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# EXACT DISCRETE REDUCTIONS FOR NESTED HARDY INEQUALITIES

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**Abstract.** Bouthat and Parisé (arXiv:2606.24044) characterized weighted Hardy inequalities for averages over nested sets and asked three questions, on function spaces beyond $L^p$, on triangular kernel operators, and on sums of integrals over different measure spaces. We answer all three by the same reduction, which shows that the measure-theoretic inequality is equivalent, with the same optimal constant, to a discrete one. This settles the multi-space question and completes the exponent picture for nested averages, the case $0 < q < p$ being resolved for all $1 \leq p < \infty$. For a Banach function space over the whole space, the right discrete object is a quotient functional on sequences, which cannot in general be recovered from the associate norms on the pieces. For direct sums of spaces over the pieces, the reduction leads exactly to these associate norms. Triangular kernels reduce, through an exact dual formula, to their column functionals. These are determined by the local norms alone when the kernel is block-factorized, and Oinarov-type criteria then hold for $1 < p \leq q < \infty$ and $1 < q < p < \infty$.

## 1. INTRODUCTION

The inequalities studied in this note descend from Hilbert's double-series theorem, which asserts that the bilinear form $\sum_{m,n\geq 1} \frac{a_m b_n}{(m+n)}$ is bounded on $\ell^2$. The theorem emerged from Hilbert's work on integral equations [18], and the sharp constant $\pi$ was determined by Schur [33]. Hardy's search for an elementary approach to Hilbert's theorem [15] led him to the inequality that now bears his name,

$$\sum_{n=1}^{\infty} \left( \frac{a_1 + \cdots + a_n}{n} \right)^p \leq \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p \qquad (a_n \geq 0,\ p > 1),$$

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which he stated in [16] with a non-optimal constant. The sharp constant $(p')^p$, where $p' = \dfrac{p}{(p-1)}$, was supplied by Landau in a letter published in [25]. An elegant elementary proof was given by Elliott [13], and the integral version appears in [17]. Much of the ensuing activity unfolded in the first volumes of the *Journal of the London Mathematical Society*. Copson [12] extended the inequality to general weighted means and proved the dual inequality now named after him, and further proofs, identities and variants were contributed in [10, 14, 19, 21]. We refer to Kufner, Maligranda and Persson [22] for a detailed account of this prehistory.

The modern era of the subject is dominated by the two-weight problem, which asks for which pairs of weights the averaging operator is bounded from $L^p$ to $L^q$. The case $p = q$ was solved by Muckenhoupt [27] and the case $p \leq q$ by Bradley [8]. In the case $q < p$, the governing condition is of integral rather than supremum type. According to the account in [34], the range $1 < q < p < \infty$ is due to Maz'ya (see also [26, Section 1.3]) and the range $0 < q < 1 < p < \infty$ to Sinnamon, while a unified proof, which also covers $p = 1$, is given in [34]. Discrete counterparts, which are not formal consequences of the continuous statements, were analyzed by Bennett [1] and by Braverman and Stepanov [9]. The theory extends in two directions. It covers Volterra-type kernel operators under Oinarov's condition [28, 37], and it reaches well beyond the Lebesgue spaces, to weighted Lorentz spaces [11], to Orlicz and modular settings [24], and to general Banach function spaces in the sense of Bennett and Sharpley [2] (for Orlicz duality, see also [32]). The monograph of Kufner, Persson and Samko [23] offers a panorama of the field.

The paper of Bouthat and Parisé [7], to which this note is devoted, emerges from a more recent line of work on *L-matrices*, that is, infinite matrices whose entries are constant along $L$-shaped sets $\{(m,n) : \max(m,n) = k\}$. The prototype is the Hilbert $L$-matrix with entries $\dfrac{1}{\max(m,n)}$. The norms, critical points and spectral properties of these matrices were investigated by Bouthat and Mashreghi [3, 4, 5] and by Štampach [35, 36]. The link with Hardy-type inequalities is immediate. Up to a diagonal term, the operator attached to the Hilbert $L$-matrix is the sum of the Cesàro averaging operator and its adjoint, so that norm bounds for $L$-matrices and Hardy–Copson-type inequalities go hand in hand. Pursuing this line, Bouthat, Mashreghi and the present author established Hardy-type inequalities for $\ell^p$ sequences [6], and the nested-average framework of [7] may be viewed as a far-reaching, measure-theoretic extension of that work. For another recent variation, in which the integration limits themselves are perturbed, see [39].

In [7], a $\sigma$-finite measure space $X$ is partitioned into measurable pieces $(X_n)_{n\geq 1}$, and the nested sets $X^{(n)} := X_1 \cup \ldots \cup X_n$ are formed. The object of study is the weighted inequality (2.1) below. On its left-hand side, averages over the nested sets, taken against a fixed density $m$ and normalized by weights $M_n$, are measured in a weighted $\ell^q$ norm. On its right-hand side, the function is measured in a blockwise weighted $L^p$ norm. For $1 \leq p \leq q \leq \infty$, the inequality holds if and only if a discrete Muckenhoupt–Bradley quantity $\beta$, built from the local norms $w_n = \|m\|_{L^{p'}(X_n)}$, is finite, and the optimal constant satisfies

$\beta \le \rho_{\mathrm{opt}} \le p^{1/q}(p')^{1/p'}\beta$ (Theorem A below). Two routes are developed there, namely a global reduction to a discrete Hardy-type inequality, and a local criterion in the spirit of Broadbent's classical argument [10]. The concluding section of [7] then raises three questions. *Question 1*: does the framework extend beyond the Lebesgue scale, to weighted $L^p$, Lorentz or Orlicz spaces, or to other Banach function spaces? As the authors note, this requires identifying the correct analogue of the quantities $w_n$. *Question 2*: can boundedness criteria be developed for kernel operators with Hardy-type kernels, especially those arising from nested sets or monotone integration regions, where Schur-type tests are known not to capture the full picture? *Question 3*: does the theory still apply when the averages are replaced by sums of integrals over a sequence of possibly different measure spaces?

The purpose of this note is to answer these three questions. Question 3 is answered completely, Question 1 through an exact general reduction, and Question 2 through an exact dual reduction together with a characterization of when it scalarizes. The same idea underlies every answer. Once the weight functions are condensed into a few discrete quantities (a single number $w_k$ per piece in the Lebesgue and direct-sum cases, a quotient functional on sequences in general), the measure-theoretic problem becomes a discrete one, and nothing is lost in the process, not even the optimal constant. The note simply runs this principle through the three questions.

For Question 3, Section 3 observes that the multi-space formulation is an *exact instance* of the original framework. Gluing the spaces into a disjoint union turns one inequality into the other, term by term, so that Theorem A applies unchanged with $w_k = \|m_k\|_{L^{p'}(\mu_k)}$. Section 4 isolates the mechanism behind this. Hölder's extremizer on each piece can be shared by all the averages at once, so the multi-space and discrete optimal constants coincide for every $0 < q \le \infty$. Section 5 uses this to settle the case $0 < q < p$ for all $1 \le p < \infty$. For $1 < p < \infty$ the discrete criterion is transferred from the continuous one of Sinnamon and Stepanov, and for $p = 1$ it is obtained directly, with no loss in the constants. Together with Theorem A, this covers every pair of exponents for the original nested-average operator.

Section 6 then answers Question 1 at two levels. For a Banach function space over the whole space, the discrete inequality is governed by a quotient functional on sequences and the reduction is exact, whereas the associate norms on the pieces cannot carry this information in general. For direct sums of spaces over the pieces, the quotient functional reduces to these associate norms, and the outer exponents become independent of the inner spaces. Section 7 finally turns to Question 2. An exact dual formula reduces a general triangular kernel to its column functionals. For block-factorized kernels these are determined by the matrix of local norms $(w_{n,k})$, and Oinarov-type criteria then hold for $1 < p \le q < \infty$ and $1 < q < p < \infty$. A simple example shows that the rowwise Hölder bound may exceed the true norm by a factor $\sqrt{N}$, and that the local norms alone cannot decide boundedness. What remains open is recorded along the way.

## 2. The setting

Unless stated otherwise, $1 \le p < \infty$ and $p \le q \le \infty$. Throughout, $p'$ denotes the conjugate exponent of $p$ (with $1' = \infty$). The endpoint $p = \infty$, which [7] handle through separate conventions, plays no role in what follows. Following [7], a *sequence of weights* is a sequence of positive real numbers. Let $(X, \Sigma, \mu)$ be a $\sigma$-finite measure space, let $X = \bigcup_{n\ge1} X_n$ be a partition of $X$ into measurable sets, and set $X^{(n)} := X_1 \cup \ldots \cup X_n$. Let $(b_n)_{n\ge1}$ and $(M_n)_{n\ge1}$ be sequences of weights, and let $m\colon X \to [0,\infty)$ be measurable. The main question of [7] asks under which conditions there exists $\rho > 0$ such that

$$\left(\sum_{n=1}^{\infty} b_n \left|\frac{1}{M_n}\int_{X^{(n)}} mf\,\mathrm{d}\mu\right|^q\right)^{1/q} \le \rho\left(\sum_{n=1}^{\infty} b_n \int_{X_n} |f|^p\,\mathrm{d}\mu\right)^{1/p} \tag{2.1}$$

holds for every complex measurable function $f$ on $X$. In words, $f$ is averaged on the left over larger and larger unions of the pieces, while on the right its size is measured piece by piece.

Set $w_n := \|m\|_{L^{p'}(X_n,\mu)} \in [0,\infty]$. The first main result of [7] may then be stated as follows. (We have substituted $u_n = \dfrac{b_n^{1/q}}{M_n}$ and $v_k = \dfrac{b_k^{1/p}}{w_k}$ in the discrete Muckenhoupt–Bradley criterion of [7, Section 2], which goes back to [8, 27], and we use the identity $-p'/p = 1 - p'$.)

