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ON SHARP RESULTS ON COEFFICIENT MULTIPLIERS OF HARMONIC FUNCTION SPACES OF SEVERAL VARIABLES AND RELATED PROBLEMS

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Abstract. The first sharp results concerning coefficient multipliers in various spaces of harmonic functions in the unit disk were obtained by A. Shields and D. Williams several decades ago. Since then, extending these theorems to higher-dimensional harmonic function spaces remained an open problem for a long time. In 1998, R. F. Shamoyan provided the first such sharp results in higher dimensions using spherical harmonics. These results were subsequently extended in various directions to harmonic function spaces on the unit ball by R. F. Shamoyan and M. Arsenović, as well as by E. Malinnikova and A. Eikrem.

The main goal of this note is to collect and survey these sharp results within a single unified framework. In the second part of this note, we also briefly review recent advances on this topic for analytic function spaces of several variables.

1. Introduction

The well-known problem of the description of spaces of coefficient multipliers of various pairs of analytic functions was considered, as far as we know, in the unit disk for the first time in mathematical literature in the classical papers of Hardy and Littlewood. The formulation of the problem is very simple. Let X, Y be two spaces of analytic functions in the unit disk. We fix an arbitrary analytic function f in X with Taylor coefficients $\{c_k\}$. We wish to describe all sequences $\{a_k\}$ such that a new analytic function g with Taylor coefficients $\{c_k \times a_k\}$ belongs to the mentioned fixed analytic function space Y in the unit disk. The description of coefficient multiplier spaces for various pairs (X, Y) can be analogously extended to several complex variables on the unit polydisc in $\mathbb{C}^n$. The details are straightforward and are left to the reader. The only required modification is to

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replace the sequence $\{c_k\}$ in the unit disc definition with a multi-index sequence $\{c_{k_1,\dots,k_n}\}$, by means of the corresponding Taylor expansion of analytic functions on the unit polydisc. Analogous problems regarding coefficient multipliers may also be studied for other analytic function spaces on the unit polydisc. Furthermore, our definition for the analytic case in $\mathbb{C}$ and $\mathbb{C}^n$ readily extends to the well-known setting of harmonic function spaces, a routine generalization left to the interested reader. Similarly, one may consider the problem of obtaining sharp estimates for coefficient multiplier spaces between various pairs of harmonic function spaces in the unit disc. A vast number of sharp one-dimensional results for harmonic and analytic spaces in the unit disc have been established over recent decades. We refer, for example, to the work of P. Duren [17] , M. Pavlović and M. Mateljević [22], [27], O. Blasco [9], [10], M. Jevtić [19], D. Williams and A. Shields [43], M. Arsenović and M. Vukotić [47], and their coauthors.

The situation concerning the description of coefficient multiplier spaces for various pairs of function spaces in $\mathbb{C}^n$ is technically much more complex, particularly for analytic and harmonic functions of several variables. Numerous such problems remain open in $\mathbb{C}^n$, opening up a promising research direction in the theory of multi-variable function spaces. In the setting of the unit polydisc, R. Shamoyan and his coauthors have established numerous sharp results for various pairs of analytic function spaces (X, Y) over the past decade. Furthermore, we highlight the important paper by M. Nawrocki [23], in which the problem of describing coefficient multipliers between area Nevanlinna-type spaces on the polydisc was resolved using non-trivial techniques.

This survey gathers recent sharp results on coefficient multipliers for harmonic and analytic spaces of several variables in $\mathbb{C}^n$. While the first one-dimensional results were established decades ago by D. Williams and A. Shields [43], the problem remained open for several variables until R. F. Shamoyan [36] obtained the first sharp result in this direction. These ideas were subsequently extended in joint work with M. Arsenović [4, 8, 5]. We also note related contributions by E. Malinnikova and K. Eikrem [18].

At the end of this section, we would like to mention several important foundational works closely related to the topic of this note. Harmonic functions and spaces of harmonic functions of several variables in various domains were extensively studied in the seminal papers by Carleson, Stein, Weiss, and others [45, 46, 14]. We also point out the important contributions of O. Oleinik and B. Pavlov [24, 25], which concern the central topic of this paper by establishing various embedding theorems in harmonic function spaces, along with extensive references therein. Furthermore, we refer the reader to [12, 20] for discussions on the boundary behavior of harmonic functions. We note that spherical harmonics—which play a fundamental role and are heavily utilized throughout this paper—have also been applied, for instance, in [16].

The theory of harmonic function spaces of several variables in various domains of $\mathbb{R}^n$ (and $\mathbb{C}^n$) originated in the foundational works of E. M. Stein and G. Weiss, and has since been further developed by numerous authors, leaving many intriguing open problems. In this paper, we also present several new discussions and pose open questions in this line of research. Finally, related results concerning coefficient multipliers in harmonic function spaces were obtained by R. Timoney and D. Stegenga.

In various spaces of analytic functions of one variable, sharp theorems on coefficient multipliers were established in classical works by Jevtić and Pavlović [19, 27], Mateljević [22], and Vukotić [47]. We refer the reader to these papers and pose the problem of extending these one-dimensional results to the unit polydisc in $\mathbb{C}^n$.

Finally, we mention the monograph [47], which provides a comprehensive collection of classical and recent sharp results concerning coefficient multipliers in analytic function spaces on the unit disk.

The study of coefficient multipliers for harmonic function spaces in several variables is a rapidly developing area with potential applications across multivariable function theory and harmonic analysis.

A wide variety of problems in harmonic function spaces of several variables have been investigated over recent decades; we refer the reader to [1, 2, 3, 11, 13] and the references therein.

We note that spherical harmonics were previously employed in [21] to construct explicit Bergman-type kernels for studying spaces of harmonic functions of several variables on the unit ball in $\mathbb{R}^n$. Using these kernels, several open problems were resolved, effectively extending classical one-dimensional results to higher dimensions. Specifically, characterizations of continuous linear functionals, bounded Bergman projections, and the action of fractional integration operators were established on A_p spaces of harmonic functions on the unit ball (we refer the reader to [21] and the references therein). However, questions concerning coefficient multipliers on these A_p harmonic Bergman spaces on the unit ball B_n, which are the main focus of this note, were not addressed there.

Throughout the paper, we generally omit detailed comments regarding the proof techniques, referring the reader instead to the cited works for full proofs and methodological details. All our proofs are based on standard and known estimates of complex and harmonic function theory of several variables in the unit ball and in the unit polydisc.

2. Main results on coefficient multipliers

The goal of this section, which is the main section of the paper, is to collect various new sharp results from papers by the first author concerning coefficient multipliers between spaces of harmonic functions of several variables. At the end of the section, we also provide certain new sharp results on coefficient multipliers in analytic spaces on the unit polydisc. Both topics represent novel and promising areas of research; accordingly, we also formulate several open questions throughout this note for interested readers.

Spherical harmonics play a central role in our work. For the reader's convenience, we briefly recall below some essential definitions and background properties concerning these functions.

We first recall the basic notation and standard harmonic function spaces on the unit ball, including Bergman, Bloch, mixed-norm, and weighted Hardy spaces.

