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ON A CLASS OF ABELIAN AND TAUBERIAN-TYPE RESULTS FOR QUASIASYMPTOTIC BEHAVIOR OF DISTRIBUTIONS VIA CERTAIN INTEGRAL TRANSFORMS

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Abstract. We review a particular class of Abelian- and Tauberian-type results relating the quasiasymptotic behavior of distributions to the asymptotic behavior of several integral transforms. The discussion covers the shearlet transform, the fractional Stockwell transform, and the short-time linear canonical transform. A unified presentation highlights the common structure of the corresponding Abelian results and the specific features of each transform. Particular attention is given to the interpretation of the associated Tauberian-type statements, with emphasis on the role of their normalization and limiting behavior.

1. INTRODUCTION

Abelian and Tauberian theorems play a fundamental role in asymptotic analysis. Abelian results describe how a prescribed asymptotic or quasiasymptotic behavior of a function or distribution is reflected in the asymptotic behavior of an associated integral transform. Tauberian results address the converse problem and seek conditions under which asymptotic information on the transform determines the asymptotic behavior of the original object. Together, Abelian and Tauberian results provide a natural framework for comparing asymptotic information in the original and transform domains and for understanding to what extent one determines the other. This is particularly relevant in distribution theory, where asymptotic behavior is understood in a weak sense through the action of distributions on test functions, and integral transforms provide a useful tool for its characterization.

Recent studies have investigated the quasiasymptotic behavior of distributions through the asymptotic properties of various integral transforms [1, 3, 4,

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6, 9, 11, 15, 17, 20, 21, 23, 24, 29]. In particular, different classes of Abelian- and Tauberian-type results have been established for the shearlet transform, the fractional Stockwell transform, and the short-time linear canonical transform. Although these transforms differ in their analytical structure and parameterization, some of the corresponding quasiasymptotic results exhibit a closely related Abelian-Tauberian pattern. This common structure provides a natural basis for their unified treatment and comparison. The present survey, therefore, focuses on this particular class of results, while the original papers [7, 8, 12] contain additional Abelian- and Tauberian-type statements of a different form. The aim of this paper is to present these results within a unified framework and to clarify the interpretation and scope of the corresponding Tauberian-type statements.

The paper is organized as follows. Section 2 recalls the notation and the basic notions of quasiasymptotic behavior of distributions. Sections 3–5 review the corresponding Abelian- and Tauberian-type results for the shearlet transform, the fractional Stockwell transform, and the short-time linear canonical transform, respectively. The final section summarizes the common features of the results and discusses the corresponding Tauberian-type statements.

2. Preliminaries

2.1. **Notation and spaces.** For $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}_0^n$, we use the standard notation $\mathbf{x}^{\mathbf{m}} = x_1^{m_1} \cdots x_n^{m_n}$, $|\mathbf{m}| = m_1 + \cdots + m_n$, and $\partial_{\mathbf{x}}^{\mathbf{m}} = \partial_{x_1}^{m_1} \cdots \partial_{x_n}^{m_n}$. The Fourier transform (FT) of an integrable function $f \in L^1(\mathbb{R}^n)$ is given by

$$\widehat{f}(\boldsymbol{\xi}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\mathbf{x}\cdot\boldsymbol{\xi}} d\mathbf{x}, \quad \boldsymbol{\xi} \in \mathbb{R}^n.$$

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ consists of all smooth functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that

$$\rho_\nu(\varphi) = \sup_{\substack{\mathbf{x}\in\mathbb{R}^n \\ |\mathbf{m}|\le\nu}} (1 + |\mathbf{x}|)^\nu \, |\partial_{\mathbf{x}}^{\mathbf{m}} \varphi(\mathbf{x})| < \infty, \tag{2.1}$$

for every $\nu \in \mathbb{N}_0$ [14, 25]. The Lizorkin space

$$\mathcal{S}_0(\mathbb{R}^n) = \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \mathbf{x}^{\mathbf{m}} \varphi(\mathbf{x}) d\mathbf{x} = 0, \forall \mathbf{m} \in \mathbb{N}_0^n\}$$

is a closed subspace of $\mathcal{S}(\mathbb{R}^n)$ equipped with the relative topology inherited from $\mathcal{S}(\mathbb{R}^n)$. Its dual $\mathcal{S}_0'(\mathbb{R}^n)$ is the space of Lizorkin distributions and is canonically isomorphic to the quotient of $\mathcal{S}'(\mathbb{R}^n)$ by the space of polynomials [14].

We recall the fractional Lizorkin test space $\mathcal{S}_0^\alpha(\mathbb{R}) = \{\varphi \in \mathcal{S}(\mathbb{R}) : e^{i\frac{c_1}{2}x^2}\varphi(x) \in \mathcal{S}_0(\mathbb{R})\}$, $\alpha \in \mathbb{R} \setminus \pi\mathbb{Z}$, $c_1 = \cot(\alpha)$, which generalizes the classical Lizorkin space by means of the fractional Fourier transform [7]. For $c \in \mathbb{R}$, we recall the shifted vanishing-moment space $\mathcal{S}_c(\mathbb{R}) = \{\varphi \in \mathcal{S}(\mathbb{R}) : e^{icx}\varphi(x) \in \mathcal{S}_0(\mathbb{R})\}$. The notation $(\mathcal{S}_0^\alpha)'(\mathbb{R})$ stands for the dual space of $\mathcal{S}_0^{-\alpha}(\mathbb{R})$, whose elements are called fractional Lizorkin distributions.

All dual spaces are equipped with the strong dual topology [27].

2.2. **Quasiasymptotic analysis of distributions.** These notions will be used in the following sections to formulate the Abelian- and Tauberian-type results associated with the integral transforms under consideration. Throughout this subsection, $\mathcal{A}$ denotes a topological vector space of test functions, and $\mathcal{A}'$ denotes its dual space.

