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NORMAL FILTER IN PSEUDO QUASI-ORDERED RESIDUATED SYSTEMS

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Abstract. The concept of quasi-ordered residuated systems is a generalization of both commutative residual lattices and hoop-algebras. On the other hand, it is a BE-algebra with respect to the residue operation. Pseudo quasi-ordered residuated systems is a noncommutative generalization of quasi-ordered residuated systems. In the present paper, we introduce the notion of normal filters in a pseudo quasi-ordered residuated system and give some characterizations of it. We state and prove some theorems which determine the relationship between this concept and the other types of filters in a pseudo quasi-ordered residuated system such as implicative and comparative filters.

1. Introduction

The filter theory of the logical algebras plays an important role in the studying of these algebras and the completeness of the corresponding non-classical logics. However, many researchers are only interested in their algebraic aspect.

Hoop algebras were presented by B. Bosbach in [9, 10]. Pseudo-hoop algebras were presented as non-commutative generalizations of hoop algebras by Georgescu, Leuştean and Preoteasa in [14], following after the notions of pseudo-MV algebras in [13] and pseudo-BL algebras ([12]). BE-algebras were introduced in [17] by S. Kim and Y. H. Kim and pseudo BE-algebras were discussed in 2013 in [4] by R. Ali Borzooei et al. The pseudo KU-algebras and the pseudo UP-algebras were studied in [19, 20] by this author. Residuated lattices were introduced by Ward and Dilworth in [27]. Commutative residuated lattices are discussed in [15]. At present, the filter theory of residuated lattices has been widely studied, and some important results have been obtained (for example, [1, 5, 7, 11, 18, 28]). Pseudo residuated lattices were in focus in [3].

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Quasi-ordered residuated system is a commutative residuated integral monoid ordered under a quasi-order, introduced by S. Bonzio and I. Chajda in [6] as a generalization of both hoop-algebras and commutative residuated lattices. (Let us note that the term 'quasi-order' covers reflexive and transitive relations.) Additionally, the residuum part of a quasi-ordered residuated system is a BE-algebra. In the last few years, the theory of quasi-ordered residuated systems was enriched with more results on filters in them (see, for example, [21, 22, 23, 25]). Here it is necessary to say that the concept of filters in this algebraic structure does not coincide with the concepts of filters neither in hoop-algebras nor in commutative residuated lattices.

Estimating that the generalization of an algebraic structure by adding the prefix 'pseudo' to its basic determination is of interest to the mathematical public, in author's papers [24, 26]. the mentioned idea is applied to the concept of quasi-ordered residuated systems. The concept of pseudo quaso-ordered residuated system was introduced and analyzed in [24]. In addition, the filters in the pseudo-algebras thus designed were discussed with special reference to the implicative and comparative filters therein. In the paper [26] associative and fantastic filters in pseudo quasi-ordered residuated systems were analyzed.

In this paper, as a continuation of previous research, in addition to the introduction of concept of normal filters in a pseudo quasi-ordered residuated system, some of the important features of normal filters in a pseudo quasi-ordered residuated system are recognized. In addition to the previous one, some connections were established between normal filters and comparative and implicative filters in this algebraic structure. In doing so, the specificities of filter determination in pseudo quasi-ordered residuated systems were taken into account.

2. Preliminaries

In this section, the necessary notions and notations and some of their interrelationships, mostly taken from paper [6, 21, 22, 23, 24], are listed in the order to enable a reader to comfortably follow the presentation in this report. It should be pointed out here that the notations for logical conjunction, logical implication and others have a literal meaning. The notation =: in the formula $A =: B$ serves to indicate that A in it is the abbreviation for the formula B.

2.1. Quasi-ordered residuated systems. In article [6], S. Bonzio and I. Chajda introduced and analyzed the concept of residual relational systems.