Let $\mathbb{B}^n$ denote the open unit ball in $\mathbb{R}^n$, and let $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$ be the unit sphere in $\mathbb{R}^n$. As usual, dx' stands for the normalized Lebesgue measure on the unit sphere $\mathbb{S}^{n-1}$. For $x \in \mathbb{R}^n$, we have $x = rx'$, where $r = |x| = \sqrt{\sum_{j=1}^{n} x_j^2}$ and $x' \in \mathbb{S}^{n-1}$. We denote by $dx = dx_1 ... dx_n = r^{n-1} dr dx'$ the normalized Lebesgue measure on $\mathbb{B}^n$ so that $\int_{\mathbb{B}^n} dx = 1$. We also denote by $h(\Omega)$ the space of all harmonic functions in an open set Ω.

Spaces of harmonic functions with mixed norm, Bergman type, Bloch type and Weighted Hardy type in the unit ball were studied by many authors in recent decades; we mention, for example, various research papers of A. Aleksandrov, P. Kargaev, A. E. Djrbashian and many others. We mention that very similar function classes of harmonic functions of several variables but in case of the upper half-plane were also studied by many researchers in recent decades. The problem of description of spaces of coefficient multipliers of these harmonic function classes of several variables was an open problem for many decades.

Harmonic weighted Bergman spaces $A^p_\alpha(\mathbb{B}^n)$ on $\mathbb{B}^n$ are defined, for $\alpha > -1$ and $0 < p \leq \infty$, by

$$A^p_\alpha(\mathbb{B}^n) = \left\{ f \in h(\mathbb{B}^n) : \|f\|_{p,\alpha} = \left(\int_{\mathbb{B}^n} |f(rx')|^p (1-r)^\alpha r^{n-1} dr dx' \right)^{\frac{1}{p}} < \infty \right\},$$

$$A^\infty_\alpha(\mathbb{B}^n) = \left\{ f \in h(\mathbb{B}^n) : \|f\|_{\infty,\alpha} = \sup_{x \in \mathbb{B}^n} |f(x)|(1-|x|)^\alpha < \infty \right\}.$$

It is easy to see that the spaces $A^p_\alpha = A^p_\alpha(\mathbb{B}^n)$ are Banach spaces for $1 \leq p \leq \infty$ and complete metric spaces for $0 < p < 1$.

Spherical harmonics, which form the core foundation of this paper, were studied in the classical work of E. Stein [46] and the well-known monograph by E. Stein and G. Weiss [44]. Furthermore, to the best of our knowledge, the Bergman spaces A_p of harmonic functions of several variables in the unit ball in $\mathbb{R}^n$ were first introduced and examined by R. Coifman and R. Rochberg [15].

For $0 < p < \infty$, $0 \leq r < 1$ and $f \in h(\mathbb{B}^n)$, we set

$$M_p(f, r) = \left(\int_{\mathbb{S}^{n-1}} |f(rx')|^p dx' \right)^{\frac{1}{p}},$$

with the usual modification in case $p = \infty$. Weighted Hardy spaces are defined, for $\alpha \geq 0$ and $0 < p \leq \infty$, by

$$H^p_\alpha(\mathbb{B}^n) = H^p_\alpha = \left\{ f \in h(\mathbb{B}^n) : \|f\|_{p,\alpha} = \sup_{r<1} M_p(f, r)(1-r)^\alpha < \infty \right\}.$$

For $\alpha = 0$, the space H^p_α is denoted simply by H^p.

For $0 < p \leq \infty$, $0 < q \leq \infty$ and $\alpha > 0$, we set mixed (quasi)-norms $\|f\|_{p,q;\alpha}$ as

$$\|f\|_{p,q;\alpha} = \left(\int_0^1 M_q(f,r)^p (1-r^2)^{\alpha p - 1} r^{n-1} dr \right)^{\frac{1}{p}}, \quad f \in h(\mathbb{B}^n), \tag{2.1}$$

with the conventional interpretation for $p = \infty$, and the corresponding spaces

$$B^{p,q}_\alpha(\mathbb{B}^n) = B^{p,q}_\alpha = \left\{ f \in h(\mathbb{B}^n) : \|f\|_{p,q;\alpha} < \infty \right\}.$$

Harmonic spaces of this type in the unit ball and the unit disk have been extensively studied in [1, 2, 3, 44, 43, 21].

Definition 2.1. *Let $0 < p, q < \infty$ and $\alpha > 0$. The harmonic Triebel-Lizorkin space $F^{p,q}_\alpha(\mathbb{B}^n) = F^{p,q}_\alpha$ consists of all functions $f \in h(\mathbb{B}^n)$ such that*

$$\|f\|_{F^{p,q}_\alpha} = \left(\int_{\mathbb{S}^{n-1}} \left(\int_0^1 |f(rx')|^p (1-r)^{\alpha p - 1} dr \right)^{\frac{q}{p}} dx' \right)^{\frac{1}{q}} < \infty$$

Next, we recall some necessary facts regarding spherical harmonics and the Poisson kernel. Let $Y^{(k)}_j$ be the spherical harmonics of order k, $1 \leq j \leq d_k$, on $\mathbb{S}^{n-1}$. Next,

$$Z^{(k)}_{x'}(y') = \sum_{j=1}^{d_k} Y^k_j(x') \overline{Y^k_j(y')}$$

are zonal harmonics of order k. Note that the spherical harmonics $Y^{(k)}_j (k \geq 0, 1 \leq j \leq d_k)$ form an orthonormal basis of $L^2(\mathbb{S}^{n-1}, dx')$. Every $f \in h(\mathbb{B}^n)$ has an expansion

$$f(x) = f(rx') = \sum_{k=0}^\infty r^k b_k \cdot Y^k(x'),$$

where $b_k = \left(b^1_k, ..., b^{d_k}_k \right)$, $Y^k = \left(Y^{(k)}_1, ..., Y^{(k)}_{d_k} \right)$ and $b_k \cdot Y^k$ denotes their scalar product, that is, $b_k \cdot Y^k = \sum_{j=1}^{d_k} b^j_k Y^{(k)}_j$. We often write $b_k = b_k(f)$ and $b^j_k = b^j_k(f)$, $f \in h(\mathbb{B}^n)$, to emphasize their dependence on f. In fact, b^j_k, $k \geq 0$, $1 \leq j \leq d_k$, are linear functionals on $h(\mathbb{B}^n)$.

The Poisson kernel of the unit ball, denoted by $P(x, y')$, is defined by

$$\begin{aligned} P(x, y') = P_{y'}(x) &= \sum_{k=0}^\infty r^k \sum_{j=1}^{d_k} Y^{(k)}_j(y') Y^{(k)}_j(x') \\ &= \frac{1}{n w_n} \frac{1 - |x|^2}{|x - y'|^n}, \quad x = rx' \in \mathbb{B}^n, \; y' \in \mathbb{S}^{n-1}, \end{aligned}$$

where w_n denotes the volume of the unit ball in $\mathbb{R}^n$. We will also use the Bergman kernel associated with the spaces A^p_β, which is defined as

$$Q_\beta(x,y) = 2\sum_{k=0}^{\infty} \frac{\Gamma(\beta+1+k+\frac{n}{2})}{\Gamma(\beta+1)\Gamma(k+\frac{n}{2})} r^k \rho^k Z^{(k)}_{x'}(y'), \quad x = rx',\ y = \rho y' \in \mathbb{B}^n,\ \beta \geq 0. \quad (2.2)$$

We recall the following standard definitions from [4]. These representations of the Bergman and Poisson kernels play a central role in the harmonic function theory of several variables. Over recent decades, they have proven essential in resolving various research problems on the unit ball, as well as in the context of harmonic function spaces on the upper half-space.