**Theorem A** ([7, Section 2])**.** *There exists a finite $\rho$ such that* (2.1) *holds for every measurable $f\colon X \to \mathbb{C}$ if and only if*

$$\beta := \sup_{N\ge1}\left(\sum_{n=N}^{\infty}\frac{b_n}{M_n^q}\right)^{1/q}\left(\sum_{k=1}^{N} b_k^{1-p'} w_k^{p'}\right)^{1/p'} < \infty, \tag{2.2}$$

*with the usual modifications when $p = 1$ or $q = \infty$. Moreover, the least such constant satisfies $\beta \le \rho_{\mathrm{opt}} \le p^{1/q}(p')^{1/p'}\beta \le 2\beta$.*

The quantity $\beta$ has the familiar Muckenhoupt shape. At every splitting point $N$, a tail of the outer weights $(b_n,\ M_n)$ is weighed against a head of the inner weights $(b_k,\ w_k)$. Note that the function $m$ enters only through the local norms $w_n$, a point worth keeping in mind throughout.

## 3. The inequality over a sequence of measure spaces

Consider now, for each $k \ge 1$, a $\sigma$-finite measure space $(X_k, \Sigma_k, \mu_k)$, a measurable function $m_k\colon X_k \to [0,\infty)$, and a measurable function $f_k\colon X_k \to \mathbb{C}$. The third of the questions raised in [7, Section 4] concerns inequalities of the form

$$\left(\sum_{n=1}^{\infty} b_n \left|\frac{1}{M_n}\sum_{k=1}^{n}\int_{X_k} m_k(x) f_k(x)\,\mathrm{d}\mu_k(x)\right|^q\right)^{1/q}$$

$$\leq \rho \left( \sum_{n=1}^{\infty} b_n \int_{X_n} |f_n(x)|^p \, \mathrm{d}\mu_n(x) \right)^{1/p}. \quad (3.1)$$

The underlying observation is elementary. A sum of integrals over different spaces is a single integral over their disjoint union. Gluing the $X_k$ side by side thus turns (3.1) into (2.1) term by term, and nothing is lost in the process, not even the optimal constant. In other words, (3.1) is not a generalization of (2.1) but an instance of it.

**Proposition 3.1.** *Let $X := \bigsqcup_{k\geq 1} X_k$ be the disjoint union of the $X_k$, equipped with the $\sigma$-algebra $\Sigma := \{A \subseteq X : A \cap X_k \in \Sigma_k \text{ for all } k \geq 1\}$ and the measure $\mu(A) := \sum_{k\geq 1} \mu_k(A \cap X_k)$, and define $m\colon X \to [0,\infty)$ by $m|_{X_k} := m_k$. Then:*

*(i) $(X, \Sigma, \mu)$ is a $\sigma$-finite measure space, and $X = \bigcup_{k\geq 1} X_k$ is a partition of $X$ into measurable sets;*

*(ii) $f \mapsto (f|_{X_k})_{k\geq 1}$ is a bijection between measurable functions $f\colon X \to \mathbb{C}$ and sequences $(f_k)_{k\geq 1}$ of measurable functions $f_k\colon X_k \to \mathbb{C}$;*

*(iii) for every $n \geq 1$,*

$$\int_{X^{(n)}} mf \, \mathrm{d}\mu = \sum_{k=1}^{n} \int_{X_k} m_k f_k \, \mathrm{d}\mu_k \qquad \text{and} \qquad \int_{X_n} |f|^p \, \mathrm{d}\mu = \int_{X_n} |f_n|^p \, \mathrm{d}\mu_n,$$

*the first identity holding whenever $mf \in L^1(X^{(n)}, \mu)$.*

*Consequently, inequality* (3.1) *for $((X_k, \mu_k, m_k))_{k\geq 1}$ coincides, optimal constant included, with inequality* (2.1) *for $(X, \mu, m)$, relative to the partition $X = \bigcup_{k\geq 1} X_k$.*

*Proof.* The direct sum of countably many measure spaces is a measure space, and it is $\sigma$-finite when each summand is. This proves (i). Point (ii) is immediate from the definition of $\Sigma$. For (iii), note that $X^{(n)}$ is the disjoint union of $X_1, \ldots, X_n$ and that $\mu|_{X_k} = \mu_k$ for each $k$. The first identity then follows from the additivity of the integral over finitely many disjoint measurable sets, whenever $mf \in L^1(X^{(n)}, \mu)$, and the second follows directly from $\mu|_{X_n} = \mu_n$. The final claim is now a term-by-term identification of the two inequalities. □

**Corollary 3.2.** *Let $1 \leq p < \infty$, $p \leq q \leq \infty$, and set $w_k := \|m_k\|_{L^{p'}(X_k,\mu_k)}$. There exists a finite $\rho$ such that* (3.1) *holds for all sequences $(f_k)_{k\geq 1}$ if and only if*

$$\widetilde{\beta} := \sup_{N\geq 1} \left( \sum_{n=N}^{\infty} \frac{b_n}{M_n^q} \right)^{1/q} \left( \sum_{k=1}^{N} b_k^{1-p'} \|m_k\|_{L^{p'}(\mu_k)}^{p'} \right)^{1/p'} < \infty,$$

*with the usual modifications when $p = 1$ or $q = \infty$, in which case*

$$\widetilde{\beta} \;\leq\; \rho_{\mathrm{opt}} \;\leq\; p^{1/q}(p')^{1/p'}\, \widetilde{\beta} \;\leq\; 2\widetilde{\beta}.$$

*Moreover, the local sufficient conditions of* [7, Section 3] *apply as they stand to* (3.1)*, once $(X, \mu, m)$ is taken to be the disjoint union of Proposition 3.1. In particular, the quantities $w_k$ are the $L^{p'}(\mu_k)$-norms of the functions $m_k$, as anticipated in* [7, Section 4].

*Proof.* Combine Proposition 3.1 with Theorem A applied to $(X, \mu, m)$, noting that $\mu|_{X_k} = \mu_k$, so that the quantities $w_k$ of Theorem A are exactly $\|m_k\|_{L^{p'}(\mu_k)}$. □

**Remark 3.3:** Two consistency checks are in order. If each $X_k = \{k\}$ is an atom of counting measure, then $w_k = m_k(k)$ and (3.1) reduces to the discrete inequality of [7]. If the $(X_k)_{k\geq 1}$ are the pieces of a partition of a single measure space, (3.1) is (2.1) itself.

## 4. Equality of the optimal constants

There are two reasons why condensing each $m_k$ into the single number $w_k$ costs nothing. First, Hölder's inequality, which produces $w_k$, is an equality for a well-chosen function on $X_k$. Second, a single function per piece serves all the averages at once. This section records the mechanism once and for all.

Running the reduction of [7, Section 2] on the disjoint union of Proposition 3.1 leads to the discrete inequality

$$\left(\sum_{n=1}^{\infty} b_n \left|\frac{1}{M_n}\sum_{k=1}^{n} w_k a_k\right|^q\right)^{1/q} \leq C \left(\sum_{k=1}^{\infty} b_k a_k^p\right)^{1/p}, \qquad (a_k)_{k\geq 1} \subseteq [0,\infty). \tag{4.1}$$

We include the short direct proof, both for completeness and because the argument makes no use of the relation $p \leq q$, a fact that the following sections exploit.

**Proposition 4.1.** *Let $1 \leq p < \infty$ and $0 < q \leq \infty$ (no relation between $p$ and $q$ is assumed), and suppose $0 < w_k < \infty$ for every $k \geq 1$, where $w_k = \|m_k\|_{L^{p'}(\mu_k)}$. Then the optimal constants of* (3.1) *and of* (4.1) *are equal.*

*Proof.* Write $\rho_{\mathrm{opt}}$ and $C_{\mathrm{opt}}$ for the two optimal constants.

$\rho_{\mathrm{opt}} \leq C_{\mathrm{opt}}$. Replacing each $f_k$ by $|f_k|$ leaves the right-hand side of (3.1) unchanged and does not decrease the left-hand side, so we may assume $f_k \geq 0$. By Hölder's inequality, $\int_{X_k} m_k f_k \,\mathrm{d}\mu_k \leq w_k \alpha_k$ with $\alpha_k := \|f_k\|_{L^p(\mu_k)}$, and applying (4.1) to $(\alpha_k)_{k\geq 1}$ yields (3.1) with $\rho = C_{\mathrm{opt}}$.

$C_{\mathrm{opt}} \leq \rho_{\mathrm{opt}}$. Assume first that $1 < p < \infty$ and set $g_k := (m_k/w_k)^{p'-1}$. Since $(p'-1)p = p'$,

$$\|g_k\|_{L^p(\mu_k)}^p = w_k^{-p'} \int_{X_k} m_k^{p'} \,\mathrm{d}\mu_k = 1,$$
$$\int_{X_k} m_k g_k \,\mathrm{d}\mu_k = w_k^{1-p'} \|m_k\|_{L^{p'}}^{p'} = w_k,$$

so $g_k$ realizes equality in Hölder's inequality. Given $(a_k)_{k\geq 1} \subseteq [0,\infty)$, the choice $f_k := a_k g_k$ turns the two sides of (3.1) into exactly the two sides of (4.1), and hence $C_{\mathrm{opt}} \leq \rho_{\mathrm{opt}}$. For $p = 1$ (so that $p' = \infty$ and $w_k = \|m_k\|_{L^\infty}$), exact extremizers are replaced by near-extremizers. Fix $\varepsilon \in (0,1)$. By the definition of the essential supremum, the set $\{x \in X_k : m_k(x) > (1-\varepsilon)w_k\}$ has positive measure, and since $\mu_k$ is $\sigma$-finite it contains a measurable subset $E$ with $0 < \mu_k(E) < \infty$. The function $g_{k,\varepsilon} := \chi_E/\mu_k(E)$ then satisfies $\|g_{k,\varepsilon}\|_{L^1} = 1$ and $\int_{X_k} m_k g_{k,\varepsilon} \,\mathrm{d}\mu_k \geq (1-\varepsilon)w_k$. Take $f_k := a_k g_{k,\varepsilon}$. Every inner integral then dominates $(1-\varepsilon)w_k a_k$, so by monotonicity

the left-hand side of (3.1) dominates $(1-\varepsilon)$ times the left-hand side of (4.1) at $(a_k)_{k\geq 1}$, while the right-hand sides coincide. Hence $(1-\varepsilon)C_{\mathrm{opt}} \leq \rho_{\mathrm{opt}}$ for every $\varepsilon \in (0,1)$, and the claim follows. □

**Remark 4.2:** In the present $L^p$-setting, the standing assumption $0 < w_k < \infty$ entails no loss of generality. If $w_k = 0$, then $m_k = 0$ $\mu_k$-a.e., so the $k$-th coordinate does not contribute to the left-hand side of either inequality. In computing the optimal constant of (4.1) one may therefore set $a_k = 0$, and the coordinate may be omitted without changing the optimal constant (its term in the second factor of $\widetilde{\beta}$ vanishes as well). If $w_k = \infty$ for some $k$, then no finite $\rho$ works. Indeed, since $\|m_k\|_{L^{p'}(\mu_k)} = \infty$, there exist $f_k \geq 0$ with $\|f_k\|_{L^p} = 1$ and $\int_{X_k} m_k f_k \,\mathrm{d}\mu_k$ arbitrarily large. For $1 < p < \infty$ this follows from the converse of Hölder's inequality on the $\sigma$-finite space $(X_k, \mu_k)$, and for $p = 1$ from the essential-supremum argument in the proof of Proposition 4.1. Setting all other coordinates to zero then defeats every finite $\rho$, and accordingly $\widetilde{\beta} = \infty$.