A positive, measurable function L, defined on $(0, A]$ (resp. $[A, \infty)$), where $A > 0$, is said to be slowly varying at the origin (resp. at infinity) if

$$\lim_{\varepsilon\to 0^+} \frac{L(\tau\varepsilon)}{L(\varepsilon)} = 1 \quad \left(\text{resp. } \lim_{\lambda\to\infty} \frac{L(\tau\lambda)}{L(\lambda)} = 1\right), \quad \text{for all } \tau > 0.$$

For a thorough treatment of slowly varying functions, we refer to [26].

We say that a distribution $f \in \mathcal{A}'(\mathbb{R}^n)$ has quasiasymptotic behavior of degree $\sigma \in \mathbb{R}$ at the origin (resp. at infinity) with respect to a slowly varying function at the origin (resp. at infinity) L if there exists a distribution $g \in \mathcal{A}'(\mathbb{R}^n)$ such that for all $\varphi \in \mathcal{A}(\mathbb{R}^n)$

$$\lim_{\varepsilon\to 0^+} \left\langle \frac{f(\varepsilon\mathbf{x})}{\varepsilon^\sigma L(\varepsilon)}, \varphi(\mathbf{x}) \right\rangle_{\mathbf{x}} = \langle g(\mathbf{x}), \varphi(\mathbf{x})\rangle_{\mathbf{x}}$$
$$\left(\text{resp. } \lim_{\lambda\to\infty} \left\langle \frac{f(\lambda\mathbf{x})}{\lambda^\sigma L(\lambda)}, \varphi(\mathbf{x}) \right\rangle_{\mathbf{x}} = \langle g(\mathbf{x}), \varphi(\mathbf{x})\rangle_{\mathbf{x}}\right).$$

We shall use the following convenient notation $f(\varepsilon\mathbf{x}) \sim \varepsilon^\sigma L(\varepsilon) g(\mathbf{x})$ as $\varepsilon \to 0^+$ in $\mathcal{A}'(\mathbb{R}^n)$ (resp. $f(\lambda\mathbf{x}) \sim \lambda^\sigma L(\lambda) g(\mathbf{x})$ as $\lambda \to \infty$ in $\mathcal{A}'(\mathbb{R}^n)$) where the limit is understood in the weak topology of $\mathcal{A}'(\mathbb{R}^n)$. The limit distribution g must be a homogeneous distribution with degree of homogeneity σ, namely, $g(\tau x) = \tau^\sigma g(x)$, for each $\tau > 0$, where the equality is understood in the distributional sense [1, 19]. For a comprehensive treatment of quasiasymptotic analysis, we refer to [5, 19, 28].

For a function h defined on $\mathbb{R}^n$ and $r \neq 0$, we use the notation $h_{1/r}(\mathbf{x}) := h(\mathbf{x}/r)$, $\mathbf{x} \in \mathbb{R}^n$.

3. Quasiasymptotic analysis via the shearlet transform

For $\mathbb{S} := \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^\times = \{(\mathbf{b}, s, a) : \mathbf{b} \in \mathbb{R}^2, s \in \mathbb{R}, a \in \mathbb{R}^\times\}$, where $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$, we recall that the natural range of the shearlet transform acting on $\mathcal{S}_0(\mathbb{R}^2)$, denoted by $\mathcal{S}(\mathbb{S})$, consists of all functions $\Phi \in C^\infty(\mathbb{S})$ such that

$$\rho^{m_1,m_2,k,l}_{r_1,r_2,\nu,q}(\Phi) = \sup_{((b_1,b_2),s,a)\in\mathbb{S}} (1+|b_1|)^{r_1} (1+|b_2|)^{r_2} (1+|s|)^{\nu} \left(|a|+|a|^{-1}\right)^q \left|\partial_a^l \partial_s^k \partial_{b_2}^{m_2} \partial_{b_1}^{m_1} \Phi\left((b_1,b_2), s, a\right)\right| < \infty, \tag{3.1}$$

for all $r_1, r_2, q, \nu, m_1, m_2, k, l \in \mathbb{N}_0$ [2]. It is topologized in the usual way. The notation $\mathcal{S}'(\mathbb{S})$ is used for the dual space of $\mathcal{S}(\mathbb{S})$ [2].

3.1. **The shearlet transform on $\mathcal{S}_0'(\mathbb{R}^2)$.** In this subsection, we recall the definition of the shearlet transform and its synthesis operator, together with the corresponding distributional framework. For a comprehensive overview, we refer to [2, 10, 16].

The shearlet transform of a Lizorkin distribution $f \in \mathcal{S}_0'(\mathbb{R}^2)$ with respect to a window $\psi \in \mathcal{S}_0(\mathbb{R}^2)$ is defined as a smooth function on the variables $(\mathbf{b}, s, a) \in \mathbb{S}$ of at most polynomial growth on $\mathbb{S}$ by ([2], Thm. 5.5)

$$\mathscr{S}_\psi f(\mathbf{b}, s, a) = \left\langle f(\mathbf{x}), \overline{S_{\mathbf{b},s,a}\psi(\mathbf{x})} \right\rangle_{\mathbf{x}}, \tag{3.2}$$

where

$$S_{\mathbf{b},s,a}\psi(\mathbf{x}) := |a|^{-3/4}\psi\left(A_a^{-1}S_s^{-1}\left(\mathbf{x}-\mathbf{b}\right)\right), \quad (\mathbf{b}, s, a) \in \mathbb{S},$$

and $S_s = \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}$ is a shearing matrix, and $A_a = a\begin{pmatrix} 1 & 0 \\ 0 & |a|^{-1/2} \end{pmatrix}$ is a parabolic scaling matrix, for $s \in \mathbb{R}, a \in \mathbb{R}^\times$ [2, 10, 16].