Definition 2.2. *For a double indexed sequence $c = \{c^j_k : k \geq 0,\ 1 \leq j \leq d_k\}$ and a harmonic function $f(rx') = \sum\limits_{k=0}^{\infty}\sum\limits_{j=1}^{d_k} r^k b^j_k(f) Y^{(k)}_j(x')$, we define*

$$(c * f)(rx') = \sum_{k=0}^{\infty}\sum_{j=1}^{d_k} r^k c^j_k b^j_k(f) Y^{(k)}_j(x'), \quad rx' \in \mathbb{B}^n,$$

provided that the series converges in $\mathbb{B}^n$. Similarly, for $f, g \in h(\mathbb{B}^n)$, we define their convolution by

$$(f * g)(rx') = \sum_{k=0}^{\infty}\sum_{j=1}^{d_k} r^k b^j_k(f) b^j_k(g) Y^{(k)}_j(x'), \quad rx' \in \mathbb{B}^n.$$

It is easily seen that $f * g$ is well defined and harmonic in $\mathbb{B}^n$.

We define an operator of fractional derivative for harmonic functions in the unit ball, which plays a crucial role in this paper. We should note that this operator was used by many authors previously in the theory of spaces of harmonic functions of several variables.

Definition 2.3. *For $t > 0$ and a harmonic function $f(x) = \sum\limits_{k=0}^{\infty} r^k b_k(f) Y^k(x')$ in the unit ball, we define the fractional derivative of order t of f by*

$$(\Lambda_t f)(x) = \sum_{k=0}^{\infty} r^k \frac{\Gamma(k+n/2+t)}{\Gamma(k+n/2)\Gamma(t)} b_k(f) \cdot Y^k(x'), \quad x = rx' \in \mathbb{B}^n.$$

Finally, based on Definition 2.2, we define the so-called spaces of coefficient multipliers for harmonic function spaces of several variables.

Definition 2.4. *Let X and Y be subspaces of $h(\mathbb{B}^n)$. We say that a double indexed sequence c is a multiplier from X to Y if $c * f \in Y$ for every $f \in X$. The vector space of all multipliers from X to Y is denoted by $M_H(X,Y)$.*

The notion of coefficient multipliers in harmonic spaces on the unit ball via spherical harmonics was introduced by the first author in [37]. We now collect the relevant sharp multiplier results from [4, 5, 8, 30].

Theorem 1. [4] *Let $1 < p \leq q \leq \infty$ and $m > \alpha - 1$. Then, for a double indexed sequence $c = \{c_k^j : k \geq 0,\ 1 \leq j \leq d_k\}$, the following conditions are equivalent:*

(1) $c \in M_H\left(B_\alpha^{p,1}, B_\beta^{q,1}\right)$.

(2) The function $g(x) = \sum_{k\geq 0} r^k \sum_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$ *is harmonic in $\mathbb{B}^n$ and satisfies the condition*

$$N_1(g) = \sup_{0\leq\rho<1} \sup_{y'\in\mathbb{S}^{n-1}} (1-\rho)^{\beta-\alpha+m+1} \int_{\mathbb{S}^{n-1}} |\Lambda_{m+1}(g * P_{x'})(\rho y')| dx' < \infty.$$

The following theorem characterizes the multiplier spaces mapping $B_\alpha^{p,1}$ into H_β^s.

Theorem 2. [4] *Let $\beta \geq 0$, $0 < p \leq 1, s \geq 1$ and $m > \alpha - 1$. Then, for a double indexed sequence $c = \{c_k^j : k \geq 0, 1 \leq j \leq d_k\}$, the following two conditions are equivalent:*

(1) $c \in M_H\left(B_\alpha^{p,1}, H_\beta^s\right)$.

(2) The function $g(x) = \sum_{k\geq 0} r^k \sum_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$ *is harmonic in $\mathbb{B}^n$ and satisfies the condition*

$$N_s(g) = \sup_{0\leq\rho<1} \sup_{y'\in\mathbb{S}^{n-1}} (1-\rho)^{m+1-\alpha+\beta} \left(\int_{\mathbb{S}^{n-1}} |\Lambda_{m+1}(g * P_{x'})(\rho y')|^s dx' \right)^{\frac{1}{s}} < \infty.$$

In the following theorem, we provide descriptions of spaces of multipliers from $B_\alpha^{p,t}$ to H_β^s.

Theorem 3. [8] *Let $0 < t, p \leq 1$, $1 \leq s \leq \infty$ and $m > \max\left(\alpha + \frac{n-1}{t} - n, -1\right)$. Then, for a double indexed sequence $c = \{c_k^j : k \geq 0, 1 \leq j \leq d_k\}$, the following conditions are equivalent:*

(1) $c \in M_H\left(B_\alpha^{p,t}, H_\beta^s\right)$.

(2) The function $g(x) = \sum_{k\geq 0} r^k \sum_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$ *is harmonic in $\mathbb{B}^n$ and satisfies the condition*

$$T_s(g) = \sup_{0\leq\rho<1} \sup_{y'\in\mathbb{S}^{n-1}} (1-\rho)^{\beta-\alpha+m+n-\frac{n-1}{t}} \left(\int_{\mathbb{S}^{n-1}} |\Lambda_{m+1}(g * P_{x'})(\rho y')|^s dx' \right)^{\frac{1}{s}} < \infty.$$

For $1 < t \leq \infty$, condition 1 implies condition 2.

We further consider multipliers in spaces $F_\alpha^{p,t}$ and H_β^s.

Theorem 4. [8] *Let $0 < t \leq p \leq 1 \leq s \leq \infty$ and $m > \max\left(\alpha + \frac{n-1}{t} - n, -1\right)$. Then, for a double indexed sequence $c = \{c_k^j : k \geq 0, 1 \leq j \leq d_k\}$, the following conditions are equivalent:*

(1) $c \in M_H\left(F_\alpha^{p,t}, H_\beta^s\right)$.

(2) The function $g(x) = \sum_{k \geq 0} r^k \sum_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$ is harmonic in $\mathbb{B}^n$ and satisfies the condition

$$T_s(g) = \sup_{0 \leq \rho < 1} \sup_{y' \in \mathbb{S}^{n-1}} (1-\rho)^{\beta - \alpha + m + n - \frac{n-1}{t}} \left(\int_{\mathbb{S}^{n-1}} |\Lambda_{m+1}(g * P_{x'})(\rho y')|^s dx' \right)^{\frac{1}{s}} < \infty.$$

Descriptions of the multiplier spaces from H_α^t to H_β^s are provided in the theorem below.

Theorem 5. [8] *Let $0 < t \leq 1 \leq s \leq \infty$, $\alpha \geq 0$, $m > \max\left(\alpha + \frac{n-1}{t} - n, -1\right)$ and $\beta > 0$. Then, for a double indexed sequence $c = \{c_k^j : k \geq 0, 1 \leq j \leq d_k\}$, the following conditions are equivalent:*

(1) $c \in M_H\left(H_\alpha^t, H_\beta^s\right)$.

(2) The function $g(x) = \sum_{k \geq 0} r^k \sum_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$ is harmonic in $\mathbb{B}^n$ and satisfies the condition

$$T_s(g) = \sup_{0 \leq \rho < 1} \sup_{y' \in \mathbb{S}^{n-1}} (1-\rho)^{\beta - \alpha + m + n - \frac{n-1}{t}} \left(\int_{\mathbb{S}^{n-1}} |\Lambda_{m+1}(g * P_{x'})(\rho y')|^s dx' \right)^{\frac{1}{s}} < \infty.$$

In the following theorem, we provide descriptions of spaces of multipliers from H^t to H^s.