Definition 2.1 ([6], Definition 2.1)**.** *A residuated relational system is a structure* $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1, \preccurlyeq \rangle$, *where* $\langle A, \cdot, \rightarrow, 1 \rangle$ *is an algebra of type* $\langle 2, 2, 0 \rangle$ *and* $\preccurlyeq$ *is a quasi-order on* A *and satisfying the following properties:*

(1) $(A, \cdot, 1)$ *is a commutative monoid;*

(2) $(\forall x \in A)(x \preccurlyeq 1)$;

(3) $(\forall x, y, z \in A)(x \cdot y \preccurlyeq z \Longleftrightarrow x \preccurlyeq y \rightarrow z)$.

We will refer to the operation $\cdot$ as (commutative) multiplication, to $\rightarrow$ as its residuum and to condition (3) as residuation.

The following proposition shows the basic properties of quasi-ordered residuated systems.

Proposition 2.1 ([6], Proposition 3.1)**.** *Let $\mathfrak{A}$ be a quasi-ordered residuated system. Then*

(4) *The operation '$\cdot$' preserves the pre-order in both positions;*

$$(\forall x, y, z \in A)(x \preccurlyeq y \implies (x \cdot z \preccurlyeq y \cdot z \wedge z \cdot x \preccurlyeq z \cdot y));$$

(5) $(\forall x, y, z \in A)(x \preccurlyeq y \implies (y \rightarrow z \preccurlyeq x \rightarrow z \wedge z \rightarrow x \preccurlyeq z \rightarrow y))$;
(6) $(\forall y, z \in A)(x \cdot (y \rightarrow z) \preccurlyeq y \rightarrow x \cdot z)$;
(7) $(\forall x, y, z \in A)(x \cdot y \rightarrow z \preccurlyeq x \rightarrow (y \rightarrow z))$;
(8) $(\forall x, y, z \in A)(x \rightarrow (y \rightarrow z) \preccurlyeq x \cdot y \rightarrow z)$;
(9) $(\forall x, y, z \in A)(x \rightarrow (y \rightarrow z) \preccurlyeq y \rightarrow (x \rightarrow z))$;
(10) $(\forall x, yz \in A)((x \rightarrow y) \cdot (y \rightarrow z) \preccurlyeq x \rightarrow z)$;
(11) $(\forall x, y \in A)((x \cdot y \preccurlyeq x) \wedge (x \cdot y \preccurlyeq y))$;
(12) $(\forall x, y, z \in A)(x \rightarrow y \preccurlyeq (y \rightarrow z) \rightarrow (x \rightarrow z))$;
(13) $(\forall x, y, z \in A)(y \rightarrow z \preccurlyeq (x \rightarrow y) \rightarrow (x \rightarrow z))$.

It is generally known that a quasi-order relation $\preccurlyeq$ on a set A generates a equivalence relation $\equiv_{\preccurlyeq} =: \preccurlyeq \cap \preccurlyeq^{-1}$ on A. Due to properties (4) and (5), this equality relation is compatible with the operations in A. Thus, $\equiv_{\preccurlyeq}$ is a congruence on A.

In the light of the previous note, it is easy to see that the following applies:

(7) and (8) give:

(14) $(\forall x, y, z \in A)(x \cdot y \rightarrow z \equiv_{\preccurlyeq} x \rightarrow (y \rightarrow z))$.

Due to the universality of formula (9) (or, due to the commutativity of the multiplication from (14)), we have:

(15) $(\forall x, y, z \in A)(x \rightarrow (y \rightarrow z) \equiv_{\preccurlyeq} y \rightarrow (x \rightarrow z))$.

Also, from (11) and (2), it follows

(16) $(\forall x \in A)(x \rightarrow x \equiv_{\preccurlyeq} 1)$

In the general case,

(17) $(\forall x, y \in A)(x \preccurlyeq y \iff x \rightarrow y \equiv_{\preccurlyeq} 1)$

is valid, which is obtained by referring to (11) and (2).

From the previous analysis it can be concluded that a quasi-ordered residuated system is a generalization of a hoop-algebra (in the sense of [8]) because the following formula

$$(\forall x, y \in A)(x \cdot (x \rightarrow y) \equiv_{\preccurlyeq} y \cdot (y \rightarrow x))$$

does not have to be a valid formula in the observed algebraic structure in the general case.