The shearlet synthesis operator $\mathscr{S}_\psi^t\Phi$ of $\Phi \in \mathcal{S}(\mathbb{S})$ with respect to $\psi \in \mathcal{S}_0(\mathbb{R}^2)$ is defined by ([2], Def. 4.1)

$$\mathscr{S}_\psi^t\Phi(\mathbf{x}) = \int_{\mathbb{R}^\times}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\Phi(\mathbf{b}, s, a)S_{\mathbf{b},s,a}\psi(\mathbf{x})\frac{d\mathbf{b}dsda}{|a|^3}, \quad \mathbf{x} \in \mathbb{R}^2.$$

Proposition 3.1. *([2], Theorems 3.1 and 4.2) The shearlet transform $\mathscr{S}_\psi : \mathcal{S}_0(\mathbb{R}^2) \to \mathcal{S}(\mathbb{S})$ and its synthesis operator $\mathscr{S}_\psi^t : \mathcal{S}(\mathbb{S}) \to \mathcal{S}_0(\mathbb{R}^2)$ are continuous linear maps for every window function $\psi \in \mathcal{S}_0(\mathbb{R}^2)$.*

By Prop. 3.1, for $\psi \in \mathcal{S}_0(\mathbb{R}^2)$, we may define the shearlet transform $\mathscr{S}_\psi : \mathcal{S}_0'(\mathbb{R}^2) \to \mathcal{S}'(\mathbb{S})$ by duality ([2], Def. 5.1)

$$\left\langle \mathscr{S}_\psi f, \Phi \right\rangle = \left\langle f, \mathscr{S}_{\overline{\psi}}^t\Phi \right\rangle, \quad f \in \mathcal{S}_0'(\mathbb{R}^2),\ \Phi \in \mathcal{S}(\mathbb{S}), \tag{3.3}$$

and its synthesis operator $\mathscr{S}_\psi^t : \mathcal{S}'(\mathbb{S}) \to \mathcal{S}_0'(\mathbb{R}^2)$ by duality ([2], Def. 5.2)

$$\left\langle \mathscr{S}_\psi^t F, \varphi \right\rangle = \left\langle F, \mathscr{S}_{\overline{\psi}}\varphi \right\rangle, \quad F \in \mathcal{S}'(\mathbb{S}),\ \varphi \in \mathcal{S}_0(\mathbb{R}^2).$$

By Prop. 3.1 and ([27], Corollary of Prop. 19.5), it follows that the shearlet transform $\mathscr{S}_\psi : \mathcal{S}_0'(\mathbb{R}^2) \to \mathcal{S}'(\mathbb{S})$ and its synthesis operator $\mathscr{S}_\psi^t : \mathcal{S}'(\mathbb{S}) \to \mathcal{S}_0'(\mathbb{R}^2)$ are continuous linear operators for every test function $\psi \in \mathcal{S}_0(\mathbb{R}^2)$ ([2], Prop. 5.3).

For $f \in \mathcal{S}_0'(\mathbb{R}^2)$, we have the Parseval identity in the distributional setting, known as the *desingularization formula* ([2], Prop. 5.6)

$$\langle f(\mathbf{x}), \varphi(\mathbf{x})\rangle = \frac{1}{C_{\psi,\eta}}\int_{\mathbb{R}^\times}\int_{\mathbb{R}}\int_{\mathbb{R}^2}\mathscr{S}_\psi f(\mathbf{b}, s, a)\mathscr{S}_{\overline{\eta}}\varphi(\mathbf{b}, s, a)\frac{d\mathbf{b}dsda}{|a|^3}, \quad \varphi \in \mathcal{S}_0(\mathbb{R}^2), \tag{3.4}$$

where $\psi, \eta \in \mathcal{S}_0(\mathbb{R}^2)$ satisfy the admissibility condition, i.e.,

$$C_{\psi,\eta} = (2\pi)^2\int_{\mathbb{R}^2}\overline{\widehat{\psi}(\boldsymbol{\xi})}\widehat{\eta}(\boldsymbol{\xi})|\xi_1|^{-2}d\boldsymbol{\xi}, \quad \boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2,$$

is finite and non-zero.

3.2. **Abelian- and Tauberian-type results through the shearlet transform.** In this subsection, we recall certain Abelian- and Tauberian-type results concerning quasiasymptotic behavior at the origin and at infinity in terms of the asymptotic behavior of the shearlet transform from [8].

We first recall the Abelian-type result at the origin.

Proposition 3.2. *([8], Prop. 3.4) Let L be a slowly varying function at the origin, $\sigma \in \mathbb{R}$ and $f \in \mathcal{S}_0'(\mathbb{R}^2)$. Suppose that $f(\varepsilon \mathbf{x}) \sim \varepsilon^\sigma L(\varepsilon) g(\mathbf{x})$ as $\varepsilon \to 0^+$ in $\mathcal{S}_0'(\mathbb{R}^2)$. Then its shearlet transform with respect to $\psi \in \mathcal{S}_0(\mathbb{R}^2)$ satisfies*

$$\frac{1}{\varepsilon^{\sigma+2} L(\varepsilon)} \mathscr{S}_{\psi_{1/\varepsilon}} f(\varepsilon^2 \mathbf{b}, s, a) \to \mathscr{S}_\psi g(\mathbf{0}, s, a) \quad \text{as} \ \ \varepsilon \to 0^+ \tag{3.5}$$

uniformly on compact subsets of $\mathbb{S}$.

We next recall the corresponding Tauberian-type statement obtained in [8].