Theorem 6. [8] *Let $0 < t < 1 \leq s \leq \infty$ and $m > \max\left(\frac{n-1}{t} - n, -1\right)$. Then, for a double indexed sequence $c = \{c_k^j : k \geq 0, 1 \leq j \leq d_k\}$, the following conditions are equivalent:*

(1) $c \in M_H\left(H^t, H^s\right)$.

(2) The function $g(x) = \sum_{k \geq 0} r^k \sum_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$ is harmonic in $\mathbb{B}^n$ and satisfies the condition

$$T_s(g) = \sup_{0 \leq \rho < 1} \sup_{y' \in \mathbb{S}^{n-1}} (1-\rho)^{m + n - \frac{n-1}{t}} \left(\int_{\mathbb{S}^{n-1}} |\Lambda_{m+1}(g * P_{x'})(\rho y')|^s dx' \right)^{\frac{1}{s}} < \infty.$$

The following result characterizes multipliers into Triebel–Lizorkin spaces.

Theorem 7. [8] *Let* $0 < p \leq 1 \leq q \leq \infty$ *and* $m > \alpha - 1$. *Then, for a double indexed sequence* $c = \{c_k^j : k \geq 0,\ 1 \leq j \leq d_k\}$, *the following conditions are equivalent:*

(1) $c \in M_H\left(B_\alpha^{p,1}, F_\beta^{q,1}\right)$.

(2) The function $g(x) = \sum_{k\geq 0} r^k \sum_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$ *is harmonic in* $\mathbb{B}^n$ *and satisfies the condition*

$$N_1(g) = \sup_{0\leq\rho<1} \sup_{y'\in\mathbb{S}^{n-1}} (1-\rho)^{\beta-\alpha+m+1} \int_{\mathbb{S}^{n-1}} |\Lambda_{m+1}(g * P_{x'})(\rho y')| dx' < \infty.$$

The subsequent theorem establishes the description of multiplier spaces from $B_\alpha^{p,1}$ to $B_\beta^{q,s}$.

Theorem 8. [8] *Let* $1 \leq p \leq q \leq \infty$, $1 \leq s \leq \infty$ *and* $m > \alpha - 1$. *Then, for a double indexed sequence* $c = \{c_k^j : k \geq 0,\ 1 \leq j \leq d_k\}$, *the following conditions are equivalent:*

(1) $c \in M_H\left(B_\alpha^{p,1}, B_\beta^{q,s}\right)$.

(2) The function $g(x) = \sum_{k\geq 0} r^k \sum_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$ *is harmonic in* $\mathbb{B}^n$ *and satisfies the condition*

$$N_s(g) = \sup_{0\leq\rho<1} \sup_{y'\in\mathbb{S}^{n-1}} (1-\rho)^{\beta-\alpha+m+1} \left(\int_{\mathbb{S}^{n-1}} |\Lambda_{m+1}(g * P_{x'})(\rho y')|^s dx' \right)^{\frac{1}{s}} < \infty.$$

The subsequent theorem establishes the description of the multiplier spaces from $B_\alpha^{p,\infty}$ to $B_\alpha^{q,\infty}$.

Theorem 9. [8] *Let* $1 < p, q < \infty$ *and* $m > \alpha - 1$. *Then, for a double indexed sequence* $c = \{c_k^j : k \geq 0,\ 1 \leq j \leq d_k\}$, *the following conditions are equivalent:*

(1) $c \in M_H\left(B_\alpha^{p,\infty}, B_\alpha^{q,\infty}\right)$.

(2) The function $g(x) = \sum_{k\geq 0} r^k \sum_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$ *is harmonic in* $\mathbb{B}^n$ *and satisfies the condition* $N_1(g) < \infty$.

Descriptions of the spaces of multipliers from B_0 to B_0 are provided in the following theorem.

Theorem 10. [8] *Let* $m > -1$. *Then, for a double indexed sequence* $c = \{c_k^j : k \geq 0,\ 1 \leq j \leq d_k\}$, *the following conditions are equivalent:*

(1) $c \in M_H(B_0, B_0)$.

(2) *The function* $g(x) = \sum_{k\geq 0} r^k \sum_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$ *is harmonic in* $\mathbb{B}^n$ *and satisfies the condition*

$$\tilde{N}_1(g) = \sup_{0\leq\rho<1} \sup_{y'\in\mathbb{S}^{n-1}} (1-\rho)^{m+1} \int_{\mathbb{S}^{n-1}} |\Lambda_{m+1}(g * P_{x'})(\rho y')| dx' < \infty.$$

We define N_∞ as usual, replacing the L^p norm in N_p by the L_∞ norm. In the following theorem, we provide descriptions of spaces of multipliers from $B_\alpha^{p,1}$ to $B_\beta^{q,\infty}$.

Theorem 11. [8] *Let* $0 < p \leq q \leq \infty$, $m > \alpha - 1$. *Then, for a double indexed sequence* $c = \{c_k^j : k \geq 0,\ 1 \leq j \leq d_k\}$, *the following conditions are equivalent:*

(1) $c \in M_H\left(B_\alpha^{p,1}, B_\beta^{q,\infty}\right)$.

(2) *The function* $g(x) = \sum_{k\geq 0} r^k \sum_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$ *is harmonic in* $\mathbb{B}^n$ *and satisfies the condition* $N_\infty(g) < \infty$.

We describe the spaces of multipliers from H^s to A_β^∞ in the theorem below.

Theorem 12. [5] *Let* $1 < s < \infty, \beta > 0$ *and let* s' *be the exponent conjugate to* s. *Then,* $c \in M_H\left(H^s, A_\beta^\infty\right)$ *if and only if the function* $g = g_c$ *satisfies the condition*

$$M_{s'}(g) = \sup_{0\leq\rho<1} \sup_{y'\in\mathbb{S}^{n-1}} (1-\rho)^\beta \left(\int_{\mathbb{S}^{n-1}} |(g * P_{x'})(\rho y')|^{s'} dx' \right)^{\frac{1}{s'}} < \infty.$$

In the following theorem, we provide descriptions of spaces of multipliers from $B_\alpha^{p,q}$ to A_β^∞.

Theorem 13. [5] *Let* $1 < p < \infty$, $1 \leq q < \infty$, α, $\beta > 0$ *and let* q' *be the exponent conjugate to* q. *Assume* $m > \max(\alpha - \beta - 1, \beta - 1)$. *Then,* $c \in M_H\left(B_\alpha^{p,q}, A_\beta^\infty\right)$ *if and only if the function* $g = g_c$ *satisfies the condition*

$$N_{q'}(g) = \sup_{0\leq\rho<1} \sup_{y'\in\mathbb{S}^{n-1}} (1-\rho)^{m+1+\beta-\alpha} \left(\int_{\mathbb{S}^{n-1}} |\Lambda_{m+1}(g * P_{x'})(\rho y')|^{q'} dx' \right)^{\frac{1}{q'}} < \infty.$$

In the following theorem, we provide descriptions of spaces of multipliers from $DB_\alpha^{p,1}$ to H_β^s.