**Remark 4.3:** The disjoint-union construction indicates that the first and third questions of [7, Section 4] are two faces of the same problem. Any extension of the framework of [7] to a class of function norms compatible with countable partitions, in the sense of the right-hand side of (2.1) (for instance, norms defined over countable direct sums of Banach function spaces), automatically contains the multi-space formulation (3.1). Conversely, (3.1) contains nothing that (2.1) does not.

## 5. The case $0 < q < p$

Proposition 4.1 makes no use of the relation $p \leq q$. In this section we exploit this fact to settle the boundedness problem for (3.1), and hence, by Remark 3.3, for the nested inequality (2.1) itself, in the entire case $0 < q < p$, $1 \leq p < \infty$, which is not covered by [7]. Theorem 5.2 treats $1 < p < \infty$, and Theorem 5.4 the remaining case $p = 1$, $0 < q < 1$. Until Theorem 5.4, we assume that $1 < p < \infty$ and $0 < q < p$, and we define $r$ by $1/r := 1/q - 1/p$. Then $0 < r < \infty$, and Hölder's inequality takes the form $\|(x_n y_n)\|_{\ell^q} \leq \|(x_n)\|_{\ell^r} \|(y_n)\|_{\ell^p}$.

By Proposition 4.1, the optimal constant of (3.1) equals the optimal constant $C_{\mathrm{d}}$ of the discrete inequality (4.1). Upon setting $u_n := b_n^{1/q}/M_n$, $v_k := b_k^{1/p}/w_k$ and $c_k := w_k a_k$, the latter takes the standard two-weight form

$$\left(\sum_{n=1}^{\infty}\left(u_n \sum_{k=1}^{n} c_k\right)^q\right)^{1/q} \leq C \left(\sum_{k=1}^{\infty}(v_k c_k)^p\right)^{1/p}, \qquad (c_k)_{k\geq 1} \subseteq [0,\infty). \tag{5.1}$$

The continuous analogue of (5.1) in the case $q < p$ is classical. As recalled in the introduction, the range $1 < q < p < \infty$ is due to Maz'ya [26] and the range $0 < q < 1 < p < \infty$ to Sinnamon, and a unified proof, which also covers $p = 1$, is given in [34]. We now state Theorem 2.4 of [34]; see also [23]. Note that our naming of the weights is the reverse of that of [34]. In Theorem B below, $u$ denotes the outer weight (their $v$) and $v$ the inner one (their $u$), and their auxiliary weight

$w = u^{1-p'}$ becomes our $v^{1-p'}$. The passage from the continuous statement to the discrete one is a transference argument with a simple plan. Sequences are encoded as step functions on the half-line. A middle-third computation shows that the discrete and continuous integral conditions agree up to a fixed factor. The inequality is then transferred back, the only loss being a diagonal term, which is estimated separately in one line.

**Theorem B** (Sinnamon–Stepanov [34, Theorem 2.4])**.** *Let $1 < p < \infty$, $0 < q < p$, $1/r = 1/q - 1/p$, and let $u, v$ be nonnegative measurable functions on $(0,\infty)$. The least constant $C \in [0,\infty]$ such that*

$$\left(\int_0^\infty \left(\int_0^x f(t)\,\mathrm{d}t\right)^q u(x)\,\mathrm{d}x\right)^{1/q} \le C\,\left(\int_0^\infty f(x)^p v(x)\,\mathrm{d}x\right)^{1/p}$$

*for all measurable $f \ge 0$ satisfies $c_1\mathcal{B} \le C \le c_2\mathcal{B}$, where*

$$c_1 := (p')^{1/p'} q^{1/p}\left(1-\frac{q}{p}\right) \qquad \text{and} \qquad c_2 := \left(\frac{r}{q}\right)^{1/r} p^{1/p}(p')^{1/p'},$$

*and*

$$\mathcal{B} := \left(\int_0^\infty \left(\int_x^\infty u\right)^{r/p} \left(\int_0^x v^{1-p'}\right)^{r/p'} u(x)\,\mathrm{d}x\right)^{1/r}.$$

*In particular, $C < \infty$ if and only if $\mathcal{B} < \infty$.*

**Proposition 5.1.** *Let $1 < p < \infty$ and $0 < q < p$, and let $(u_n)_{n\ge1}$, $(v_n)_{n\ge1}$ be sequences of weights. Set*

$$U_N := \sum_{n=N}^{\infty} u_n^q, \qquad V_N := \sum_{k=1}^{N} v_k^{-p'} \ \ (V_0 := 0),$$

$$B_{\mathrm{d}} := \left(\sum_{N=1}^{\infty} U_N^{r/p}\, V_N^{r/p'}\, u_N^q\right)^{1/r}.$$

*Then the least constant $C_{\mathrm{d}}$ in* (5.1) *satisfies*

$$3^{-1-1/r} c_1\, B_{\mathrm{d}} \;\le\; C_{\mathrm{d}} \;\le\; \widetilde{c}_2\, B_{\mathrm{d}}, \qquad \text{where } \widetilde{c}_2 := \begin{cases} c_2 + 1, & q \ge 1, \\ (c_2^q + 1)^{1/q}, & 0 < q < 1, \end{cases}$$

*and $c_1, c_2$ are the constants of Theorem B. In particular, $C_{\mathrm{d}} < \infty$ if and only if $B_{\mathrm{d}} < \infty$. (The discrete Hardy inequality in the case $q < p$ is classical, see for instance* [1, 9]*. We include the transference argument in order to keep the constants explicit and the note self-contained.)*

*Proof.* Let $I_n := [n-1, n)$ and define step weights on $(0,\infty)$ by $\widetilde{u} := \sum_{n\ge1} u_n^q \chi_{I_n}$ and $\widetilde{v} := \sum_{n\ge1} v_n^p \chi_{I_n}$, so that $\widetilde{v}^{1-p'} = \sum_{n\ge1} v_n^{-p'}\chi_{I_n}$ because $p(1-p') = -p'$. Write $C_{\mathrm{c}}$ and $\mathcal{B}$ for the quantities of Theorem B associated with $(\widetilde{u}, \widetilde{v})$.

*Step 1: $C_{\mathrm{c}} \le C_{\mathrm{d}}$.* Let $f \ge 0$ be measurable and set $c_k := \int_{I_k} f$. We may assume that every $c_k$ is finite, since otherwise the right-hand side is infinite. For $x \in I_n$ we

have $\int_0^x f \le c_1 + \cdots + c_n$, while $|I_k| = 1$ and Jensen's inequality give $c_k^p \le \int_{I_k} f^p$. Hence

$$\left(\int_0^\infty \left(\int_0^x f\right)^q \widetilde{u}\,\mathrm{d}x\right)^{1/q} \le \left(\sum_n u_n^q \left(\sum_{k\le n} c_k\right)^q\right)^{1/q} \le C_{\mathrm{d}} \left(\sum_k v_k^p c_k^p\right)^{1/p} \le C_{\mathrm{d}} \left(\int_0^\infty f^p \widetilde{v}\right)^{1/p}.$$

*Step 2:* $\mathcal{B} \le B_{\mathrm{d}} \le 3^{1+1/r}\mathcal{B}$. For $x \in I_N$ we have $\int_x^\infty \widetilde{u} \le U_N$ and $\int_0^x \widetilde{v}^{1-p'} \le V_N$, and integrating over each $I_N$ yields $\mathcal{B}^r \le B_{\mathrm{d}}^r$. Conversely, for $x \in [N - \frac{2}{3}, N - \frac{1}{3}] \subseteq I_N$,

$$\int_x^\infty \widetilde{u} = U_{N+1} + u_N^q (N - x) \ge \frac{1}{3} U_N,$$
$$\int_0^x \widetilde{v}^{1-p'} = V_{N-1} + v_N^{-p'} (x - (N-1)) \ge \frac{1}{3} V_N,$$

Since the interval has length $\frac{1}{3}$ and $r/p + r/p' = r$, it follows that

$$\mathcal{B}^r \ge \frac{1}{3} \sum_{N\ge 1} (\tfrac{1}{3} U_N)^{r/p} (\tfrac{1}{3} V_N)^{r/p'} u_N^q = 3^{-1-r} B_{\mathrm{d}}^r.$$

*Step 3: necessity.* If $C_{\mathrm{d}} < \infty$, then by Step 1 and Theorem B,

$$B_{\mathrm{d}} \le 3^{1+1/r}\mathcal{B} \le 3^{1+1/r} c_1^{-1} C_{\mathrm{c}} \le 3^{1+1/r} c_1^{-1} C_{\mathrm{d}}.$$