On the other hand, it is obvious that the quasi-ordered residuated system is a BE-algebra (according to [17], Definition 1) with respect to the residuum operation.

The determination of logical algebras, which were previously mentioned and are not defined in this text, can be found in [16].

Since the axioms that determine the concept of hoop-algebras are mutually independent, we conclude that there is (at least one) model that does not satisfy the previous equality. So, there is (at least one) example of a quasi-ordered residuated system that it is not a hoop-algebra. Apart from the mentioned, many examples of quasi-ordered residual system can be found in the previously published texts of this author (for example, [21, 22, 23, 25]).

Definition 2.2 ([21], Definition 3.1)**.** *For a subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that it is a* filter *of $\mathfrak{A}$ if it satisfies conditions*

(F2) $(\forall u, v \in A)((u \in F \wedge u \preccurlyeq v) \implies v \in F)$, *and*

(F3) $(\forall u, v \in A)((u \in F \wedge u \to v \in F) \implies v \in F)$.

Let it note that the empty subset of A satisfies the conditions (F2) and (F3). Therefore, $\emptyset$ is a filter in $\mathfrak{A}$. It is shown ([21], Proposition 3.4 and Proposition 3.2), that if a non-empty subset F of a quasi-ordered system $\mathfrak{A}$ satisfies the condition (F2), then it also satisfies the following conditions

(F0) $1 \in F$ and

(F1) $(\forall u, v \in A)((u \cdot v \in F \implies (u \in F \wedge v \in F))$.

Also, it can be seen without difficulty that $((F3) \wedge F \neq \emptyset) \implies (F2)$ is valid. Indeed, if (F3) holds, then the formula $u \in F \wedge u \preccurlyeq v$, can be transformed into the formula $u \in F \wedge u \to v \equiv_{\preccurlyeq} 1 \in F$ by (F0) so from here, according to (F3) it can be demonstrate the validity of implications (F2). However, the reverse does not have to be valid.

If $\mathfrak{F}(A)$ is the family of all filters in a QRS $\mathfrak{A}$, then $\mathfrak{F}(A)$ is a complete lattice ([21], Theorem 3.1).

2.2. Pseudo quasi-ordered residuated systems.

Pseudo quasi-ordered residuated system is a non-commutative generalization of the concept of quasi-ordered residuated systems. This concept is given in the following definition.

Definition 2.3 ([24], Definition 2.1)**.** *A* pseudo quasi-ordered relational system *is a structure $\mathfrak{pA} =: \langle A, \cdot, \to, \rightsquigarrow, 1, \preccurlyeq \rangle$, where $\langle A, \cdot, \to, \rightsquigarrow, 1 \rangle$ is an algebra of type $\langle 2, 2, 2, 0 \rangle$ and $\preccurlyeq$ is a quasi-order on A satisfying the following properties:*

(1) $(A, \cdot, 1)$ *is a monoid;*

(2) $(\forall x \in A)(x \preccurlyeq 1)$;

(3L) $(\forall x, y, z \in A)(x \cdot y \preccurlyeq z \iff x \preccurlyeq y \to z)$;

(3R) $(\forall x, y, z \in A)(x \cdot y \preccurlyeq z \iff y \preccurlyeq x \rightsquigarrow z)$.

We will refer to the operation $\cdot$ as (non-commutative) multiplication, to $\to$ as its left residuum and to $\rightsquigarrow$ as right residuum.

The following examples allow to a reader to better understand the concept of pseudo quasi-ordered residuated systems:

Example 2.1: For a (non-commutative) monoid A, let $\mathfrak{P}(A)$ be denote the powerset of A ordered by set inclusion and '$\cdot$' the usual multiplication of subsets of A:

$$(\forall X, Y \in \mathfrak{P}(A)))(X \cdot Y =: \{xy : x \in X \wedge y \in Y\}).$$

Then $\langle \mathfrak{P}(A), \cdot, \rightarrow, A, \subseteq \rangle$ is a quasi-ordered residuated system in which the residuum are given by

$$(\forall X, Y \in \mathfrak{P}(A))(Y \rightarrow X := \{z \in A : Yz \subseteq X\})$$
$$(\forall X, Y \in \mathfrak{P}(A))(Y \rightsquigarrow X := \{z \in A : zY \subseteq X\}).$$