Theorem 14. [5] *Let* $1 < s < \infty$, $\alpha, \beta > 0$, $0 < p \leq 1$ *and let* s' *be the exponent conjugate to* s. *Then,* $c \in M_H\left(DB_\alpha^{p,1}, H_\beta^s\right)$ *if and only if the function* $g = g_c$ *satisfies* *the condition*

$$L_{s'}(g) = \sup_{0\le\rho<1} \sup_{y'\in\mathbb{S}^{n-1}} (1-\rho)^{m+2+\beta-\alpha} \left(\int_{\mathbb{S}^{n-1}} |\Lambda_{m+1}(g * P_{x'})(\rho y')|^{s'} dx' \right)^{\frac{1}{s'}} < \infty.$$

In the following theorem, we provide descriptions of spaces of multipliers from $A^{p,1}_{\alpha}$ to $A^{p,1}_{\beta}$.

Theorem 15. [30] *Let* $0 < p \le 1$, $\alpha \in (0,1)$, $m > \max\{\alpha - 1, \frac{1}{p} - 1\}$, $\beta > 0$. *Suppose that* $g(x) = \sum_{k\ge0} r^k \sum_{j=1}^{d_k} c_k^{(j)} Y_j^{(k)}(x')$ *is a harmonic function in* $\mathbb{B}^n$. *Then, for a double indexed sequence* $c = \{c_k^j : k \ge 0,\ 1 \le j \le d_k\}$, *the following conditions are equivalent:*

(1) $c \in M_H\left(A^{p,1}_{\alpha}, A^{p,1}_{\beta}\right)$.

(2) $\sup_{0<\rho<1} \sup_{y'\in\mathbb{S}^{n-1}} \left(\int_{\mathbb{S}^{n-1}} |\Lambda_{m+1}(g * P_{x'})(\rho y')| dx' \right) (1-\rho)^{m+1-\alpha+\beta} < \infty.$

In the new sharp assertions presented below, we establish similar multiplier theorems within analytic function spaces of several variables—specifically on the unit polydisc in $\mathbb{C}^n$—thereby providing novel sharp results in this active area of research. We also pose several related open questions. In the one-dimensional case, such sharp results constitute a classical topic in complex function theory on the unit disk. The first sharp multiplier results on the unit disk were obtained by Hardy, Littlewood, Flett, Duren, Pavlović, and their coauthors. We first recall the standard definitions of the main analytic function spaces on the polydisc and then formulate our sharp results in this setting.

All theorems presented in this note on coefficient multipliers in analytic spaces on the unit polydisc are due to R. Shamoyan. These relatively recent results systematically extend previously known one-dimensional theorems on the unit disk obtained by various authors.

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in $\mathbb{C}$, $T = \partial U = \{z \in \mathbb{C} : |z| = 1\}$, U^n is the unit polydisc in $\mathbb{C}^n$, $T^n \subset \partial U^n$ is the distinguished boundary of U^n, $\mathbb{Z}_+ = \{n \in \mathbb{Z} : n \ge 0\}$, $\mathbb{Z}^n_+$ is the set of all multi-indexes and $I = [0,1)$.

We use the following notation: for $z = (z_1, ..., z_n) \in \mathbb{C}^n$ and $k = (k_1, ..., k_n) \in \mathbb{Z}^+_n$, we set $z^k = z_1^{k_1} ... z_n^{k_n}$, and for $(z_1, .., z_n) \in U^n$ and $\gamma \in \mathbb{R}$, we set $(1-|z|)^\gamma = (1-|z_1|)^\gamma ... (1-|z_n|)^\gamma$ and $(1-z)^\gamma = (1-z_1)^\gamma ... (1-z_n)^\gamma$. Next, for $z \in \mathbb{R}^n, w \in \mathbb{C}^n$, we set $wz = (w_1 z_1, ..., w_n z_n)$. Also, for $k \in \mathbb{Z}^n_+$ and $a \in \mathbb{R}$, we set $k + a = (k_1 + a, ..., k_n + a)$. For $z \in \mathbb{C}^n$, $\bar{z} = (\bar{z}_1, ..., \bar{z}_n)$. For $k \in (0,\infty)^n$, $\Gamma(k) = \Gamma(k_1) ... \Gamma(k_n)$.

The Lebesgue measure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ is denoted by $dV(z)$, the normalized Lebesgue measure on T^n is denoted by $d\xi = d\xi_1 ... d\xi_n$ and $dR = dR_1 ... dR_n$ is the Lebesgue measure on $[0,\infty)^n$.

The space of all functions holomorphic in U^n is denoted by $H(U^n)$. Every $f \in H(U^n)$ admits an expansion $f(z) = \sum_{k \in \mathbb{Z}_+^n} a_k z^k$. For $\beta > -1$, the operator of fractional differentiation is defined by

$$D^\beta f(z) = \sum_{k \in \mathbb{Z}_+^n} \frac{\Gamma(k+\beta+1)}{\Gamma(\beta+1)\Gamma(k+1)} a_k z^k, \ \ z \in U^n.$$

The fractional derivative operator corresponding to $\beta = (\beta_1, \ldots, \beta_n)$, where $\beta_j > -1, j = 1, \ldots, n$, is defined by

$$D^{\overrightarrow{\beta}} f(z) = \sum_{k \in \mathbb{Z}_+^n} \frac{\Gamma(k_j+\beta_j+1)}{\Gamma(\beta_j+1)\Gamma(k_j+1)} a_{k_1 \ldots k_n} z_1^{k_1} \ldots z_n^{k_n}.$$

For $f \in H(U^n)$, $0 < p < \infty$ and $r \in I^n$, we have

$$M_p(f, r) = \left(\int_{\mathbb{T}^n} |f(r\xi)|^p d\xi \right)^{\frac{1}{p}},$$

with the conventional modification in the case $p = \infty$. For $0 < p \leq \infty$, we define the standard analytic Hardy classes in the polydisc as follows:

$$H^p(U^n) = \left\{ f \in H(U^n) : \|f\|_{H^p} = \sup_{r \in I^n} M_p(f, r) < \infty \right\}.$$

For $0 < p \leq \infty$, $0 < q < \infty$ and $\alpha > 0$, we define the mixed (quasi-) norm spaces as follows:

$$A_\alpha^{p,q}(U^n) = \left\{ f \in H(U^n) : \|f\|^q_{A_\alpha^{p,q}} = \int_{I^n} M_p^q(f, r)(1-r)^{\alpha q - 1} dr < \infty \right\}.$$

If we replace the integration over I^n above by integration over the unit interval I, we obtain similar spaces to the analytic spaces $A_\alpha^{p,q}$, but defined on a subframe, denoted by $B_\alpha^{p,q}$.

For $0 < p$, $q < \infty$ and $\alpha > 0$, the Lizorkin–Triebel space $F_\alpha^{p,q}(U^n)$, denoted simply by $F_\alpha^{p,q}$, consists of all functions $f \in H(U^n)$ such that

$$\|f\|^p_{F_\alpha^{p,q}} = \int_{T^n} \left(\int_{I^n} |f(r\xi)|^q (1-r)^{\alpha q - 1} dr \right)^{\frac{p}{q}} d\xi < \infty.$$

If we replace the integration over I^n above by integration over the unit interval I, then we obtain similar spaces to the analytic spaces $F_\alpha^{p,q}$, but defined on a subframe, denoted by $T_\alpha^{p,q}$.