*Step 4: sufficiency.* Fix $(c_k)_{k\ge 1} \subseteq [0,\infty)$, put $F_n := \sum_{k\le n} c_k$ (with $F_0 := 0$) and $f := \sum_{k\ge 1} c_k \chi_{I_k}$, so that $\int_0^\infty f^p \widetilde{v} = \sum_k (v_k c_k)^p$ and $\int_0^x f \ge F_{n-1}$ for $x \in I_n$. Hence

$$\begin{aligned}\left(\sum_n u_n^q F_{n-1}^q\right)^{1/q} &\le \left(\int_0^\infty \left(\int_0^x f\right)^q \widetilde{u}\,\mathrm{d}x\right)^{1/q} \\ &\le C_{\mathrm{c}} \,\|(v_k c_k)\|_{\ell^p} && \text{(definition of } C_{\mathrm{c}}) \\ &\le c_2\, B_{\mathrm{d}} \,\|(v_k c_k)\|_{\ell^p} && \text{(Theorem B and Step 2).}\end{aligned}$$

Moreover, since $r - q = qr/p$ and $r = p' \cdot r/p'$, we have

$$(u_n/v_n)^r = (u_n^q)^{r/p} (v_n^{-p'})^{r/p'} u_n^q \le U_n^{r/p} V_n^{r/p'} u_n^q,$$

whence $\|(u_n/v_n)\|_{\ell^r} \le B_{\mathrm{d}}$ and, by Hölder's inequality with $1/q = 1/r + 1/p$,

$$\left(\sum_n (u_n c_n)^q\right)^{1/q} \le B_{\mathrm{d}} \,\|(v_k c_k)\|_{\ell^p}.$$

Since $F_n = F_{n-1} + c_n$, the last two displays combine into $C_{\mathrm{d}} \le \widetilde{c}_2 B_{\mathrm{d}}$, by Minkowski's inequality in $\ell^q$ when $q \ge 1$ and by the inequality $\|x+y\|_{\ell^q}^q \le \|x\|_{\ell^q}^q + \|y\|_{\ell^q}^q$ when $0 < q < 1$. □

Combining Propositions 4.1 and 5.1, we obtain the announced characterization.

**Theorem 5.2.** *Let $1 < p < \infty$, $0 < q < p$, $1/r := 1/q - 1/p$, and suppose $0 < w_k := \|m_k\|_{L^{p'}(X_k,\mu_k)} < \infty$ for every $k \geq 1$. Then there exists a finite $\rho$ such that* (3.1) *holds for all sequences $(f_k)_{k\geq 1}$ if and only if*

$$B := \left( \sum_{N=1}^{\infty} \left( \sum_{n=N}^{\infty} \frac{b_n}{M_n^q} \right)^{r/p} \left( \sum_{k=1}^{N} b_k^{1-p'} w_k^{p'} \right)^{r/p'} \frac{b_N}{M_N^q} \right)^{1/r} < \infty,$$

*in which case*

$$3^{-1-1/r} c_1 \, B \;\leq\; \rho_{\mathrm{opt}} \;\leq\; \widetilde{c}_2 \, B,$$

*with $c_1$ and $\widetilde{c}_2$ as in Proposition 5.1. By Remark 3.3, the same statement characterizes the nested inequality* (2.1) *on a single measure space in the case $q < p$, with $w_k = \|m\|_{L^{p'}(X_k,\mu)}$.*

*Proof.* By Proposition 4.1 (valid for every $0 < q \leq \infty$) the optimal constant of (3.1) equals the optimal constant of (5.1) with $u_n = b_n^{1/q}/M_n$ and $v_k = b_k^{1/p}/w_k$, for which $u_n^q = b_n/M_n^q$ and $v_k^{-p'} = b_k^{1-p'} w_k^{p'}$. The conclusion is then Proposition 5.1 in these variables. □

**Remark 5.3:** Formally, as $q \uparrow p$ one has $r \to \infty$, and the $\ell^r$-type quantity $B$ degenerates into the supremum defining $\widetilde{\beta}$ in Corollary 3.2 with $q = p$. The two characterizations are thus consistent along the diagonal.

The endpoint $p = 1$ calls for a separate argument. Theorem B assumes $p > 1$, and in the continuous criterion of Sinnamon and Stepanov for $p = 1$ [34, Theorem 3.3] the integral $\int_0^x v^{1-p'}$, which degenerates as $p \downarrow 1$, is replaced by the lower envelope $\operatorname{ess\,inf}_{0<t<x} v(t)$ of the inner weight. Two features simplify the transference. Since the right-hand side is now an $L^1$-norm, the mass of each $c_k$ may be concentrated near the left endpoint of $I_k$ at no cost, and the discrete and continuous optimal constants coincide exactly. Moreover, for a step weight built from the normalized inner weights $\nu_k = b_k/w_k$, the lower envelope is the running minimum $\underline{\nu}_N = \min_{1\leq k\leq N} \nu_k$, which is the quantity appearing in the criterion below.

**Theorem 5.4.** *Let $p = 1$ and $0 < q < 1$, and suppose $0 < w_k := \|m_k\|_{L^\infty(\mu_k)} < \infty$ for every $k \geq 1$. Set*

$$\alpha_n := \frac{b_n}{M_n^q}, \quad A_N := \sum_{n=N}^{\infty} \alpha_n, \quad \nu_k := \frac{b_k}{w_k}, \quad \underline{\nu}_N := \min_{1\leq k\leq N} \nu_k, \quad s := \frac{q}{1-q},$$

*and*

$$D_1 := \left[ (1-q) \sum_{N=1}^{\infty} \underline{\nu}_N^{-s} \left( A_N^{1/(1-q)} - A_{N+1}^{1/(1-q)} \right) \right]^{1/s},$$

*with $D_1 := \infty$ if some $A_N$ is infinite. Then* (3.1) *admits a finite $\rho$ if and only if $D_1 < \infty$, in which case*

$$q(1-q)\,D_1 \;\le\; \rho_{\mathrm{opt}} \;\le\; (1-q)^{-(1-q)/q}\,D_1.$$

*Proof.* By Proposition 4.1, $\rho_{\mathrm{opt}}$ equals the least constant $C_{\mathrm{d}}$ in

$$\left(\sum_{n=1}^{\infty}\alpha_n\left(\sum_{k=1}^{n}c_k\right)^{q}\right)^{1/q} \le C\sum_{k=1}^{\infty}\nu_k c_k, \qquad (c_k)_{k\ge1}\subseteq[0,\infty),$$

obtained from (5.1) with $p = 1$, $u_n^q = \alpha_n$ and $v_k = \nu_k$. Encode the weights as the step functions $\widetilde{u} := \sum_n \alpha_n \chi_{I_n}$ and $\widetilde{v} := \sum_n \nu_n \chi_{I_n}$ on $(0,\infty)$, with $I_n = [n-1, n)$, and write $C_{\mathrm{c}}$ for the least constant in the continuous inequality $\left(\int_0^\infty (\int_0^x f)^q \widetilde{u}\,\mathrm{d}x\right)^{1/q} \le C\int_0^\infty f\widetilde{v}\,\mathrm{d}x$, over measurable $f \ge 0$.

This time the discrete and continuous constants are *equal*. On the one hand, given $f \ge 0$, the choice $c_k := \int_{I_k} f$ gives $\int_0^x f \le \sum_{k\le n} c_k$ for $x \in I_n$ and $\int_0^\infty f\widetilde{v} = \sum_k \nu_k c_k$, and hence $C_{\mathrm{c}} \le C_{\mathrm{d}}$. On the other hand, given $(c_k)$ and $\varepsilon \in (0,1)$, the spikes $f_\varepsilon := \varepsilon^{-1}\sum_k c_k \chi_{[k-1,\,k-1+\varepsilon)}$ satisfy $\int_0^\infty f_\varepsilon \widetilde{v} = \sum_k \nu_k c_k$, while $\int_0^x f_\varepsilon = \sum_{k\le n} c_k$ for $x \in [n-1+\varepsilon, n)$. Hence, for each $n$, $\int_{I_n}(\int_0^x f_\varepsilon)^q \widetilde{u}\,\mathrm{d}x \to \alpha_n(\sum_{k\le n} c_k)^q$ as $\varepsilon \to 0^+$, and Fatou's lemma applied to the sum over $n$ yields $C_{\mathrm{d}} \le C_{\mathrm{c}}$.

By the $p = 1$ theorem of Sinnamon and Stepanov [34, Theorem 3.3], $C_{\mathrm{c}}$ is controlled from both sides, with the constants displayed above, by the quantity

$$\left[\int_0^\infty \left(\int_x^\infty \widetilde{u}\right)^{s} \widetilde{u}(x)\,\underline{\widetilde{v}}(x)^{-s}\,\mathrm{d}x\right]^{1/s},$$

where, following [34, Definition 3.1], $\underline{\widetilde{v}}(x) := \operatorname{ess\,inf}_{0<t<x}\widetilde{v}(t)$, a nonincreasing function dominated by $\widetilde{v}$ almost everywhere. It remains to identify this quantity with $D_1$. For our step weight, $\underline{\widetilde{v}} = \underline{\nu}_N$ almost everywhere on $I_N$, and $\int_x^\infty \widetilde{u} = A_{N+1} + \alpha_N(N-x)$. Substituting $t = \alpha_N(N-x)$, we obtain

$$\begin{aligned}\int_{I_N}\left(\int_x^\infty \widetilde{u}\right)^{s}\widetilde{u}(x)\,\mathrm{d}x &= \int_0^{\alpha_N}(A_{N+1}+t)^s\,\mathrm{d}t = \frac{A_N^{s+1}-A_{N+1}^{s+1}}{s+1}\\ &= (1-q)\left(A_N^{1/(1-q)} - A_{N+1}^{1/(1-q)}\right),\end{aligned}$$

since $s+1 = 1/(1-q)$. Summing over $N$ gives exactly $D_1^s$. □

**Remark 5.5:** Under the natural right-hand side $\sup_{k\ge1}\|f_k\|_{L^\infty(\mu_k)}$, and with $w_k := \|m_k\|_{L^1(\mu_k)}$, the choice $f_k \equiv 1$ simultaneously maximizes every inner integral. Hence, for every $0 < q \le \infty$,

$$\rho_{\mathrm{opt}} = \left(\sum_{n=1}^{\infty} b_n\left(\frac{1}{M_n}\sum_{k=1}^{n} w_k\right)^{q}\right)^{1/q},$$

with the evident supremum when $q = \infty$, both sides being read in $[0,\infty]$.