□

Example 2.2: Let $G = (G, +, -, 0, \vee, \wedge)$ be an arbitrary lattice ordered group. For an arbitrary element $u \in G$ such that $u \geqslant 0$ define on the set $G[u] = [0, u]$ the following operation

$$x \cdot y = (x - u + y) \wedge 0,\ x \rightarrow y = (y - x + u) \wedge u \text{ and } x \rightsquigarrow y = (u - x + y) \wedge u.$$

Then $\langle G[u], \cdot, \rightarrow, \rightsquigarrow, u \rangle$ is a pseudo quasi-ordered residuated system. □

Example 2.3: Let G be an arbitrary lattice ordered group and $nG = \{g \in G : g \leqslant 0\}$. On nG we define the following operations

$$x \cdot y = x + y,\ x \rightarrow y = (y - x) \wedge 0,\ x \rightsquigarrow y = (-x = y) \wedge 0.$$

Then $\langle nG, \cdot, \rightarrow, \rightsquigarrow, 0 \rangle$ is a pseudo quasi-ordered residuated system. □

This system of axioms is hereinafter referred to as pQRS.

Let $\mathfrak{pA} = \langle A, \cdot, \rightarrow, \rightsquigarrow, 1 \rangle$ be a pseudo quasi-ordered residuated system. Then the operation $\cdot$ in $A/\equiv_{\preccurlyeq}$ is commutative if and only if $\rightarrow \equiv \rightsquigarrow$. Indeed, for arbitrary elements $x, y, z \in A$ the following extended equivalence

$$x \cdot y \preccurlyeq z \iff x \preccurlyeq y \rightarrow z \iff x \preccurlyeq y \rightsquigarrow z \iff y \cdot x \preccurlyeq z$$

is valid. In particular, for $z \equiv y \cdot x$ (or for $z \equiv x \cdot y$) we get $x \cdot y \preccurlyeq y \cdot x \preccurlyeq x \cdot y$, i.e. we get $x \cdot y \equiv_{\preccurlyeq} y \cdot x$. Conversely, for arbitrary elements $x, y, z \in A$ the following extended equivalence

$$x \preccurlyeq y \rightarrow z \iff x \cdot y \preccurlyeq z \iff y \cdot x \preccurlyeq z \iff x \preccurlyeq y \rightsquigarrow z$$

is valid. From here, for $x \equiv y \rightarrow z$, or for $x \equiv y \rightsquigarrow z$, we get $x \rightarrow y \preccurlyeq x \rightsquigarrow y \preccurlyeq x \rightarrow y$. In this case, $\mathfrak{pA}/\equiv_{\preccurlyeq}$ is a quasi-ordered residuated system. Thus, we can conclude that the pseudo quasi-ordered residuated system is a generalization of a quasi-ordered residuated system.

In the following theorem, we collect and reformulate some results proved in [6].

Theorem 2.1 ([24], Theorem 2.1)**.** *Let $\mathfrak{pA} =: \langle A, \cdot, \rightarrow, \rightsquigarrow, 1, \preccurlyeq \rangle$ be a pseudo quasi-ordered residuated system. Then:*

(a) $(\forall x, y \in A)((x \preccurlyeq y \iff x \rightarrow y \equiv_{\preccurlyeq} 1) \wedge (x \preccurlyeq y \iff x \rightsquigarrow y \equiv_{\preccurlyeq} 1))$
(b) $(\forall x \in A)((x \rightarrow x \equiv_{\preccurlyeq} 1) \wedge (x \rightsquigarrow x \equiv_{\preccurlyeq} 1))$
(cL) $(\forall x, y, z \in A)(x \cdot y \rightarrow z \equiv_{\preccurlyeq} x \rightarrow (y \rightarrow z))$
(cR) $(\forall x, y, z \in A)(x \cdot y \rightsquigarrow z \equiv_{\preccurlyeq} y \rightsquigarrow (x \rightsquigarrow z))$
(d) $(\forall x, y, z \in A)(x \preccurlyeq y \implies ((x \cdot z \preccurlyeq y \cdot z) \wedge (z \cdot x \preccurlyeq z \cdot y)))$