For all $\alpha \geq 0, \beta \geq 0$, we set

$$A_{\alpha,\beta}^{\infty,\infty}(U^n) = \left\{ f \in H(U^n) : \|f\|_{A_{\alpha,\beta}^{\infty,\infty}} = \sup_{r \in I^n} (M_\infty(D^\alpha f, r))(1-r)^\beta < \infty \right\}.$$

We note that replacing q by ∞ in the usual way gives us some other spaces $F^{p,\infty}$. The limit space case $F^{p,\infty,s}(U^n)$ is defined as the space of all analytic functions f in the polydisc such that the function $\phi(\xi) = \sup_{r\in I^n}|f(r\xi)|(1-r)^s$, $\xi \in T^n$ is in $L^p(T^n, d\xi)$, $s \geq 0$, $0 < p \leq \infty$.

Ultimately, the limit space case $A^{p,\infty,s}(U^n)$ is the space of all analytic functions f in the polydisc such that

$$\sup_{r\in I^n} M_p(f,r)(1-r)^s < \infty, \quad s \geq 0,\ 0 < p \leq \infty.$$

It is clear that the limit $F^{p,\infty,s}$ space is embedded in the last space we defined.

The following definition of coefficient multipliers is well known in the unit disk.

Definition 2.5. *Let X and Y be quasi-normed subspaces of $H(U^n)$. A sequence $c = \{c_k\}_{k\in\mathbb{Z}_+^n}$ is said to be a coefficient multiplier from X to Y if, for any function $f(z) = \sum_{k\in\mathbb{Z}_+^n} a_k z^k$ in X, the function $h = M_c f$, defined by $h(z) = \sum_{k\in\mathbb{Z}_+^n} c_k a_k z^k$, is in Y. The set of all multipliers from X to Y is denoted by $M_T(X,Y)$.*

This concept and the definition of coefficient multipliers in analytic spaces on the polydisc are well established; related problems were previously investigated in the works of M. Nawrocki, M. Gvaradze, and others.

Let $dm_2(\xi)$ denote the Lebesgue measure on the unit disk $\mathbb{D}$. Analogously, by replacing 2 with $2n$, $dm_{2n}(\xi)$ denotes the Lebesgue measure on the unit polydisc U^n.

Let $L^{\overrightarrow{p}}(\overrightarrow{\alpha})$ be the space of all measurable functions in U^n such that

$$\|f\|_{L^{\overrightarrow{p}}(\overrightarrow{\alpha})} = \Bigg(\int_U (1-|\xi_n|)^{\alpha_n}\Bigg(\int_U (1-|\xi_{n-1}|)^{\alpha_{n-1}} \dots \Bigg(\int_U (1-|\xi_1|)^{\alpha_1}|f(\xi_1,\dots,\xi_n)|^{p_1}\times$$

$$\times dm_2(\xi_1)\Bigg)^{\frac{p_2}{p_1}} \dots dm_2(\xi_{n-1})\Bigg)^{\frac{p_n}{p_{n-1}}} dm_2(\xi_n)\Bigg)^{\frac{1}{p_n}} < +\infty,$$

where all p_j are positive and all $\alpha_j > -1$ for any $j = 1,\dots,n$. Let $H^{\overrightarrow{p}}(\overrightarrow{\alpha}) = L^{\overrightarrow{p}}(\overrightarrow{\alpha}) \cap H(U^n)$.

We now define new mixed-norm weighted Hardy analytic spaces on the polydisc in the following natural way. Let

$$H^{\overrightarrow{p}}_{\overrightarrow{\alpha}} = \sup_{r_j<1}\left(\int_T \dots \Bigg(\int_T \Bigg(\int_T |f(r\overrightarrow{\xi})|^{p_1} d\xi_1\Bigg)^{\frac{p_2}{p_1}} d\xi_2\Bigg)^{\frac{p_3}{p_2}} \dots d\xi_n\right)^{\frac{1}{p_n}} \prod_{j=1}^n (1-r_j)^{\alpha_j}$$

where $p_j \in (0,\infty]$, $\alpha_j \geq 0$, $j = 1,\dots,n$.

Let $F^{\infty,q}_{s,\alpha}$ denote the space of analytic functions on the unit polydisc U^n such that

$$\int_{U^n} \frac{|f(z)|^q(1-|z|)^s(1-|w|)^\alpha}{|1-wz|^{\alpha+1}} dm_{2n}(z)$$

is bounded on U^n as a function of w. Here, we assume that $s > -1$, while the remaining parameters q and α are positive.

Below, we present a series of sharp, recent results on coefficient multipliers in analytic function spaces on the polydisc due to R. Shamoyan. These theorems systematically generalize several well-known one-dimensional results established by various authors over recent decades.

The following sharp theorems provide a complete characterization of the spaces of coefficient multipliers mapping $F^{q,\infty}_{s,\alpha}$ into $A^{p,\nu}_{\gamma}$, as well as between other related pairs of function spaces. For the formal definitions of the various function spaces that appear in these statements but are not explicitly defined here, we refer the reader to the works of R. Shamoyan listed in the references (see, e.g., [31, 32, 34, 35, 38, 40, 41, 42]).

Analytic function spaces of this type, along with their pointwise and coefficient multipliers on the unit disk and unit ball, have been extensively studied over recent decades by Ortega, Fàbrega, Pavlović, Jevtić, Blasco, Yaroslavtseva, among many others.

In the following theorem, we provide a complete characterization of the multiplier spaces mapping $F^{\infty,q}_{sq-1,\alpha}$ into $A^{p,\nu}_{\gamma}(U^n)$.

Theorem 16. [31] *Let* $0 < p,\ v < \infty$, $\gamma > 0$, *and* $g \in H(U^n)$.

(1) If $g(z) = \sum\limits_{|k|\geq 0}^{\infty} c_{k_1,\ldots,k_n} z^{k_1} \ldots z^{k_n}$ *is a multiplier from* $F^{\infty,q}_{sq-1,\alpha}$ *to* $A^{p,v}_{\gamma}(U^n)$, *then*

$$\sup_{r \in I^n} M_p(D^\beta g, r) \prod_{j=1}^{n} (1-r_j)^{\gamma+\beta_j+1-s} < \infty, \quad \text{for all } \beta_j > s-1-\gamma,\ j = 1, \ldots, n.$$

(2) If $g(z) = \sum\limits_{k_1 \geq 0, \ldots, k_n \geq 0}^{\infty} c_{k_1,\ldots,k_n} z^{k_1} \ldots z^{k_n}$ *is a multiplier from* $F^{\infty,q}_{sq-1,\alpha}$ *to* $F^{p,v}_{\gamma}(U^n)$, *then for* $v \leq p$,

$$\sup_{r \in I^n} M_p(D^\beta g, r) \prod_{j=1}^{n} (1-r_j)^{\gamma+\beta_j+1-s} < \infty, \quad \text{for all } \beta_j > s-1-\gamma,\ j = 1, \ldots, n.$$

Below, we present an extensive series of recent, sharp results concerning coefficient multipliers in analytic spaces on the unit polydisc, derived from the works of R. Shamoyan cited in the references. For the definitions of the function spaces appearing in the next theorem, we refer the reader to the earlier papers of the first author.

A complete characterization of the multipliers from BMOA into $A^{p,q}_{\gamma}(U^n)$ and $F^{p,\nu}_{\gamma}(U^n)$ is given in the following theorem.

Theorem 17. [31] *Let* $0 < p, q < \infty$, *and* $g \in H(U^n)$.