Proposition 3.3. *([8], Prop. 3.5) Let L be a slowly varying function at the origin, $\sigma \in \mathbb{R}$, $f \in \mathcal{S}_0'(\mathbb{R}^2)$ and $\psi \in \mathcal{S}_0(\mathbb{R}^2)$ be a function admitting a reconstruction shearlet. Suppose that the following two conditions hold:*

(i) The limit

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^\sigma L(\varepsilon)} \mathscr{S}_{\psi_{1/\varepsilon}} f(\varepsilon^2 \mathbf{b}, s, a) = M_{\mathbf{b},s,a}, \tag{3.6}$$

exists and is finite for all $(\mathbf{b}, s, a) \in \mathbb{S}$, and

(ii) There exist constants l, $m \geq 0$, $k > 2$ and $C > 0$ such that

$$\frac{1}{\varepsilon^\sigma L(\varepsilon)} \left| \mathscr{S}_{\psi_{1/\varepsilon}} f(\varepsilon^2 \mathbf{b}, s, a) \right| \leq C \frac{(1+|s|)^l \left(|a|+|a|^{-1}\right)^m}{(1+|\mathbf{b}|)^k}, \tag{3.7}$$

for all $(\mathbf{b}, s, a) \in \mathbb{S}$ and $0 < \varepsilon < 1$.

Then there exists a homogeneous distribution g such that

$$f(\varepsilon \mathbf{x}) \sim \varepsilon^\sigma L(\varepsilon) g(\mathbf{x}) \quad \text{as} \ \ \varepsilon \to 0^+ \ \ \text{in} \ \ \mathcal{S}_0'(\mathbb{R}^2).$$

Remark 3.1: Under the assumptions (3.6) and (3.7), Proposition 3.3 yields

$$f(\varepsilon \mathbf{x}) \sim \varepsilon^\sigma L(\varepsilon) g(\mathbf{x}) \quad \text{as } \varepsilon \to 0^+ \ \text{ in } \ \mathcal{S}_0'(\mathbb{R}^2).$$

Hence, Proposition 3.2 and relation (3.5) imply

$$\frac{1}{\varepsilon^\sigma L(\varepsilon)} \mathscr{S}_{\psi_{1/\varepsilon}} f(\varepsilon^2 \mathbf{b}, s, a) = \varepsilon^2 \frac{1}{\varepsilon^{\sigma+2} L(\varepsilon)} \mathscr{S}_{\psi_{1/\varepsilon}} f(\varepsilon^2 \mathbf{b}, s, a) \longrightarrow 0 \quad \text{as} \ \ \varepsilon \to 0^+,$$

for all $(\mathbf{b}, s, a) \in \mathbb{S}$. Consequently, the limiting function $M_{\mathbf{b},s,a}$ in (3.6) is necessarily zero.

Analogous arguments yield the following Abelian-type result for quasiasymptotics at infinity.

Proposition 3.4. *([8], Prop. 3.6) Let L be a slowly varying function at infinity, $\sigma \in \mathbb{R}$ and $f \in \mathcal{S}_0'(\mathbb{R}^2)$. Suppose that $f(\lambda \mathbf{x}) \sim \lambda^\sigma L(\lambda) g(\mathbf{x})$ as $\lambda \to \infty$ in $\mathcal{S}_0'(\mathbb{R}^2)$. Then its shearlet transform with respect to $\psi \in \mathcal{S}_0(\mathbb{R}^2)$ satisfies*

$$\frac{1}{\lambda^{\sigma+2} L(\lambda)} \mathscr{S}_{\psi_{1/\lambda}} f(\mathbf{b}, s, a) \to \mathscr{S}_\psi g(\mathbf{0}, s, a) \quad \text{as } \lambda \to \infty \tag{3.8}$$

uniformly on compact subsets of $\mathbb{S}$.

A corresponding Tauberian-type statement was also established in [8].

Proposition 3.5. *([8], Prop. 3.7) Let L be a slowly varying function at infinity, $\sigma \in \mathbb{R}$, $f \in \mathcal{S}_0'(\mathbb{R}^2)$ and $\psi \in \mathcal{S}_0(\mathbb{R}^2)$ be a function admitting a reconstruction shearlet. Suppose that the following two conditions hold:*

(i) The limit

$$\lim_{\lambda \to \infty} \frac{1}{\lambda^{\sigma+4} L(\lambda)} \mathscr{S}_{\psi_{1/\lambda}} f(\mathbf{b}, s, a) = M_{\mathbf{b},s,a}, \tag{3.9}$$

exists and is finite for all $(\mathbf{b}, s, a) \in \mathbb{S}$, and

(ii) There exist constants l, $m \geq 0$, $k > 2$ and $C > 0$ such that

$$\frac{1}{\lambda^{\sigma+4} L(\lambda)} \left| \mathscr{S}_{\psi_{1/\lambda}} f(\mathbf{b}, s, a) \right| \leq C \frac{(1 + |s|)^l \left(|a| + |a|^{-1} \right)^m}{(1 + |\mathbf{b}|)^k}, \tag{3.10}$$

for all $(\mathbf{b}, s, a) \in \mathbb{S}$ and $\lambda > 1$.

Then there exists a homogeneous distribution g such that

$$f(\lambda \mathbf{x}) \sim \lambda^\sigma L(\lambda) g(\mathbf{x}) \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}_0'(\mathbb{R}^2).$$

Remark 3.2: If the conclusion of Proposition 3.5 holds, then Proposition 3.4 gives

$$\frac{1}{\lambda^{\sigma+2} L(\lambda)} \mathscr{S}_{\psi_{1/\lambda}} f(\mathbf{b}, s, a) \longrightarrow \mathscr{S}_\psi g(\mathbf{0}, s, a).$$

Therefore,

$$\frac{1}{\lambda^{\sigma+4} L(\lambda)} \mathscr{S}_{\psi_{1/\lambda}} f(\mathbf{b}, s, a) = \frac{1}{\lambda^2} \frac{1}{\lambda^{\sigma+2} L(\lambda)} \mathscr{S}_{\psi_{1/\lambda}} f(\mathbf{b}, s, a) \longrightarrow 0 \quad \text{as} \quad \lambda \to \infty,$$

for all $(\mathbf{b}, s, a) \in \mathbb{S}$. Hence, the limiting function $M_{\mathbf{b},s,a}$ in (3.9) is necessarily zero.