(e) $(\forall x,y,z \in A)(x \preccurlyeq y \implies ((z \to x \preccurlyeq z \to y) \wedge (y \to z \preccurlyeq x \to z)))$
(f) $(\forall x,y,z \in A)(x \preccurlyeq y \implies ((z \rightsquigarrow x \preccurlyeq z \rightsquigarrow y) \wedge (y \rightsquigarrow z \preccurlyeq x \rightsquigarrow z)))$
(g) $(\forall x,y \in A)(x \cdot y \preccurlyeq x \ \wedge \ x \cdot y \preccurlyeq y)$
(h) $(\forall x,y \in A)(x \preccurlyeq y \to x \wedge x \preccurlyeq y \rightsquigarrow x)$
(i) $(\forall x \in A)(x \equiv_{\preccurlyeq} 1 \to x \wedge x \equiv_{\preccurlyeq} 1 \rightsquigarrow x)$
(j) $(\forall x,y,z \in A)((y \to z) \cdot (x \to y) \preccurlyeq x \to z)$
(k) $(\forall x,y,z \in A)((x \rightsquigarrow y) \cdot (y \rightsquigarrow z) \preccurlyeq x \rightsquigarrow z)$.

Having regard to (3L) and (3R), it follows directly from statements (j) and (k):
(*l*) $(\forall x,y,z \in A)(x \to y \preccurlyeq (z \to x) \to (z \to y))$,
(m) $(\forall x,y,z \in A)(x \to y \preccurlyeq (y \to z) \rightsquigarrow (x \to z))$,
(n) $(\forall x,y,z \in A)(x \rightsquigarrow y \preccurlyeq (y \rightsquigarrow z) \to (x \rightsquigarrow z))$,
(p) $(\forall x,y,z \in A)(x \rightsquigarrow y \preccurlyeq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y))$.

Definition 2.4 ([24], Definition 2.3)**.** *A pseudo quasi-ordered residuated system* $\mathfrak{pA}$ *is called a* strong pseudo quasi-ordered residuated system *if additionally it satisfies the following conditions:*

(S1) $(\forall x,y \in A)((x \to y) \rightsquigarrow y \equiv_{\preccurlyeq} (y \to x) \rightsquigarrow x)$, *and*

(S2) $(\forall x,y \in A)((x \rightsquigarrow y) \to y \equiv_{\preccurlyeq} (y \rightsquigarrow x) \to x)$.

Filters in pseudo-hoop algebra have been studied in [14] and [2], for example. In this algebraic structure, a filter F is a non-empty sub-semigroup that satisfies the standard condition for filters. In addition, in [14], Proposition 3.1 it is shown that the condition

$$1 \in F \wedge (\forall a,b \in A)((a \in F \wedge a \to b \in F) \implies b \in F)$$

is equivalent to the condition

$$1 \in F \wedge (\forall a,b \in A)((a \in F \wedge a \rightsquigarrow b \in F) \implies b \in F).$$

In pseudo quasi-ordered residuated systems, the concept of filters in them is determined somewhat differently.

Definition 2.5 ([24], Definition 2.8)**.** *Let* $\mathfrak{pA} =: \langle A, \cdot, \to, \rightsquigarrow, 1, \preccurlyeq \rangle$ *be a pseudo quasi-ordered residuated system. A subset F of A is a filter of* $\mathfrak{pA}$ *if it satisfies the following conditions:*

(F2) $(\forall x,y \in A)((x \in F \wedge x \preccurlyeq y) \implies v \in F)$,

(F3L) $(\forall x,y \in A)((x \in F \wedge x \to y \in F) \implies y \in F)$,

(F3R) $(\forall x,y \in A)((x \in F \wedge x \rightsquigarrow y \in F) \implies y \in F)$.