(1) *If* $\gamma > 0$ *and* $g(z) = \sum\limits_{|k|\geq 0}^{\infty} c_{k_1,\dots,k_n} z^{k_1} \dots z^{k_n}$ *is a multiplier from BMOA to* $A_{\gamma}^{p,q}(U^n)$,

$$\sup_{r\in I^n} M_p(D^{\beta}g, r)\prod_{j=1}^{n}(1-r_j)^{\gamma+\beta_j+2+v} < \infty, \text{ where } \beta > -v-2 \text{ for any } v > -1.$$

(2) *If* $g(z) = \sum\limits_{k_1\geq 0,\dots,k_n\geq 0}^{\infty} c_{k_1,\dots,k_n} z^{k_1} \dots z^{k_n}$ *is a multiplier from BMOA to* $F_{\gamma}^{p,q}(U^n)$, *then for* $q \leq p$,

$$\sup_{r\in I^n} M_p(D^{\beta}g, r)\prod_{j=1}^{n}(1-r_j)^{\gamma+\beta_j+2+v} < \infty, \text{ where } \beta > -v-2 \text{ for any } v > -1.$$

In the following theorem, we provide complete descriptions of multipliers from Y to $F_{\gamma}^{p,\infty}$ and $A_{\gamma}^{p,\infty}$.

Theorem 18. [31] *Let* $p, \gamma > 0$, *and* $g(z) = \sum\limits_{|k|\geq 0} c_k z^k \in H(U^n)$.

(1) *If* $g \in M_T\left(Y, F_{\gamma}^{p,\infty}\right)$, *then*

$$\sup_{r\in I^n} M_p(D^{\beta}g, r)\prod_{j=1}^{n}(1-r_j)^{\gamma+\beta_j+1-s} < \infty, \quad \text{for all } \beta_j > s-1-\gamma,\ j = 1,\dots,n.$$

(2) *If* $g \in M_T\left(Y, A_{\gamma}^{p,\infty}\right)$, *then*

$$\sup_{r\in I^n} M_p(D^{\beta}g, r)\prod_{j=1}^{n}(1-r_j)^{\gamma+\beta_j+1-s} < \infty, \quad \text{for all } \beta_j > s-1-\gamma,\ j = 1,\dots,n.$$

Throughout what follows, we use the convention $\frac{a}{b} = 0$ when $b = \infty$. We establish a full description of the multipliers mapping $A_{p,q}^{\gamma}$, $F_{p,q}^{\gamma}$, $A_{p,\infty}^{\gamma}$, $F_{p,\infty}^{\gamma}$, and $A_{\infty,q}^{\gamma}$ into Y in the following theorem.

Theorem 19. [31] *Let* $p > 0$, $q > 0$, $\gamma > 0$, *and* $g(z) = \sum\limits_{k\in\mathbb{Z}_+^n} c_k z^k \in H(U^n)$.

(1) *If* $g \in M_T\left(A_{\gamma}^{p,q}, Y\right)$, *then*

$$\sup_{r\in I^n} M_{\infty}(D^{\beta}g, r_1,\dots,r_n)\prod_{j=1}^{n}(1-r_j)^{\tau_j} < \infty,$$

where $\tau_j = v_j + \beta - \gamma - \frac{1}{p} + 1$, $j = 1,\dots,n$, *for* $\beta > \max\limits_{j}\left(\gamma + \frac{1}{p} - 1 - v_j\right)$.

(2) *If* $g \in M_T\left(F_{\gamma}^{p,q}, Y\right)$, *then*

$$\sup_{r\in I^n} M_{\infty}(D^{\beta}g, r_1,\dots,r_n)\prod_{j=1}^{n}(1-r_j)^{\tau_j} < \infty,$$

where $\tau_j = v_j + \beta - \gamma - \frac{1}{p} + 1,\ j = 1, ..., n,$ *for* $\beta > \max_j \left(\gamma + \frac{1}{p} - 1 - v_j\right).$

(3) *If* $g \in M_T\left(F_\gamma^{p,\infty}, Y\right)$, *then*

$$\sup_{r \in I^n} M_\infty(D^\beta g, r_1, ..., r_n) \prod_{j=1}^n (1 - r_j)^{\tau_j} < \infty,$$

where $\tau_j = v_j + \beta - \gamma - \frac{1}{p} + 1,\ j = 1, ..., n,$ *for* $\beta > \max_j \left(\gamma + \frac{1}{p} - 1 - v_j\right).$

(4) *If* $g \in M_T\left(A_\gamma^{p,\infty}, Y\right)$, *then*

$$\sup_{r \in I^n} M_\infty(D^\beta g, r_1, ..., r_n) \prod_{j=1}^n (1 - r_j)^{\tau_j} < \infty,$$

where $\tau_j = v_j + \beta - \gamma - \frac{1}{p} + 1,\ j = 1, ..., n,$ *for* $\beta > \max_j \left(\gamma + \frac{1}{p} - 1 - v_j\right).$

(5) *If* $g \in M_T\left(A_\gamma^{\infty,q}, Y\right)$, *then*

$$\sup_{r \in I^n} M_\infty(D^\beta g, r_1, ..., r_n) \prod_{j=1}^n (1 - r_j)^{\tau_j} < \infty,$$

where $\tau_j = v_j + \beta - \gamma - \frac{1}{p} + 1,\ j = 1, ..., n,$ *for* $\beta > \max_j \left(\gamma + \frac{1}{p} - 1 - v_j\right)$ *and* $p = \infty$.

We again assume that Y is the same BMOA space as in the previous theorem. In the following theorems on multipliers, we provide descriptions of spaces of coefficient multipliers from $M_\gamma^{p,q}$, $M_\gamma^{p,\infty}$, and $M_\gamma^{\infty,q}$ to the function space Y.

Theorem 20. [31] *Let* $p,\ q,\ \gamma > 0, v > 0,$ *and* $g(z) = \sum_{k \in \mathbb{Z}_+^n} c_k z^k \in H(U^n)$.

(1) *If* $g \in (M_T)\left(M_\gamma^{p,q}, Y\right)$, *then*

$$\sup_{r \in I} M_\infty(D^\beta g, r)(1 - r)^{v + n(\beta+1) - \gamma n - \frac{1}{p}} < \infty, \quad \text{for } \beta > \gamma + \frac{1}{np} - \frac{v}{n} - 1.$$

(2) *If* $g \in (M_T)\left(M_\gamma^{p,\infty}, Y\right)$, *then*

$$\sup_{r \in I} M_\infty(D^\beta g, r)(1 - r)^{v + n(\beta+1) - \gamma n - \frac{1}{p}} < \infty, \quad \text{for } \beta > \gamma + \frac{1}{np} - \frac{v}{n} - 1.$$

(3) *If* $g \in (M_T)\left(M_\gamma^{\infty,q}, Y\right)$, *then*

$$\sup_{r \in I} M_\infty(D^\beta g, r)(1 - r)^{v + n(\beta+1) - \gamma n} < \infty, \quad \text{for } \beta > \gamma - \frac{v}{n} - 1.$$

The theorem below completely characterizes the multiplier spaces mapping $T_\gamma^{p,q}$, $T_\gamma^{p,\infty}$, and $T_\gamma^{\infty,q}$ into Y.

Theorem 21. [31] *Let* $p,\ q,\ \gamma > 0$ *and* $g(z) = \sum\limits_{k\in\mathbb{Z}_+^n} c_k z^k \in H(U^n)$.

(1) *If* $g \in (M_T)\left(T_\gamma^{p,q}, Y\right)$, *then*

$$\sup_{r\in I^n} M_\infty(D^\beta g, r)\prod_{j=1}^n (1-r_j)^{\tau_j+\beta+1-\frac{\gamma}{n}-\frac{1}{p}} < \infty,$$

for $\beta > \max\limits_j \left(\frac{1}{p}+\frac{\gamma}{n}-\tau_j-1\right)$, $\tau_j > 0$, $j = 1, ..., n$.