**Remark 5.6:** Theorems 5.2 and 5.4, together with Theorem A and the elementary treatment of degenerate coordinates in Remark 4.2, settle the boundedness problem for (3.1) in the entire range $1 \le p < \infty$, $0 < q \le \infty$, and Remark 5.5 disposes of the endpoint $p = \infty$. For the nested-average problem considered here, what remains open in the case $q < p$ is the finer constant problem. The two-sided bounds above are inherited from Theorem B and from [34], and they are far from optimal. It would also be of interest to develop local criteria in the spirit of [7, Section 3] adapted to this case.

## 6. Beyond $L^p$: inner Banach function spaces

The first question of [7, Section 4] asks whether the framework extends beyond $L^p$ to weighted $L^p$, Lorentz, Orlicz, or other Banach function spaces, and observes that this requires identifying "the correct analogue of the quantities $w_n$". In this section we answer this question at two levels. At the global level, where $E$ is an arbitrary Banach function space over $X$ itself, the correct analogue of the $w_n$ is not a sequence of numbers but an induced sequence gauge, the *quotient gauge* of Theorem 6.1, and the reduction is exact. An elementary example shows that no criterion based on the local associate norms alone can cover this generality. At the blockwise level, where the space is a direct sum of spaces over the pieces, the quotient gauge scalarizes. The correct analogue of $w_n$ is then the associate-space norm, and the reduction persists with equal optimal constants for arbitrary monotone, positively homogeneous outer gauges (Theorem 6.3). In both cases the function-space structure disappears from the problem entirely, exactly as in the Lebesgue case.

The mechanism behind the blockwise reduction (as opposed to the global quotient-gauge argument of Theorem 6.1, which needs no duality at all) is already visible in the proof of Proposition 4.1. That proof uses exactly two facts about $L^p$, namely Hölder's inequality and the existence of near-extremizers for it. Both are encoded in the associate norm. Hölder's inequality follows directly from its definition, and whenever $0 < \|m_k\|_{E_k'} < \infty$, the defining supremum provides arbitrarily good near-extremizers.

We briefly recall the standard framework; see Bennett and Sharpley [2, Chapter 1]. A *Banach function space* (BFS) over a $\sigma$-finite measure space $(\Omega, \nu)$ is determined by a function norm $\|\cdot\|_E$, defined on all nonnegative measurable functions with values in $[0, \infty]$. Its *associate norm* is

$$\|g\|_{E'} := \sup \left\{ \int_\Omega g\varphi \, \mathrm{d}\nu : \ \varphi \ge 0, \ \|\varphi\|_E \le 1 \right\},$$

so that Hölder's inequality $\int_\Omega g\varphi \, \mathrm{d}\nu \le \|g\|_{E'} \|\varphi\|_E$ holds, while near-extremizers exist by the very definition of the supremum.

Call a map $\Phi$, assigning to each sequence $x = (x_n)_{n \ge 1} \subseteq [0, \infty]$ a value $\Phi(x) \in [0, \infty]$, a *gauge* if it is monotone ($x \le y$ coordinatewise implies $\Phi(x) \le \Phi(y)$) and positively homogeneous ($\Phi(\lambda x) = \lambda \Phi(x)$ for $\lambda > 0$). Call it *faithful* if $\Phi(x) = \infty$ whenever some coordinate $x_k$ is infinite. Weighted $\ell^q$ quasi-norms ($0 < q \le$

$\infty$, positive weights), and more generally weighted Lorentz and Orlicz sequence norms, are faithful gauges.

We first treat the global level. Let $X := \bigsqcup_{k\geq 1} X_k$ be the disjoint union of Proposition 3.1, let $m|_{X_k} := m_k$, let $E$ be a BFS over $(X, \mu)$, and set

$$L_k(f) := \int_{X_k} mf \, \mathrm{d}\mu \qquad (f \geq 0 \text{ measurable on } X).$$

The *quotient gauge* induced by $E$ and $m$ is

$$\|a\|_{\mathcal{Q}} := \inf\{\|f\|_E : \ f \geq 0, \ L_k(f) \geq a_k \text{ for every } k \geq 1\}$$

for every $a = (a_k)_{k\geq 1} \subseteq [0, \infty]$, with $\inf \emptyset := \infty$. Since the $L_k$ are linear and the feasible set shrinks as $a$ grows, $\|\cdot\|_{\mathcal{Q}}$ is itself monotone, positively homogeneous and subadditive, with values in $[0, \infty]$. We call such a map an *extended sequence gauge*.

**Theorem 6.1.** *Let $\Phi$ be a monotone gauge. The least constant $\rho_{\mathrm{opt}}$ such that*

$$\Phi((\textstyle\sum_{k=1}^n L_k(f))_{n\geq 1}) \leq \rho \, \|f\|_E \qquad \textit{for all measurable } f \geq 0 \tag{6.1}$$

*equals the least constant $C$ such that*

$$\Phi((\textstyle\sum_{k=1}^n a_k)_{n\geq 1}) \leq C \, \|a\|_{\mathcal{Q}} \qquad \textit{for all } (a_k)_{k\geq 1} \subseteq [0, \infty].$$

*Only the monotonicity of $\Phi$ is used here, and its positive homogeneity plays no role.*

*Proof.* If $f \geq 0$, then $f$ itself is feasible for $a := (L_k(f))_{k\geq 1}$, so $\|a\|_{\mathcal{Q}} \leq \|f\|_E$. The left-hand side of (6.1) therefore equals $\Phi((\sum_{k\leq n} a_k)) \leq C\|a\|_{\mathcal{Q}} \leq C\|f\|_E$, and hence $\rho_{\mathrm{opt}} \leq C$. Conversely, let $a$ be given with $\|a\|_{\mathcal{Q}} < \infty$ (otherwise there is nothing to prove), let $\varepsilon > 0$, and choose $f \geq 0$ feasible for $a$ with $\|f\|_E \leq \|a\|_{\mathcal{Q}} + \varepsilon$. Since $L_k(f) \geq a_k$, monotonicity of $\Phi$ gives

$$\Phi((\textstyle\sum_{k\leq n} a_k)) \leq \Phi((\sum_{k\leq n} L_k(f))) \leq \rho_{\mathrm{opt}}\|f\|_E \leq \rho_{\mathrm{opt}}(\|a\|_{\mathcal{Q}} + \varepsilon).$$

Letting $\varepsilon \to 0^+$, we conclude that $C \leq \rho_{\mathrm{opt}}$. □

**Example 6.2.** Let $X = \{1, 2, 3, \dots\}$ carry the counting measure, with $X_k := \{k\}$ and $m \equiv 1$, so that $L_k(f) = f(k)$. For $E = \ell^1$ and for $E = \ell^\infty$, every local quantity attached to the pieces coincides. The norm of $\chi_{\{k\}}$ is 1 in both spaces, as is the norm of the coordinate functional $f \mapsto f(k)$. Yet

$$\|a\|_{\mathcal{Q}(\ell^1)} = \sum_{k\geq 1} a_k \qquad \text{and} \qquad \|a\|_{\mathcal{Q}(\ell^\infty)} = \sup_{k\geq 1} a_k,$$

and the optimal constants in (6.1) differ accordingly. For $\Phi(x) = \sup_n x_n$, they are 1 and $\infty$, respectively. No characterization based solely on the norms of the block coordinate functionals (equivalently, on the local associate norms) can therefore cover arbitrary global Banach function spaces. The quotient gauge is the correct object.

We now turn to the blockwise level, where the quotient gauge scalarizes. For each $k \geq 1$, let $E_k$ be a BFS over $(X_k, \mu_k)$ and set $w_k := \|m_k\|_{E_k'}$. We assume throughout this section that $0 < w_k < \infty$ for every $k$. (Unlike in Remark 4.2,

the finiteness of $w_k$ is now a genuine hypothesis. A coordinate with $w_k = 0$ may still be discarded, since then $m_k = 0$ a.e., but an arbitrary gauge $\Phi$ need not see every coordinate, so $w_k = \infty$ can no longer be shown to force unboundedness.) Given gauges $\Phi$ (on the left) and $\Psi$ (on the right, assumed faithful), consider the inequality

$$\Phi\left(\left(\sum_{k=1}^{n}\int_{X_k} m_k f_k \,\mathrm{d}\mu_k\right)_{n\ge 1}\right) \;\le\; \rho\,\Psi((\|f_k\|_{E_k})_{k\ge 1}), \tag{6.2}$$

required to hold for all sequences $(f_k)_{k\ge 1}$ of nonnegative measurable functions on the $X_k$. (For complex-valued $f_k$ with absolutely convergent inner integrals, the corresponding inequality follows by monotonicity of $\Phi$.) The case where $\Phi(x) = (\sum_n b_n (x_n/M_n)^q)^{1/q}$, $\Psi(a) = (\sum_k b_k a_k^p)^{1/p}$ and $E_k = L^p(\mu_k)$ is (3.1).

The right-hand side of (6.2) sees $f$ only through the sequence of blockwise norms $\|f_k\|_{E_k}$. When the outer gauge $\Psi$ is a Banach sequence lattice norm for which the direct-sum functional

$$\|f\|_E := \Psi((\|f|_{X_k}\|_{E_k})_{k\ge 1})$$

defines a Banach function space over $X$, this construction falls within the global framework of Theorem 6.1. For an arbitrary gauge $\Psi$ (which may fail to be subadditive, as with weighted $\ell^q$, $0 < q < 1$), the displayed formula still defines a functional on nonnegative measurable functions. Although Theorem 6.1 was stated for a Banach function space, its proof uses only the resulting nonnegative functional and the definition of the quotient gauge, and therefore carries over verbatim to this setting. In either reading, the quotient gauge scalarizes exactly as follows.

