot ... \cdot B_n[g_1(f(\lambda))g_2(f(\lambda)) \cdot ... \cdot g_n(f(\lambda))f(\lambda)] = A_1 A_2 \cdot ... \cdot A_n f(\lambda)$ for all $\lambda \in \Omega$.

2.1. **Topologically Mixing Solutions of Abstract n-Parameter Fractional Differential Cauchy Inclusion (P).** In this subsection, we will briefly analyze the existence of $\tilde{E}$-(strongly) topologically mixing solutions to abstract n-parameter fractional differential Cauchy inclusion (P). We introduce the notion of a $\tilde{E}$-(strong) topologically mixing property of inclusion (P) in the same way as in Definition 2.1. Here, by a solution of (P), we mean any function $u \in \mathcal{C}([0,+\infty)^n : E)$ such that $\mathbf{D}_{t_i}^{\alpha_i} u(t) \in \mathcal{A}_i u(t)$, $t \in [0,+\infty)^n$, $1 \leq i \leq n$ and the initial conditions in (P) hold; if, additionally, for each $i \in \mathbb{N}_n$ there exists a continuous function $u_{\mathcal{A}_i} \in C([0,+\infty)^n : E)$ such that $u_{\mathcal{A}_i}(t) \in \mathcal{A}_i u(t)$ for all $t \in [0,+\infty)^n$, then we say that $u(\cdot)$ is a strong solution of (P). By Z we denote the collection of all initial values $x \in \bigcap_{i \in \mathbb{N}_n} D(\mathcal{A}_i)$ for which there exists a strong solution of (P).

Now we will introduce the following notion:

Definition 2.2. *Suppose that $n \in \mathbb{N}$, $\tilde{E}$ is a closed linear subspace of E, the operator family $(R(t))_{t \in [0,+\infty)^n} \subseteq L(E)$ is given by* (1.1)*, where $(R_i(t_i))_{t_i \geq 0} \subseteq L(E)$ is strongly continuous and $R_i(0) = C_i$ is injective for $1 \leq i \leq n$. Then we say that $(R(t))_{t \in [0,+\infty)^n}$ is $\tilde{E}$-strongly topologically mixing if for any pair of open non-empty subsets U, V of $\tilde{E}$ there exists a positive real number $M > 0$ such that, for every point $t = (t_1, ..., t_n) \in [0,+\infty)^n$ with $t_1 + ... + t_n \geq M$, there exists an element $x_t \in U$ such that the mapping $s \mapsto C_1^{-1} R_1(s_1) \cdot ... \cdot C_n^{-1} R_n(s_n) x_t$, $s = (s_1, ..., s_n) \in [0,+\infty)^n$ is continuous and $C_1^{-1} R_1(t_1) \cdot ... \cdot C_n^{-1} R_n(t_n) x_t \in V$.*

We introduce the notion of $\tilde{E}$-topologically mixing property for $(R(t))_{t \in [0,+\infty)^n}$ in the same way as above, by replacing the term "$t_1 + ... + t_n \geq M$" in the above definition with the term "$t_1 \geq M, ..., t_n \geq M$". Finally, we omit the term "$\tilde{E}$-" from the notation if $\tilde{E} = E$.

It is worth noting that many important examples of strong topologically mixing multiparameter semigroups of operators and topologically mixing fractional solution operator families of form (1.1) can be constructed similarly as in [14, Example 3.3.12]; cf. also the last paragraph on [14, p. 412]. The possible applications in the study of existence of $\tilde{E}$-(strongly) topologically mixing solutions to problem (P) can be simply given by using Lemma 1 and Theorem 2 below, which extends the statement of [15, Theorem 3.6.3] to the multidimensional setting (the proof is similar to that of Theorem 1 and therefore omitted):

Theorem 2. *Suppose that $\alpha_i \in (0,2)$ for $1 \leq i \leq n$, Ω is an open connected subset of $\mathbb{C}$, $f : \Omega \to E \setminus \{0\}$ is an analytic mapping, $g_i : R(f) \to \mathbb{C} \setminus \{0\}$ and $g_i(f(\lambda))f(\lambda) \in \mathcal{A}_i f(\lambda)$ for all $\lambda \in \Omega$ and $1 \leq i \leq n$. Set $\tilde{E} := \overline{span\{f(\lambda) : \lambda \in \Omega\}}$. If there exist*

two non-empty subsets Ω_∞ and Ω_0 of Ω, both of which admit a cluster point in Ω, such that the inclusions (2.1)-(2.2) *hold, then the problem (P) is $\tilde{E}$-topologically mixing; furthermore, if $\mathcal{A}_i f(\lambda) \in \tilde{E}$ for all $\lambda \in \Omega$ and $1 \leq i \leq n$, then the problem $(P)_{\tilde{E}}$, obtained by replacing the multivalued linear operators $\mathcal{A}_i$ in (P) with the operators $\mathcal{A}_{i|\tilde{E}}$ for $1 \leq i \leq n$, respectively, is topologically mixing in the Fréchet space $\tilde{E}$.*