(2) *If* $g \in (M_T)\left(T_\gamma^{p,\infty}, Y\right)$, *then*

$$\sup_{r\in I^n} M_\infty(D^\beta g, r)\prod_{j=1}^n (1-r_j)^{\tau_j+\beta+1-\frac{\gamma}{n}-\frac{1}{p}} < \infty,$$

for $\beta > \max\limits_j \left(\frac{1}{p}+\frac{\gamma}{n}-\tau_j-1\right)$, $\tau_j > 0$, $j = 1, ..., n$.

(3) *If* $g \in (M_T)\left(T_\gamma^{\infty,q}, Y\right)$, *then*

$$\sup_{r\in I^n} M_\infty(D^\beta g, r)\prod_{j=1}^n (1-r_j)^{\tau_j+\beta+1-\frac{\gamma}{n}-\frac{1}{p}} < \infty,$$

for $\beta > \max\limits_j \left(\frac{1}{p}+\frac{\gamma}{n}-\tau_j-1\right)$, $\tau_j > 0, j = 1, ..., n$, *and* $p = \infty$.

In the following theorems, we provide complete descriptions of multipliers from $F_\alpha^{p,q}(U^n)$ to $A_\beta^{t,s}(U^n)$ and from $A_\alpha^{p,q}(U^n)$ to $A_\beta^{t,s}(U^n)$.

Theorem 22. [32] *Let* $g(z) = \sum\limits_{k\in\mathbb{Z}_+^n} c_k z^k \in H(U^n)$. *Assume that* $\frac{t}{2} \le s \le t < \infty$, $0 < p, q \le 1$, $0 < \max(p, q) \le s \le 1$, $t\left(\alpha + \frac{1}{p}\right) < 2$, $m \in \mathbb{N}$ *and* $m > \frac{2}{t} - 1$.

(1) $\{c_k\}_{k\in\mathbb{Z}_+^n} \in M_T\left(F_\alpha^{p,q}(U^n), A_\beta^{t,s}(U^n)\right)$ *if and only if*

$$\sup_{r\in I^n} M_t(D^m g, r)(1-r)^{m+1-\frac{1}{p}+\beta-\alpha} < \infty.$$

(2) $\{c_k\}_{k\in\mathbb{Z}_+^n} \in M_T\left(A_\alpha^{p,q}(U^n), A_\beta^{t,s}(U^n)\right)$ *if and only if*

$$\sup_{r\in I^n} M_t\left(D^m g, r\right)(1-r)^{m+1-\frac{1}{p}+\beta-\alpha} < \infty.$$

Descriptions of the multiplier spaces from $F_\alpha^{p,q}(U^n)$ and $A_\alpha^{p,q}(U^n)$ to $A_\beta^{t,s}(U^n)$ are provided in the next theorem.

Theorem 23. [33] *Let* $g(z) = \sum\limits_{k\in\mathbb{Z}_+^n} c_k z^k \in H(U^n)$. *Assume that* $t \le 1$, $\frac{t}{2} < s \le t < \infty$, $\beta + \frac{1}{t} < \alpha + \frac{1}{p} < \frac{2}{t}$, $m \in \mathbb{N}$, *and* $m > \frac{2}{t} - 1$.

(1) $\{c_k\}_{k\in\mathbb{Z}_+^n} \in M_T\left(F_\alpha^{p,q}(U^n), A_\beta^{t,s}(U^n)\right)$ *if and only if*

$$\sup_{r\in I^n} M_t(D^m g, r)(1-r)^{m+1-\frac{1}{p}+\beta-\alpha} < \infty.$$

(2) $\{c_k\}_{k\in\mathbb{Z}_+^n} \in M_T\left(A_\alpha^{p,q}(U^n), A_\beta^{t,s}(U^n)\right)$ *if and only if*

$$\sup_{r\in I^n} M_t(D^m g, r)(1-r)^{m+1-\frac{1}{p}+\beta-\alpha} < \infty.$$

In the following theorem, we provide complete descriptions of multipliers from $F_\alpha^{p,q}(U^n)$ and $A_\alpha^{p,q}(U^n)$ to $F_\beta^{t,s}(U^n)$.

Theorem 24. [33] *Let* $g(z) = \sum_{k\in\mathbb{Z}_+^n} c_k z^k \in H(U^n)$. *Assume that* $t \leq 1$, $\frac{t}{2} < s \leq t < \infty$, $0 < \max(p,q) \leq s < \infty$, $\beta + \frac{1}{t} < \alpha + \frac{1}{p} < \frac{2}{t}$, $m \in \mathbb{N}$ *and* $m > \frac{2}{t} - 1$.

(1) $\{c_k\}_{k\in\mathbb{Z}_+^n} \in M_T\left(F_\alpha^{p,q}(U^n), F_\beta^{t,s}(U^n)\right)$ *if and only if*

$$\sup_{r\in I^n} M_t(D^m g, r)(1-r)^{m+1-\frac{1}{p}+\beta-\alpha} < \infty.$$

(2) $\{c_k\}_{k\in\mathbb{Z}_+^n} \in M_T\left(A_\alpha^{p,q}(U^n), F_\beta^{t,s}(U^n)\right)$ *if and only if*

$$\sup_{r\in I^n} M_t(D^m g, r)(1-r)^{m+1-\frac{1}{p}+\beta-\alpha} < \infty.$$

While our primary focus has been on the characterization of coefficient multipliers in multi-variable harmonic function spaces of Bergman, mixed-norm, and Bloch types, analogous questions naturally arise for harmonic Hardy and weighted Hardy spaces. These directions remain largely open, and we present them here as open problems for the reader.

Harmonic Hardy spaces in higher dimensions have been investigated in connection with various other problems by A. Aleksandrov, P. Kargaev [1, 2, 3], T. Wolff [48], among others, over recent decades. This area continues to offer many unresolved questions.

Furthermore, sharp estimates for Taylor coefficients, the action of fractional integration and differentiation operators, and related coefficient multipliers in one-dimensional Nevanlinna-type analytic classes were recently obtained in papers by E. Rodikova [28, 29], R. Meštrović, V. Gavrilov, and their coauthors. Extending these sharp one-dimensional results to higher dimensions represents another intriguing open problem that we pose to interested readers.

Similar results on coefficient multipliers are probably also valid for harmonic function spaces in the upper half-space R_+^{n+1}. We leave this open problem to the interested reader. We refer the reader to research papers [21, 30, 31, 32, 33, 34] of R. Shamoyan.

Recently, novel mixed-norm Bergman-type spaces of analytic functions on the unit polydisc were introduced by O. Yaroslavtseva [49], where complete characterizations of certain coefficient multiplier spaces were established. We refer the interested reader to [49] for these sharp results.

In this note, we intentionally omit problems related to pointwise multipliers in harmonic and analytic spaces in higher dimensions, as this constitutes a distinct and active area of research; for relevant findings in this direction, see [26].

Furthermore, we do not address coefficient multipliers for analytic function spaces on the unit ball or on more general complex domains in higher dimensions. Although one can define such multiplier spaces in analogous ways via Taylor series expansions, these issues remain largely unexplored. We pose the systematic study of coefficient multipliers in such higher-dimensional domains as an open problem for future research.

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