It is easy to conclude that the sets $\emptyset$ and A are filters in $\mathfrak{pA}$. These are trivial filters in $\mathfrak{pA}$. A filter F of $\mathfrak{pA}$ is proper if $F \neq A$. Besides, if F is a non-empty filter in $\mathfrak{pA}$, then $1 \in F$. Indeed, if $F \neq \emptyset$, then there exists an element $x \in F$. Since $x \preccurlyeq 1$, according to (2), we conclude that $1 \in F$ according to (F2). Thus, $(F \neq \emptyset \wedge (F2)) \implies 1 \in F$.

In addition to the above, the following implications are valid $(F3L) \implies (F2)$ and $(F3R) \implies (F2)$ assuming $F \neq \emptyset$. Assume that (F3L) (i.e. (3FR), respectively) is a valid formula and let elements $x,y \in A$ be such that $x \in F \wedge x \preccurlyeq y$. Then

$x \in F \wedge x \to y \equiv_{\preccurlyeq} 1 \in F$ and $x \in F \wedge x \rightsquigarrow y \equiv_{\preccurlyeq} 1 \in F$. Hence $y \in F$ according to (3FL) or (3FR) respectively.

Also, for a non-empty filter F in a pseudo quasi-ordered residuated system $\mathfrak{pA}$, conditions (F3L) and (F3R) are equivalent. Indeed. Let the non-empty set F in a pseudo quasi-ordered residuated system $\mathfrak{pA}$, satisfy the condition (3FR). The assumption $F \neq \emptyset$, implies $1 \in F$. Let $u, v \in A$ be arbitrary elements such that $u \to v \in F$. Then $u \to v \preccurlyeq (v \rightsquigarrow z) \to (u \rightsquigarrow z)$. If we take $z = v$, we get $u \to v \preccurlyeq (v \rightsquigarrow v) \to (u \rightsquigarrow v)$. This means $u \to v \preccurlyeq 1 \to (u \rightsquigarrow v)$ by (b) and $u \to v \preccurlyeq u \rightsquigarrow v$, due to (i). Thus $u \rightsquigarrow v \in F$ by (F2). Therefore, $v \in F$ by (F3R). This means that the implication $F \neq \emptyset \wedge (F3R) \implies (F3L)$ is proved. The implication $F \neq \emptyset \wedge (3FL) \implies (3FR)$ can be proved analogously to the previous proof.

Example 2.4: Let G be as in Example 2.2. If K is a convex lattice ordered subgroup of G, then the set $F = \{x \in G[u] : u - x \in K\}$ is a filter in $G[u]$. □

Example 2.5: Let G be as in Example 2.3. If K is a convex lattice ordered subgroup of a lattice ordered group G. The $F = K \cap nG$ is a filter in nG. □

The concepts of implicative filters and comparative filters can be determined in pseudo-quasi-ordered residuated systems by looking at the corresponding concepts in pseudo-hoop algebras as done in [2]:

Definition 2.6 ([24], Definition 2.9). *Let $\mathfrak{pA}$ be a pseudo quasi-ordered residuated system and F be a non-empty subset of A. F is said to be an* implicative filter *in $\mathfrak{pA}$ if the following holds:*

(F2) $(\forall x, y \in A)((x \in F \wedge x \preccurlyeq y) \implies v \in F)$,

(IF3) $(\forall x, y, z \in A)((x \in F \wedge x \to ((y \to z) \rightsquigarrow y) \in F) \implies y \in F)$ *and*

(IF4) $(\forall x, y, z \in A)((x \in F \wedge x \rightsquigarrow ((y \rightsquigarrow z) \to y) \in F) \implies y \in F)$.

Definition 2.7 ([24], Definition 2.11). *Let $\mathfrak{pA}$ be a pseudo quasi-ordered residuated system and F be a non-empty subset of A. A subset F of A is called a* comparative filter *of $\mathfrak{pA}$ if the following holds:*

(F2) $(\forall x, y \in A)((x \in F \wedge x \preccurlyeq y) \implies v \in F)$,

(CF3) $(\forall x, y, z \in A)((x \rightsquigarrow y \in F \wedge x \to (y \to z) \in F) \implies x \to z \in F)$,

(CF4) $(\forall x, y, z \in A)((x \to y \in F \wedge x \rightsquigarrow (y \rightsquigarrow z) \in F) \implies x \rightsquigarrow z \in F)$.