4. Quasiasymptotic analysis via the fractional Stockwell transform

For $\mathbb{Y} = \mathbb{R} \times \mathbb{R}^\times$, we recall that the natural range of the fractional Stockwell transform (FRST) acting on $\mathcal{S}_0^\alpha(\mathbb{R})$, denoted by $\mathcal{S}(\mathbb{Y})$, consists of all functions $\Phi \in C^\infty(\mathbb{Y})$ such that for all $q, r, m, l \in \mathbb{N}_0$

$$\rho_{q,r}^{l,m}(\Phi) = \sup_{(b,a) \in \mathbb{Y}} (1 + |b|)^r \left(|a|^q + |a|^{-q} \right) \left| \frac{\partial^l}{\partial a^l} \frac{\partial^m}{\partial b^m} \left(\Phi(b, a) \right) \right| < \infty.$$

It is topologized in the usual way. The notation $\mathcal{S}'(\mathbb{Y})$ is used for the dual space of $\mathcal{S}(\mathbb{Y})$ [7, 22].

4.1. **The fractional Stockwell transform.** We briefly present the distributional framework for the fractional Stockwell transform (FRST), together with its definition and synthesis operator. For further details, we refer to [7, 30].

The FRST of a fractional Lizorkin distribution $f \in (\mathcal{S}_0^\alpha)'(\mathbb{R})$ with respect to a window function $\psi \in \mathcal{S}_{c_2}(\mathbb{R})$ is defined as a smooth function on the variables $(b,a) \in \mathbb{Y}$ of at most polynomial growth on $\mathbb{Y}$ by ([7], Prop. 7)

$$S_\psi^\alpha f(b,a) := \left\langle f(x), \psi_{\alpha,b,a}(x) \right\rangle_{\mathbf{x}}, \tag{4.1}$$

where $c_1 = \cot(\alpha),\ c_2 = \csc(\alpha)$, and

$$\psi_{\alpha,b,a}(x) = (2\pi)^{-1/2}|a|\psi(a(x-b))e^{i\left(axc_2 - \frac{c_1}{2}(x^2-b^2)\right)},\ (b,a) \in \mathbb{Y},\ x \in \mathbb{R}$$

([30], Rel. 10). The fractional Stockwell synthesis operator $(S_\psi^\alpha)^*\Phi$ of $\Phi \in \mathcal{S}(\mathbb{Y})$ with respect to a window function $\psi \in \mathcal{S}_{c_2}(\mathbb{R})$ is defined by ([7], Def. 2)

$$(S_\psi^\alpha)^*\Phi(x) := \int_{\mathbb{R}^\times}\int_{\mathbb{R}} \Phi(b,a)\psi_{\alpha,b,a}(x)|a|^{-1}dbda,\ \ x \in \mathbb{R}. \tag{4.2}$$

Proposition 4.1. *([7], Theorems 3 and 4) The FRST $S_\psi^\alpha : \mathcal{S}_0^\alpha(\mathbb{R}) \to \mathcal{S}(\mathbb{Y})$ and its synthesis operator $(S_\psi^\alpha)^* : \mathcal{S}(\mathbb{Y}) \to \mathcal{S}_0^\alpha(\mathbb{R})$ are continuous linear maps for every window function $\psi \in \mathcal{S}_{c_2}(\mathbb{R})$.*

By Prop. 4.1, for $\psi \in \mathcal{S}_{c_2}(\mathbb{R})$, we may define the FRST $S_\psi^\alpha : (\mathcal{S}_0^\alpha)'(\mathbb{R}) \to \mathcal{S}'(\mathbb{Y})$ by duality ([7], Def. 3)

$$\left\langle S_\psi^\alpha f, \Phi \right\rangle := \left\langle f, \overline{(S_\psi^\alpha)^*\overline{\Phi}} \right\rangle, \quad f \in (\mathcal{S}_0^\alpha)'(\mathbb{R}), \Phi \in \mathcal{S}(\mathbb{Y}), \tag{4.3}$$

and its synthesis operator $(S_\psi^\alpha)^* : \mathcal{S}'(\mathbb{Y}) \to (\mathcal{S}_0^\alpha)'(\mathbb{R})$ by duality ([7], Def. 4)

$$\left\langle (S_\psi^\alpha)^* F, \varphi \right\rangle := \left\langle F, \overline{S_\psi^\alpha \overline{\varphi}} \right\rangle, \quad F \in \mathcal{S}'(\mathbb{Y}),\ \varphi \in \mathcal{S}_0^{-\alpha}(\mathbb{R}).$$

By Prop. 4.1 and ([27], Corollary of Prop. 19.5), it follows that the FRST $S_\psi^\alpha : (\mathcal{S}_0^\alpha)'(\mathbb{R}) \to \mathcal{S}'(\mathbb{Y})$ and its synthesis operator $(S_\psi^\alpha)^* : \mathcal{S}'(\mathbb{Y}) \to (\mathcal{S}_0^\alpha)'(\mathbb{R})$ are continuous linear maps for every window function $\psi \in \mathcal{S}_{c_2}(\mathbb{R})$ ([7], Prop. 5).