**Theorem 6.3.** *Assume $0 < w_k < \infty$ for every $k \ge 1$. Then the least constant $\rho_{\mathrm{opt}}$ in* (6.2) *equals the least constant $C$ such that*

$$\Phi\left(\Big(\sum_{k=1}^{n} w_k a_k\Big)_{n\ge 1}\right) \le C\,\Psi((a_k)_{k\ge 1}) \qquad \text{for all } (a_k)_{k\ge 1} \subseteq [0,\infty). \tag{6.3}$$

*In particular, the analogue of the quantities $w_k$ asked for in* [7, Section 4] *is the associate norm $\|m_k\|_{E_k'}$, and no duality beyond the one available in every Banach function space is required.*

*Proof.* Write $C_{\mathrm{d}}$ for the least constant in (6.3).

($\rho_{\mathrm{opt}} \le C_{\mathrm{d}}$.) Let $(f_k)_{k\ge 1}$ be admissible and set $\alpha_k := \|f_k\|_{E_k}$. If some $\alpha_k = \infty$, then $\Psi((\alpha_k)) = \infty$ by faithfulness and there is nothing to prove. Otherwise Hölder's inequality gives $\int_{X_k} m_k f_k \,\mathrm{d}\mu_k \le w_k \alpha_k < \infty$ for every $k$, and by monotonicity of $\Phi$ followed by (6.3) applied to $(\alpha_k)$, the left-hand side of (6.2) is at most $C_{\mathrm{d}}\,\Psi((\alpha_k))$.

($C_{\mathrm{d}} \le \rho_{\mathrm{opt}}$.) Let $(a_k)_{k\ge 1} \subseteq [0,\infty)$ and $\varepsilon \in (0,1)$. By definition of the associate norm, for each $k$ there is $g_{k,\varepsilon} \ge 0$ with $\|g_{k,\varepsilon}\|_{E_k} \le 1$ and $\int_{X_k} m_k g_{k,\varepsilon} \,\mathrm{d}\mu_k \ge (1-\varepsilon) w_k$. Taking $f_k := a_k g_{k,\varepsilon}$, we obtain

$$(1-\varepsilon)\,\Phi((\textstyle\sum_{k\le n} w_k a_k)_{n\ge 1}) \le \Phi\left(\left(\sum_{k\le n} a_k \int_{X_k} m_k g_{k,\varepsilon}\,\mathrm{d}\mu_k\right)_{n\ge 1}\right)$$

$$\leq \rho_{\mathrm{opt}}\,\Psi((\|f_k\|_{E_k})_{k\geq 1}) \;\leq\; \rho_{\mathrm{opt}}\,\Psi((a_k)_{k\geq 1}),$$

where the first inequality uses the homogeneity and monotonicity of $\Phi$, the second is (6.2), and the last uses $\|f_k\|_{E_k} \leq a_k$ and the monotonicity of $\Psi$. Let $\varepsilon \to 0^+$. □

**Proposition 6.4.** *Let $\Psi$ be a faithful gauge, let $\|\cdot\|_E$ be the direct-sum functional $\|f\|_E = \Psi((\|f|_{X_k}\|_{E_k})_{k\geq 1})$, define its quotient functional by*

$$\|a\|_{\mathcal{Q}} := \inf\{\|f\|_E :\ f \geq 0,\ L_k(f) \geq a_k \text{ for every } k \geq 1\},$$

*and let $w_k = \|m_k\|_{E_k'} \in (0,\infty)$. Then*

$$\|a\|_{\mathcal{Q}} = \Psi((a_k/w_k)_{k\geq 1}) \qquad \text{for every } (a_k)_{k\geq 1} \subseteq [0,\infty).$$

*When the direct-sum functional defines a Banach function space, this identifies Theorem 6.3 as the direct-sum specialization of Theorem 6.1. For a general faithful gauge $\Psi$, the same formula follows from the identical quotient-gauge argument, although the resulting direct-sum functional need not be a Banach function-space norm.*

*Proof.* If $f \geq 0$ is feasible for $a$, then Hölder's inequality gives $w_k\|f_k\|_{E_k} \geq L_k(f) \geq a_k$, so $\|f\|_E = \Psi((\|f_k\|_{E_k})) \geq \Psi((a_k/w_k))$ by monotonicity. Hence $\|a\|_{\mathcal{Q}} \geq \Psi((a_k/w_k))$. Conversely, with the near-extremizers $g_{k,\varepsilon}$ of the proof of Theorem 6.3, the function $f$ with $f_k := a_k\, g_{k,\varepsilon}/((1-\varepsilon)w_k)$ is feasible for $a$, and $\|f\|_E \leq \Psi((a_k/w_k))/(1-\varepsilon)$ by homogeneity of $\Psi$. Letting $\varepsilon \to 0^+$ completes the proof. □

**Corollary 6.5.** *Let $w_k = \|m_k\|_{E_k'}$, let $0 < q \leq \infty$ and $1 \leq s < \infty$, and consider*

$$\Big(\sum_{n=1}^{\infty} b_n \Big|\frac{1}{M_n}\sum_{k=1}^{n}\int_{X_k} m_k f_k \,\mathrm{d}\mu_k\Big|^q\Big)^{1/q} \;\leq\; \rho\,\Big(\sum_{k=1}^{\infty} b_k \|f_k\|_{E_k}^{s}\Big)^{1/s}. \tag{6.4}$$

*If $s \leq q$, then (6.4) admits a finite $\rho$ if and only if $\widetilde{\beta}_s < \infty$, where $\widetilde{\beta}_s$ is the quantity of Corollary 3.2 with $p$ replaced by $s$ (usual modifications when $s = 1$ or $q = \infty$), and in that case $\widetilde{\beta}_s \leq \rho_{\mathrm{opt}} \leq s^{1/q}(s')^{1/s'}\widetilde{\beta}_s$. If $1 < s < \infty$ and $0 < q < s$, then (6.4) admits a finite $\rho$ if and only if $B_s < \infty$, where $B_s$ is the quantity of Theorem 5.2 with $p$ replaced by $s$, with the corresponding two-sided estimates. If $s = 1$ and $0 < q < 1$, then (6.4) admits a finite $\rho$ if and only if $D_1 < \infty$, where $D_1$ is the quantity of Theorem 5.4 with $w_k = \|m_k\|_{E_k'}$, and in that case $q(1-q)\,D_1 \leq \rho_{\mathrm{opt}} \leq (1-q)^{-(1-q)/q}\,D_1$. In particular, the outer exponent $s$ and the inner spaces $E_k$ decouple completely. (The endpoint $s = \infty$, whose right-hand side requires a separate reading, is left aside.)*

*Proof.* Theorem 6.3, with $\Phi$ and $\Psi$ the indicated weighted $\ell^q$ and $\ell^s$ gauges, reduces (6.4) to the discrete inequality (4.1) with $p$ replaced by $s$. The conclusion then follows from the discrete criterion underlying Theorem A when $s \leq q$, from Proposition 5.1 when $1 < s < \infty$ and $q < s$, and from the discrete characterization established in the proof of Theorem 5.4 when $s = 1$ and $0 < q < 1$. □

**Remark 6.6:** (i) *Weighted Lebesgue.* For $E_k = L^p(v_k\,\mathrm{d}\mu_k)$ with $1 < p < \infty$, a direct computation gives $w_k = (\int_{X_k} m_k^{p'} v_k^{1-p'}\,\mathrm{d}\mu_k)^{1/p'}$. (ii) *Orlicz.* For $E_k = L^{\varphi}(\mu_k)$

with the Luxemburg norm, the associate space is $L^{\varphi^*}(\mu_k)$ with the Orlicz norm, isometrically [32], so $w_k$ is the Orlicz norm of $m_k$ with respect to the complementary Young function $\varphi^*$ (equivalently, up to a factor 2, its Luxemburg norm). (iii) *Lorentz.* For $E_k = L^{p,s_0}(\mu_k)$ with $1 < p < \infty$ and $1 \le s_0 \le \infty$, the associate space is $L^{p',s_0'}(\mu_k)$ up to equivalence of norms [2], and the characterizations above hold with $w_k \asymp \|m_k\|_{L^{p',s_0'}(X_k)}$ and constants incorporating the equivalence.

**Proposition 6.7.** *Let $E = L^{\varphi}(X,\mu)$ carry the Luxemburg norm, where $\varphi$ is a Young function, and for $t \ge 0$ set*

$$\theta_k(t) := \inf\left\{ \int_{X_k} \varphi(g)\,\mathrm{d}\mu : \ g \ge 0,\ \int_{X_k} mg\,\mathrm{d}\mu \ge t \right\}.$$

*Then, for every $(a_k)_{k\ge1} \subseteq [0,\infty)$,*

$$\|a\|_{\mathcal{Q}} = \inf\left\{ \lambda > 0 : \ \sum_{k\ge1} \theta_k(a_k/\lambda) \le 1 \right\},$$

*so that even the genuinely global Orlicz case admits an exact discrete modular description.*

*Proof.* Write $\Lambda$ for the right-hand side, with $\inf\emptyset := \infty$.

($\Lambda \le \|a\|_{\mathcal{Q}}$.) Let $f \ge 0$ be feasible for $a$ and let $\lambda > \|f\|_E$. The definition of the Luxemburg norm then gives $\int_X \varphi(f/\lambda)\,\mathrm{d}\mu \le 1$. For each $k$, the function $f|_{X_k}/\lambda$ is admissible in the infimum defining $\theta_k(a_k/\lambda)$. Decomposing the modular over the pieces, we obtain

$$\sum_{k\ge1} \theta_k(a_k/\lambda) \le \sum_{k\ge1} \int_{X_k} \varphi(f/\lambda)\,\mathrm{d}\mu = \int_X \varphi(f/\lambda)\,\mathrm{d}\mu \le 1,$$

and hence $\Lambda \le \lambda$. Taking infima over $\lambda$ and over feasible $f$ gives $\Lambda \le \|a\|_{\mathcal{Q}}$.