Assuming additionally that $\alpha_i = 1$ for all $i \in \mathbb{N}_n$ as well as that there exists $\epsilon > 0$ such that $\Re(g_i(f(\lambda))) \leq -\epsilon$ for all $\lambda \in \Omega_0$ and $1 \leq i \leq n$ as well as $\Re(g_i(f(\lambda))) \geq \epsilon$ for all $\lambda \in \Omega_\infty$ and $1 \leq i \leq n$, then the problem (P) is $\tilde{E}$-strongly topologically mixing; furthermore, if $\mathcal{A}_i f(\lambda) \in \tilde{E}$ for all $\lambda \in \Omega$ and $1 \leq i \leq n$, then the problem $(P)_{\tilde{E}}$, obtained by replacing the operator $\mathcal{A}_i$ in (P) with the operators $\mathcal{A}_{i|\tilde{E}}$, respectively, is strongly topologically mixing in the Fréchet space $\tilde{E}$.

We close this section by providing the following illustrative application of Theorem 2:

Example 2. *(cf. also* [15, Example 3.6.13(i)]*) Let $E := C^2(\mathbb{R})$ be equipped with the usual Fréchet topology, $0 < \alpha_i < 2$ for $1 \leq i \leq n$, $m \in C(\mathbb{R})$ and $m(x) > 0$, $x \in \mathbb{R}$. If $\lambda \in \mathbb{C}$, then we denote by $\{f_\lambda^1(x), f_\lambda^2(x)\}$ the fundamental set of solutions of equation $y'' = \lambda m(x) y$. Then we know that, for every open connected subset Ω of $\mathbb{C} \setminus (-\infty, 0]$ such that $\Omega \cap \{e^{\pm it\alpha_1/2} : t \geq 0\} \neq \emptyset$, the mappings $\lambda \mapsto f_\lambda^1(x) \in E$ and $\lambda \mapsto f_\lambda^2(x) \in E$ are analytic. Let Ω be such a set, and let additionally $\Omega \cap \{e^{\pm it\alpha_i/2} : t \geq 0\} \neq \emptyset$ for all $2 \leq i \leq n$. Set $\tilde{E} := \overline{span}\{f_\lambda^i(x) : \lambda \in \Omega,\ i = 1, 2\}$. Arguing as in Example 1(i), with the operator A replaced therein by the operator $\Delta \cdot m(x)^{-1}$, we can simply prove with the help of Theorem 2 that there exist sufficiently small real numbers $\epsilon_i > 0$, for $2 \leq i \leq n$, such that the following system of abstract partial fractional Poisson heat equations:*

$$\mathbf{D}_{t_1}^{\alpha_1}[m(x)u(t,x)] = \Delta u(t,x), \quad t \in [0,+\infty)^n,\ x \in \mathbb{R};$$
$$\mathbf{D}_{t_i}^{\alpha_i}[m(x)u(t,x)] = \Delta u(t,x) + \epsilon_i m(x)u(t,x), \quad t \in [0,+\infty)^n,\ x \in \mathbb{R},\ 2 \leq i \leq n,$$

endowed with the corresponding boundary conditions, is $\tilde{E}$-topologically mixing.

3. Conclusions and Final Remarks

In this paper, we have investigated topologically mixing properties of abstract partial fractional Cauchy inclusions. We have introduced the notion of strong topologically mixing property in the multidimensional setting and provided several new results and illustrative examples in this direction.

We will analyze topologically dynamical properties of strongly continuous semigroups over topological monoids and topologically mixing properties of abstract multi-term partial fractional differential inclusions in our follow-up research studies. We will also consider the existence and uniqueness of weak solutions to the abstract n-parameter fractional Sobolev problem (FP) and the abstract n-parameter fractional differential Cauchy inclusion (P) somewhere else.

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