For $f \in (\mathcal{S}_0^\alpha)'(\mathbb{R})$, we have the following *desingularization formula* ([7], Cor. 8)

$$\langle f, \varphi \rangle = \frac{1}{K_{\psi,\eta}^\alpha}\int_{\mathbb{R}^\times}\int_{\mathbb{R}} S_\psi^\alpha f(b,a)\overline{S_\eta^\alpha \overline{\varphi}(b,a)}|a|^{-1}dbda, \quad \varphi \in \mathcal{S}_0^{-\alpha}(\mathbb{R}), \tag{4.4}$$

where $\eta \in \mathcal{S}_{c_2}(\mathbb{R})$ is a reconstruction window for $\psi \in \mathcal{S}_{c_2}(\mathbb{R}) \setminus \{0\}$, i.e.,

$$K_{\psi,\eta}^\alpha = \int_{\mathbb{R}} \overline{\widehat{\psi}(\xi - c_2)}\widehat{\eta}(\xi - c_2)|\xi|^{-1}d\xi$$

is finite and non-zero.

4.2. **Abelian- and Tauberian-type results through the FRST.** This subsection reviews a selected set of Abelian- and Tauberian-type results for the FRST obtained in [7], concerning quasiasymptotic behavior at the origin and at infinity.

We first consider the Abelian-type result corresponding to quasiasymptotic behavior at the origin.

Proposition 4.2. *([7], Prop. 13) Let $f \in \mathcal{S}'(\mathbb{R})$ and $\psi \in \mathcal{S}_{c_2}(\mathbb{R})$. If $e^{i\frac{c_1}{2}(\varepsilon x)^2} f(\varepsilon x) \in \mathcal{S}'(\mathbb{R})$ has quasiasymptotic behavior $e^{i\frac{c_1}{2}(\varepsilon x)^2} f(\varepsilon x) \sim \varepsilon^\sigma L(\varepsilon) g(x)$ as $\varepsilon \to 0^+$ in $\mathcal{S}'_0(\mathbb{R})$, then*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^\sigma L(\varepsilon)} S^\alpha_\psi f\left(\varepsilon^2 b, \frac{a}{\varepsilon}\right) = |c_2|^{-1} S_{\psi_{1/c_2}} g\,(0, ac_2)$$

uniformly for (b, a) in compact subsets of $\mathbb{Y}$.

We next recall the corresponding Tauberian-type statement obtained in [7].

Proposition 4.3. *([7], Thm. 14) Let $\psi \in \mathcal{S}_{c_2}(\mathbb{R}) \setminus \{0\}$ and $f \in (\mathcal{S}^\alpha_0)'(\mathbb{R})$. Suppose that the following two conditions hold:*

i) The limit

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{\sigma-1} L(\varepsilon)} S^\alpha_\psi f\left(\varepsilon^2 b, \frac{a}{\varepsilon}\right) = M_{b,a} \tag{4.5}$$

exists and is finite for all $(b, a) \in \mathbb{Y}$, and

ii) There exist constants $s \geq 0, r > 1$ and $C > 0$ such that

$$\frac{1}{\varepsilon^{\sigma-1} L(\varepsilon)} \left| S^\alpha_\psi f\left(\varepsilon^2 b, \frac{a}{\varepsilon}\right)\right| \leq C(1 + |b|^2)^{-r/2}(|a|^s + |a|^{-s}) \tag{4.6}$$

for all $(b, a) \in \mathbb{Y}$ and $0 < \varepsilon < 1$.

Then, there exists a homogeneous distribution $g \in \mathcal{S}'_0(\mathbb{R})$ such that $e^{i\frac{c_1}{2}(\varepsilon x)^2} f(\varepsilon x) \sim \varepsilon^\sigma L(\varepsilon) g(x)$ as $\varepsilon \to 0^+$ in $\mathcal{S}'_0(\mathbb{R})$.

Analogous arguments yield the following Abelian-type result for quasiasymptotics at infinity.

Proposition 4.4. *([7], Prop. 15) Let $f \in \mathcal{S}'(\mathbb{R})$ and $\psi \in \mathcal{S}_{c_2}(\mathbb{R})$. If $e^{i\frac{c_1}{2}(\lambda x)^2} f(\lambda x) \in \mathcal{S}'(\mathbb{R})$ has quasiasymptotic behavior $e^{i\frac{c_1}{2}(\lambda x)^2} f(\lambda x) \sim \lambda^\sigma L(\lambda) g(x)$ as $\lambda \to \infty$ in $\mathcal{S}'_0(\mathbb{R})$, then*

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^\sigma L(\lambda)} S^\alpha_\psi f\left(b, \frac{a}{\lambda}\right) = e^{-i\frac{c_1}{2}b^2} |c_2|^{-1} S_{\psi_{1/c_2}} g\,(0, ac_2)$$

uniformly for (b, a) in compact subsets of $\mathbb{Y}$.

The corresponding Tauberian-type statement at infinity was also obtained in [7].

Proposition 4.5. *([7], Thm. 16) Let $\psi \in \mathcal{S}_{c_2}(\mathbb{R}) \setminus \{0\}$ and $f \in (\mathcal{S}^\alpha_0)'(\mathbb{R})$. Suppose that the following two conditions hold:*

i) The limit

$$\lim_{\lambda\to\infty}\frac{1}{\lambda^{\sigma+1}L(\lambda)}S_\psi^\alpha f\left(b,\frac{a}{\lambda}\right)=M_{b,a} \tag{4.7}$$

exists and is finite for all $(b,a)\in\mathbb{Y}$, *and*

ii) There exist constants $s\geq 0, r>1$ *and* $C>0$ *such that*

$$\frac{1}{\lambda^{\sigma+1}L(\lambda)}\left|S_\psi^\alpha f\left(b,\frac{a}{\lambda}\right)\right|\leq C(1+|b|^2)^{-r/2}(|a|^s+|a|^{-s}) \tag{4.8}$$

for all $(b,a)\in\mathbb{Y}$ *and* $\lambda\geq 1$.