($\|a\|_{\mathcal{Q}} \le \Lambda$.) Let $\lambda > 0$ satisfy $\sum_k \theta_k(a_k/\lambda) \le 1$ and let $\varepsilon \in (0,1)$. Each $\theta_k(a_k/\lambda)$ is then finite, so each defining admissible class is nonempty. For each $k$ we may therefore choose $g_k \ge 0$ on $X_k$ with $\int_{X_k} mg_k\,\mathrm{d}\mu \ge a_k/\lambda$ and $\int_{X_k} \varphi(g_k)\,\mathrm{d}\mu \le \theta_k(a_k/\lambda) + \varepsilon 2^{-k}$. Let $f \ge 0$ be the function with $f|_{X_k} = \lambda g_k$. Then $L_k(f) \ge a_k$ for every $k$, so $f$ is feasible for $a$, and $\int_X \varphi(f/\lambda)\,\mathrm{d}\mu = \sum_k \int_{X_k} \varphi(g_k)\,\mathrm{d}\mu \le 1+\varepsilon$. Since $\varphi$ is convex with $\varphi(0) = 0$, we have $\varphi(x/(1+\varepsilon)) \le \varphi(x)/(1+\varepsilon)$, and hence $\int_X \varphi(f/(\lambda(1+\varepsilon)))\,\mathrm{d}\mu \le 1$ and $\|f\|_E \le \lambda(1+\varepsilon)$. Therefore $\|a\|_{\mathcal{Q}} \le \lambda(1+\varepsilon)$. Letting $\varepsilon \to 0^+$ and taking the infimum over $\lambda$ completes the proof. □

**Remark 6.8:** Theorems 6.1 and 6.3 absorb the measure-theoretic content of the first question entirely. For a global Banach function space the correct object is the quotient gauge, for a direct sum it is the sequence of associate norms, and Example 6.2 shows that the former cannot in general be recovered from the local associate norms. What remains is purely a sequence-space problem, namely the discrete inequality between $\Phi$ and $\|\cdot\|_{\mathcal{Q}}$ (or $\Psi$), which is well studied in many cases; see, e.g., [11] for weighted Lorentz norms and [24] for modular (Orlicz) inequalities. Together with Remark 4.3, this also confirms that the first and third questions of [7, Section 4] are two faces of the same reduction.

## 7. TRIANGULAR KERNEL OPERATORS

The second question of [7, Section 4] concerns kernel operators $Tf(x) = \int_X k(x,y)f(y)\,\mathrm{d}\mu(y)$ and asks for boundedness criteria adapted to Hardy-type kernels, especially those arising from nested sets or monotone integration regions. We treat it in three steps. First, an exact dual formula (Theorem 7.2) reduces the boundedness problem for a *general* triangular kernel, with no structural hypothesis, to the family of its column functionals. Second, a strict-convexity argument (Theorem 7.3) shows that these column functionals are represented exactly by their coordinate norms $w_{n,k} = \|m_{n,k}\|_{L^{p'}(\mu_k)}$, through the triangle inequality, precisely when the kernel is *block-factorized* (for $1 < p < \infty$ and columns of finite $L^{p'}$-norm). Block factorization is thus not merely a convenient sufficient hypothesis but the exact frontier of universal scalarization, and Example 7.5 measures what lies beyond it. Third, on the block-factorized class the reduction of the previous sections applies at full strength and yields explicit Oinarov-type criteria for $1 < p \le q < \infty$ and $1 < q < p < \infty$ (Theorems 7.6 and 7.7).

We work in the multi-space setting of Section 3, which, by Proposition 3.1, entails no loss of generality. A *nested kernel operator* is determined by measurable functions $m_{n,k}\colon X_k \to [0,\infty)$, $1 \le k \le n$, via

$$(Tf)_n := \sum_{k=1}^{n} \int_{X_k} m_{n,k} f_k \,\mathrm{d}\mu_k, \qquad n \ge 1.$$

When $m_{n,k} = m_k$ is independent of $n$, this is the operator of [7]. We consider the inequality

$$\left(\sum_{n=1}^{\infty} b_n \left|\frac{(Tf)_n}{M_n}\right|^q\right)^{1/q} \le \rho \left(\sum_{k=1}^{\infty} b_k \int_{X_k} |f_k|^p \,\mathrm{d}\mu_k\right)^{1/p}. \tag{7.1}$$

**Proposition 7.1.** *Set $w_{n,k} := \|m_{n,k}\|_{L^{p'}(\mu_k)}$ and let $C_{\mathrm{mat}}$ denote the least constant $C$ such that*

$$\left(\sum_{n=1}^{\infty} b_n \left(\frac{1}{M_n}\sum_{k=1}^{n} w_{n,k} a_k\right)^q\right)^{1/q} \le C \left(\sum_{k=1}^{\infty} b_k a_k^p\right)^{1/p} \tag{7.2}$$

*for all sequences $(a_k)_{k\ge1} \subseteq [0,\infty)$. Then:*

- *(i)* (Rowwise Hölder bound) $\rho_{\mathrm{opt}} \le C_{\mathrm{mat}}$ *always;*
- *(ii)* (Exactness for block-factorized kernels) *if $m_{n,k} = d_{n,k} m_k$ with scalars $d_{n,k} \in [0,\infty)$ and functions $m_k$ independent of $n$, and if $0 < w_k := \|m_k\|_{L^{p'}(\mu_k)} < \infty$, then $\rho_{\mathrm{opt}} = C_{\mathrm{mat}}$, with $w_{n,k} = d_{n,k} w_k$.*

*The same statements hold, with the same proof, for inner Banach function spaces and arbitrary gauges as in Section 6.*

*Proof.* Part (i) follows from the rowwise Hölder inequality

$$|(Tf)_n| \le \sum_{k\le n} w_{n,k} \|f_k\|_{L^p(\mu_k)}$$

and from (7.2) applied to $\alpha_k = \|f_k\|_{L^p(\mu_k)}$, exactly as in Proposition 4.1. For (ii), the point is that the Hölder (near-)extremizer of Proposition 4.1 on $X_k$ depends only on $m_k$, and hence is shared by all rows. Since $d_{n,k} \geq 0$, taking $f_k := a_k g_k$ (respectively $f_k := a_k g_{k,\varepsilon}$ when $p = 1$) turns both sides of (7.1) into the corresponding sides of (7.2), up to a factor $(1 - \varepsilon)$. □

**Theorem 7.2.** *Let $1 < p < \infty$ and $1 < q < \infty$, and let $q' := q/(q-1)$. The least constant in* (7.1)*, namely $\rho_{\mathrm{opt}}$, is*

$$\sup \left\{ \left[ \sum_{k=1}^{\infty} b_k^{1-p'} \left\| \sum_{n=k}^{\infty} \frac{g_n}{M_n}\, m_{n,k} \right\|_{L^{p'}(\mu_k)}^{p'} \right]^{\frac{1}{p'}} :\ g_n \geq 0,\ \left( \sum_{n=1}^{\infty} b_n^{1-q'} g_n^{q'} \right)^{\frac{1}{q'}} \leq 1 \right\}.$$

*All sums and norms in this formula are interpreted in $[0, \infty]$. In particular, the pointwise sums $\sum_{n \geq k} (g_n/M_n)\, m_{n,k}$ are extended-valued measurable functions, and the $m_{n,k}$ are not assumed to lie in $L^{p'}(\mu_k)$. The relevant information about the k-th column is thus the functional $g \mapsto \|\sum_{n \geq k} g_n m_{n,k}\|_{L^{p'}(\mu_k)}$, and not merely the sequence of numbers $w_{n,k}$.*

*Proof.* We may restrict to $f_k \geq 0$. By weighted $\ell^q$–$\ell^{q'}$ duality, for every sequence $x_n \geq 0$,

$$\left( \sum_n b_n x_n^q \right)^{1/q} = \sup \left\{ \sum_n g_n x_n :\ g_n \geq 0,\ \left( \sum_n b_n^{1-q'} g_n^{q'} \right)^{1/q'} \leq 1 \right\}.$$

(The dual of $\ell^q(b)$ under the unweighted pairing $\sum_n g_n x_n$ is thus $\ell^{q'}(b^{1-q'})$, and similarly, below, the dual of $\ell^p(b)$ is $\ell^{p'}(b^{1-p'})$.) Applying this to $x_n = (Tf)_n/M_n$ and interchanging the order of summation and integration by Tonelli's theorem, we obtain

$$\sum_n \frac{g_n}{M_n} (Tf)_n = \sum_{k=1}^{\infty} \int_{X_k} \left( \sum_{n \geq k} \frac{g_n}{M_n} m_{n,k} \right) f_k \, \mathrm{d}\mu_k.$$

For fixed $g$, take the supremum of the right-hand side over $f \geq 0$ with

$$\sum_k b_k \|f_k\|_{L^p(\mu_k)}^p \leq 1.$$

By blockwise $L^p$–$L^{p'}$ duality followed by weighted $\ell^p$–$\ell^{p'}$ duality, this supremum equals the quantity in square brackets. Exchanging the two suprema, which is legitimate for suprema of nonnegative quantities, gives the formula. □

**Theorem 7.3.** *Let $1 < p < \infty$, fix $k \geq 1$, and let $(m_{n,k})_{n \geq k}$ be nonnegative functions in $L^{p'}(\mu_k)$. The following are equivalent:*

- *(i)* $\| \sum_{n \geq k} \lambda_n m_{n,k} \|_{L^{p'}(\mu_k)} = \sum_{n \geq k} \lambda_n \| m_{n,k} \|_{L^{p'}(\mu_k)}$ *for every finitely supported $(\lambda_n) \subseteq [0, \infty)$;*
- *(ii)* $m_{n,k} = d_{n,k} m_k$ *for all $n \geq k$, for some nonnegative $m_k \in L^{p'}(\mu_k)$ and scalars $d_{n,k} \geq 0$.*

*Proof.* (ii)⇒(i) is immediate. For the converse, since $1 < p' < \infty$, the space $L^{p'}(\mu_k)$ is strictly convex, so $\|x+y\| = \|x\| + \|y\|$ with $x, y \neq 0$ forces $x/\|x\| = y/\|y\|$. Applying (i) to pairs, we see that any two nonzero members of the column lie on a common ray. Taking $m_k$ to be any fixed nonzero member of the column (or $m_k := 0$ if they all vanish) then yields (ii). □

**Corollary 7.4.** *Let $1 < p < \infty$ and $1 < q < \infty$, and suppose that $m_{n,k} \in L^{p'}(\mu_k)$ for all $1 \le k \le n$. In dual form, the rowwise Hölder bound $\rho_{\mathrm{opt}} \le C_{\mathrm{mat}}$ of Proposition 7.1(i) (the scalar majorant underlying Schur-type tests) is the columnwise triangle inequality*

$$\left\| \sum_{n \ge k} \frac{g_n}{M_n}\, m_{n,k} \right\|_{L^{p'}(\mu_k)} \le \sum_{n \ge k} \frac{g_n}{M_n}\, w_{n,k}.$$

*Indeed, the dual expression of $C_{\mathrm{mat}}$ is obtained from Theorem 7.2 by replacing each column functional by its coordinatewise majorant. By Theorem 7.3, this inequality is an equality for every $g \ge 0$ if and only if the kernel is block-factorized. The block-factorized class is therefore exactly the class for which every column functional is represented by its coordinate norms through equality in the triangle inequality. For particular weights, the equality $\rho_{\mathrm{opt}} = C_{\mathrm{mat}}$ may of course also occur accidentally.*

**Example 7.5.** Let $p = q = 2$ and $b_n = M_n = 1$, let $X_1 = \{1, \dots, N\}$ carry the counting measure, and let all other pieces be empty. Take $m_{n,1} := \chi_{\{n\}}$ for $1 \le n \le N$ and $m_{n,1} := 0$ for $n > N$. Then $(Tf)_n = f(n)$ for $n \le N$, so $T$ is an isometry and $\rho_{\mathrm{opt}} = 1$. On the other hand, $w_{n,1} = 1$ for $n \le N$, so (7.2) reads $\|(a_1, \dots, a_1, 0, \dots)\|_{\ell^2} \le C a_1$ and $C_{\mathrm{mat}} = \sqrt{N}$. The same mechanism separates bounded from unbounded operators. On $X_1 = \{1, 2, 3, \dots\}$ with counting measure (the other pieces being still empty), the kernels $m_{n,1} := \chi_{\{n\}}$ and $m'_{n,1} := \chi_{\{1\}}$, $n \ge 1$, have the same norms $w_{n,1} = w'_{n,1} = 1$, yet the first gives the identity operator on $\ell^2$ and the second the unbounded map $f \mapsto (f(1), f(1), \dots)$. Thus no condition depending on the kernel only through the norms $w_{n,k}$ can characterize (7.1) in general. The numbers $w_{n,k}$ record only the radii of the column functionals of Theorem 7.2 and forget their geometry. In the finite example, $N$ mutually orthogonal directions are recorded as $N$ copies of the same radius. This quantifies the observation of [7, Section 4] that Schur-type tests do not capture the full picture.

**Theorem 7.6.** *Let $1 < p \le q < \infty$, let $m_{n,k} = d_{n,k} m_k$ be block-factorized with $0 < w_k < \infty$, and suppose there is $h \ge 1$ such that*

$$h^{-1}(d_{n,j} + d_{j,k}) \le d_{n,k} \le h(d_{n,j} + d_{j,k}) \qquad \text{for all } 1 \le k \le j \le n.$$

*Then (7.1) admits a finite $\rho$ if and only if $\max(A_0, A_1) < \infty$, where*

$$A_0 := \sup_{N \ge 1} \left( \sum_{n=N}^{\infty} b_n \left(\frac{d_{n,N}}{M_n}\right)^q \right)^{1/q} \left( \sum_{k=1}^{N} b_k^{1-p'} w_k^{p'} \right)^{1/p'},$$

$$A_1 := \sup_{N\geq 1} \left( \sum_{n=N}^{\infty} \frac{b_n}{M_n^q} \right)^{1/q} \left( \sum_{k=1}^{N} d_{N,k}^{p'} \, b_k^{1-p'} \, w_k^{p'} \right)^{1/p'},$$

*in which case* $\rho_{\mathrm{opt}} \asymp \max(A_0, A_1)$ *with implicit constants depending only on* $p$, $q$ *and* $h$.

*Proof.* By Proposition 7.1(ii), $\rho_{\mathrm{opt}}$ equals the optimal constant of the matrix inequality (7.2) with entries $d_{n,k} w_k$. Upon substituting $u_n = b_n^{1/q}/M_n$, $v_k = b_k^{1/p}/w_k$ and $c_k = w_k a_k$, this is the discrete Volterra-type inequality with matrix $(d_{n,k})$. The characterization is then the theorem of Oinarov, Persson and Temirkhanova [30, Theorem 2.1]. Their Assumption 1.1 is more general than the Oinarov condition used here. Under the latter one may take, in their notation,

$$a_{i,j} = d_{i,j}, \qquad b_{i,j} = d_{i,j}, \qquad \omega_j \equiv 1.$$

The required almost-monotonicity follows directly from the Oinarov condition. For $i_2 \geq i_1 \geq j$ we have $d_{i_2,j} \geq h^{-1}(d_{i_2,i_1} + d_{i_1,j}) \geq h^{-1} d_{i_1,j}$, and similarly $d_{i,j_2} \geq h^{-1} d_{i,j_1}$ for $i \geq j_1 \geq j_2$. Moreover, $d_{i,j} \asymp d_{i,k} + d_{k,j}$ is precisely their structural condition under this specialization. Their quantities $F_0$ and $F_1$ then become exactly $A_0$ and $A_1$ after the indicated change of weights. □

**Theorem 7.7** (Oinarov kernels, $1 < q < p < \infty$)**.** *Let* $1 < q < p < \infty$ *and* $1/r := 1/q - 1/p$, *let* $m_{n,k} = d_{n,k} m_k$ *be block-factorized with* $0 < w_k < \infty$, *and let* $(d_{n,k})$ *satisfy the condition of Theorem 7.6. With* $\alpha_n := b_n / M_n^q$ *and* $\delta_k := b_k^{1-p'} w_k^{p'}$, *set*

$$\mathcal{B}_0 := \left[ \sum_{N=1}^{\infty} \delta_N \left( \sum_{n=N}^{\infty} d_{n,N}^{q} \alpha_n \right)^{r/q} \left( \sum_{k=1}^{N} \delta_k \right)^{r/q'} \right]^{1/r},$$

$$\mathcal{B}_1 := \left[ \sum_{N=1}^{\infty} \alpha_N \left( \sum_{k=1}^{N} d_{N,k}^{p'} \delta_k \right)^{r/p'} \left( \sum_{n=N}^{\infty} \alpha_n \right)^{r/p} \right]^{1/r}.$$

*Then* (7.1) *admits a finite* $\rho$ *if and only if* $\max(\mathcal{B}_0, \mathcal{B}_1) < \infty$, *in which case* $\rho_{\mathrm{opt}} \asymp \max(\mathcal{B}_0, \mathcal{B}_1)$, *with implicit constants depending only on* $p$, $q$ *and* $h$.

*Proof.* As in the proof of Theorem 7.6, the exact reduction of Proposition 7.1(ii) identifies $\rho_{\mathrm{opt}}$ with the optimal constant of the discrete Volterra-type inequality with matrix $(d_{n,k})$. The characterization for $1 < q < p < \infty$ is Theorem 2.1 of Oinarov, Okpoti and Persson [29]. Their condition (4) involves two auxiliary sequences, denoted $(b_i)$ and $(c_i)$ in their notation, and taking both identically equal to 1 reduces it exactly to the Oinarov condition above. With this specialization and the substitution $u_n = b_n^{1/q}/M_n$, $v_k = b_k^{1/p}/w_k$, $c_k = w_k a_k$, their two characterizing quantities ($B_1$ and $B_2$ in their notation) reduce to $\mathcal{B}_0$ and $\mathcal{B}_1$. Indeed, we have

$$\frac{r}{q} = \frac{p}{p-q}, \qquad \frac{r}{q'} = \frac{p(q-1)}{p-q}, \qquad \frac{r}{p'} = \frac{q(p-1)}{p-q}, \qquad \frac{r}{p} = \frac{q}{p-q},$$

while $u_n^q = b_n/M_n^q = \alpha_n$ and $v_k^{-p'} = b_k^{-p'/p} w_k^{p'} = \delta_k$, so that their quantities become $\mathcal{B}_0$ and $\mathcal{B}_1$, respectively. For $d_{n,k} \equiv 1$, the quantity $\mathcal{B}_1$ coincides with the quantity $B$ of Theorem 5.2. □

**Remark 7.8:** Constant matrices $d_{n,k} \equiv 1$ satisfy the condition with $h = 2$ and give $A_0 = A_1 = \widetilde{\beta}$, recovering Corollary 3.2. Separable matrices $d_{n,k} = \varphi_n \psi_k$ reduce to this case by absorbing $\varphi_n$ into $M_n$ and $\psi_k$ into $m_k$. The first genuinely new kernels covered by Theorems 7.6 and 7.7 are therefore the non-separable ones, such as the discrete Riemann–Liouville coefficients $d_{n,k} = n - k + 1$ (for which $h = 2$). The matrix theorems invoked in their proofs are not new. The contribution here is the exact reduction, through which every discrete result of this type becomes a measure-theoretic kernel theorem the moment the column profiles factor. The discrete theory continues to develop. The hypotheses of [29, 30] are already weaker than the classical Oinarov condition, further criteria in the case $1 < q < p < \infty$ appear in [38], alternative criteria for broader matrix classes have been obtained in [20], and very recently Oinarov and Temirkhanova [31] established necessary and sufficient boundedness conditions for a wide class of weighted matrix operators in the case $1 < q < p < \infty$. Each such result, whenever applicable to the scalar matrix obtained after factorization, transfers immediately through the exact reduction. What remains open is of a different nature, namely the sharp-constant problem in the case $q < p$, and simple Muckenhoupt-style criteria for genuinely row-dependent kernels, which, by Example 7.5, cannot be built from the numbers $w_{n,k}$ alone.

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