Then, there exists a homogeneous distribution $g\in\mathcal{S}_0'(\mathbb{R})$ *such that* $e^{i\frac{c_1}{2}(\lambda x)^2}f(\lambda x)\sim\lambda^\sigma L(\lambda)g(x)$ *as* $\lambda\to\infty$ *in* $\mathcal{S}_0'(\mathbb{R})$.

Remark 4.1: The Tauberian-type statements in Propositions 4.3 and 4.5 exhibit the same feature as in the shearlet case. If the corresponding quasiasymptotic conclusions hold, then Propositions 4.2 and 4.4 imply that the normalized expressions appearing in the Tauberian assumptions differ from the Abelian ones by the factors ε and λ^{-1}, respectively. Hence, in both cases the limiting function $M_{b,a}$ is necessarily zero. Therefore, Propositions 4.3 and 4.5 do not yield nontrivial direct Tauberian converses of the corresponding Abelian results.

Remark 4.2: Related Abelian- and Tauberian-type results for the classical Stockwell transform were obtained in [22]. The same normalization pattern occurs in that setting: under the corresponding quasiasymptotic conclusions, the normalized expressions in the Tauberian statements differ from the Abelian ones by factors tending to zero. Consequently, the associated limiting functions are necessarily zero, so the corresponding Tauberian-type statements do not yield nontrivial direct converses of the Abelian results.

5. Quasiasymptotic analysis via the short-time linear canonical transform

5.1. **The short-time linear canonical transform.** We first present the short-time linear canonical transform (STLCT) and its synthesis operator, together with the distributional framework required for the subsequent analysis. For further details, we refer to [12, 13, 18].

The STLCT of a distribution $f\in\mathcal{S}'(\mathbb{R})$ with respect to a window function $\psi\in\mathcal{S}(\mathbb{R})$ is defined as a smooth function on the variables $(b,a)\in\mathbb{R}\times\mathbb{R}$ of at most polynomial growth on $\mathbb{R}\times\mathbb{R}$ by ([12], Prop. 5)

$$SL_{\psi,A}f(b,a)=\left\langle f(x),\overline{\psi(x-b)}K_A(x,a)\right\rangle_x,\quad (b,a)\in\mathbb{R}\times\mathbb{R}, \tag{5.1}$$

where the kernel $K_A(x,a)$ is given by

$$K_A(x,a)=\frac{1}{\sqrt{i2\pi\beta}}e^{i\frac{1}{2}\left(\frac{\alpha}{\beta}x^2-\frac{2}{\beta}xa+\frac{\delta}{\beta}a^2\right)},\quad (x,a)\in\mathbb{R}\times\mathbb{R},$$

for $\beta\neq 0$ and the matrix $A=(\alpha,\beta;\gamma,\delta)=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\mathbb{R}^{2\times 2}$ with $det(A)=1$.

The short-time linear canonical synthesis operator $SL^*_{\psi,A}\Phi$ of $\Phi \in \mathcal{S}(\mathbb{R}\times\mathbb{R})$ with respect to a window $\psi \in \mathcal{S}(\mathbb{R})$ and the matrix A is defined by ([12], Rel. 3.1)

$$SL^*_{\psi,A}\Phi(x) := \int_{\mathbb{R}}\int_{\mathbb{R}} \Phi(b,a)\psi(x-b)K_{A^{-1}}(a,x)dbda, \quad x \in \mathbb{R}. \tag{5.2}$$

Proposition 5.1. *([12], Theorems 1 and 2) The STLCT $SL_{\psi,A} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}^2)$ and its synthesis operator $SL^*_{\psi,A} : \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}(\mathbb{R})$ are continuous linear maps for every window function $\psi \in \mathcal{S}(\mathbb{R})$.*

By Prop. 5.1, for $\psi \in \mathcal{S}(\mathbb{R})$, we may define the STLCT $SL_{\psi,A} : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R}^2)$ by duality ([12], Def. 1)

$$\left\langle SL_{\psi,A}f, \Phi\right\rangle := \left\langle f, \overline{SL^*_{\psi,A}\overline{\Phi}}\right\rangle, \quad f \in \mathcal{S}'(\mathbb{R}),\ \Phi \in \mathcal{S}(\mathbb{R}^2), \tag{5.3}$$

and its synthesis operator $SL^*_{\psi,A} : \mathcal{S}'(\mathbb{R}^2) \to \mathcal{S}'(\mathbb{R})$ by duality ([12], Def. 2)

$$\left\langle SL^*_{\psi,A}F, \varphi\right\rangle := \left\langle F, \overline{SL_{\psi,A}\overline{\varphi}}\right\rangle, \quad F \in \mathcal{S}'(\mathbb{R}^2),\ \varphi \in \mathcal{S}(\mathbb{R}).$$

By Prop. 5.1 and ([27], Corollary of Prop. 19.5), it follows that the STLCT $SL_{\psi,A} : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R}^2)$ and its synthesis operator $SL^*_{\psi,A} : \mathcal{S}'(\mathbb{R}^2) \to \mathcal{S}'(\mathbb{R})$ are continuous linear maps for every test function $\psi \in \mathcal{S}(\mathbb{R})$ ([12], Prop. 3).

For $f \in \mathcal{S}'(\mathbb{R})$, we have the following *desingularization formula* ([12], Cor. 1)

$$\langle f, \varphi\rangle = \frac{1}{(\eta,\psi)}\int_{\mathbb{R}}\int_{\mathbb{R}} SL_{\psi,A}f(b,a)\overline{SL_{\eta,A}\overline{\varphi}(b,a)}dbda, \quad \varphi \in \mathcal{S}(\mathbb{R}), \tag{5.4}$$

where $\eta \in \mathcal{S}(\mathbb{R})$ is a reconstruction window for $\psi \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$, i.e., $(\eta,\psi) = \int_{\mathbb{R}} \overline{\widehat{\psi}(\xi)}\widehat{\eta}(\xi)d\xi$ is finite and non-zero.

5.2. **Abelian- and Tauberian-type results through the STLCT.** This subsection focuses on the STLCT-based description of quasiasymptotic behavior at the origin, following the results obtained in [12].

We begin with the Abelian-type result, which links the quasiasymptotic behavior of the distribution to the limiting behavior of its STLCT.

Proposition 5.2. *([12], Prop. 9) Let L be a slowly varying function at the origin, $\sigma \in \mathbb{R}$, $f \in \mathcal{S}'(\mathbb{R})$ and $\psi \in \mathcal{S}(\mathbb{R})$. Suppose that $f(\varepsilon x) \sim \varepsilon^\sigma L(\varepsilon)g(x)$ as $\varepsilon \to 0^+$ in $\mathcal{S}'(\mathbb{R})$. Then,*

$$\frac{1}{\varepsilon^{\sigma+1}L(\varepsilon)}e^{-i\frac{\delta}{2\beta}(a/\varepsilon)^2}SL_{\psi_{1/\varepsilon},A}f\left(\varepsilon^2 b, \frac{a}{\varepsilon}\right) \to \beta^{-1/2}\, SL_{\psi,(0,1;-1,0)}g\left(0, \frac{a}{\beta}\right), \tag{5.5}$$

as $\varepsilon \to 0^+$, uniformly for (b,a) in bounded subsets of $\mathbb{R}\times\mathbb{R}$.

A related Tauberian-type statement is also available in [12].

Proposition 5.3. *([12], Prop. 10) Let L be a slowly varying function at the origin, $\sigma \in \mathbb{R}$, $f \in \mathcal{S}'(\mathbb{R})$ and $\psi \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$. Suppose that the following two conditions hold:*

(i) The limit

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^\sigma L(\varepsilon)} e^{-i\frac{\delta}{2\beta}(a/\varepsilon)^2} SL_{\psi_{1/\varepsilon},A} f\left(\varepsilon^2 b, \frac{a}{\varepsilon}\right) = M_{b,a} \tag{5.6}$$

exists and is finite for all $(b,a) \in \mathbb{R}\times\mathbb{R}$, *and*

(ii) There exist constants $l > 1, k \geq 0$ *and* $C > 0$ *such that*

$$\frac{1}{\varepsilon^\sigma L(\varepsilon)} \left| SL_{\psi_{1/\varepsilon},A} f\left(\varepsilon^2 b, \frac{a}{\varepsilon}\right)\right| \leq C(1+|b|)^{-l}(1+|a|)^k \tag{5.7}$$

for all $(b,a) \in \mathbb{R}\times\mathbb{R}$ *and* $0 < \varepsilon < 1$.

Then, there exists a homogeneous distribution $g \in \mathcal{S}'(\mathbb{R})$ *such that* $f(\varepsilon x) \sim \varepsilon^\sigma L(\varepsilon) g(x)$ *as* $\varepsilon \to 0^+$ *in* $\mathcal{S}'(\mathbb{R})$.

Remark 5.1: The same phenomenon as in the shearlet and fractional Stockwell transform settings occurs here. If the quasiasymptotic conclusion of Proposition 5.3 holds, then Proposition 5.2 implies that the normalized expression in the Tauberian assumption is ε times the corresponding Abelian one. Hence, the limiting function is necessarily zero, and Proposition 5.3 does not provide a nontrivial direct Tauberian converse of Proposition 5.2.

Remark 5.2: Similar Abelian- and Tauberian-type results for the short-time Fourier transform were obtained in [21]. In particular, comparing Theorems 3.4 and 3.5 therein shows the same normalization phenomenon: under the corresponding quasiasymptotic conclusion, the normalized expression in the Tauberian assumption is ε^2 times the Abelian-normalized one. Hence, the limiting function $M_{x,\omega}$ is necessarily zero, and the Tauberian-type statement does not yield a nontrivial direct converse of the corresponding Abelian result.

6. Discussion and concluding remarks

The results reviewed in the previous sections concern a particular class of Abelian- and Tauberian-type statements relating the quasiasymptotic behavior of distributions to the asymptotic behavior of the corresponding integral transforms. Additional Abelian- and Tauberian-type results were obtained in the original papers, but the present survey is restricted to this specific family of implications for the shearlet transform, the fractional Stockwell transform, and the short-time linear canonical transform.

Within this framework, the Abelian-type results display a clear common pattern. Starting from a prescribed quasiasymptotic behavior of the underlying distribution, one obtains the corresponding asymptotic behavior of the transform after an appropriate scaling and normalization. Although the precise scaling, normalization, and limiting expressions depend on the transform under consideration, the underlying mechanism is analogous in the three settings considered here.

The corresponding Tauberian-type statements require a more careful interpretation. As observed in the preceding sections, for the particular class of results considered here, the normalization used in the Tauberian statements forces the

associated limiting function to vanish whenever the stated quasiasymptotic conclusion holds. Thus, these statements do not provide nontrivial direct Tauberian converses of the corresponding Abelian relations. The comparison therefore indicates that obtaining nontrivial Tauberian characterizations for the corresponding asymptotic regimes would require an appropriate reformulation and further analysis. These observations may also be useful when formulating analogous Abelian and Tauberian-type results for other integral transforms of distributions.

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