In the literature, this filter often appears under the name 'positive implicative filter' (see, for example, [2]). On the other hand, the literature related to residual lattices (for example, [1, 5, 7, 18]), hoop-algebras ([8]) and quasi-ordered residuated systems ([22, 23]) suggests that the terms in Definitions 2.6 and 2.7 should be replaced. However, in order to harmonize our terminology with the terminology appearing in the text [2], we have decided to accept this way of determining special filters in pseudo QRSs. This was done in our previous papers [24, 26].

3. THE MAIN RESULTS

Thus section is central part of this report. It contains three subsections. While in the first subsection we prove a proposition useful for further work, in Subsection 3.2 we introduce the concept of a normal filter in a pseudo quasi-ordered residuated system. In addition, we show some of the important features of this filter. In the following two subsections, we discuss the relationship between the concept of the normal filter and the concept of the implicative filter (Subsection 3.3) and the concept of the comparative filter (Subsection 3.4) in pseudo QRSs.

3.1. **An additional property of pseudo QRSs.** In this subsection, we show an additional property of pseudo QRSs that we use it in the next subsection.

Proposition 3.1. *Let $\mathfrak{pA}$ be a pseudo quasi-ordered residuated system. Then:*

(q) $(\forall x, y, z \in A)(x \rightsquigarrow (y \rightarrow z) \equiv_{\preccurlyeq} y \rightarrow (x \rightsquigarrow z))$,

(r) $(\forall x, y, z \in A)(x \rightarrow (y \rightsquigarrow z) \equiv_{\preccurlyeq} y \rightsquigarrow (x \rightarrow z))$.

Proof. We prove the formula (q). The formula (r) can be proved in a similar way.

We start from the formula $x \rightsquigarrow (y \rightarrow z) \preccurlyeq x \rightsquigarrow (y \rightarrow z)$ which is valid due to the transitivity of the order relation. Then $x \cdot (x \rightsquigarrow (y \rightarrow z)) \preccurlyeq y \rightarrow z$ according (3R) and $(x \cdot (x \rightsquigarrow (y \rightarrow z))) \cdot y \preccurlyeq z$ by (3L). Hence $x \cdot (x \rightsquigarrow (y \rightarrow z)) \cdot y \preccurlyeq z$, since the multiplication in $\mathfrak{pA}$ is associative. Thus $(x \rightsquigarrow (y \rightarrow z)) \cdot y \preccurlyeq x \rightsquigarrow z$ by (3R) and $x \rightsquigarrow (y \rightarrow z) \preccurlyeq y \rightarrow (x \rightsquigarrow z)$ in according to (3L).

On the contrary, if we start from a valid formula $y \rightarrow (x \rightsquigarrow z) \preccurlyeq y \rightarrow (x \rightsquigarrow z)$, we have $(y \rightarrow (x \rightsquigarrow z)) \cdot y \preccurlyeq x \rightsquigarrow z$ by (3L) and $x \cdot (y \rightarrow (x \rightsquigarrow z)) \cdot y \preccurlyeq z$ by (3R). From here, we get $x \cdot (y \rightarrow (x \rightsquigarrow z)) \preccurlyeq y \rightarrow z$ by applying (3L) first and then we get $y \rightarrow (x \rightsquigarrow z) \preccurlyeq x \rightsquigarrow (y \rightarrow z)$ by applying (3R).

The validity of formula (q) has been proven. The formula (r) can be obtained from the formula (q) by replacing the variables x and y in it. □

3.2. **Normal filters in pseudo QRSs.** In this subsection, we introduce the concept of normal filters in a pseudo quasi-ordered residuated system and then analyze its properties.

Definition 3.1. *Let F be a non-empty subset of a pseudo quasi-ordered residuated system $\mathfrak{pA}$. F is called a normal filter in $\mathfrak{pA}$ if he satisfies the requirements (F2), (F3L), (F3R) and*

(NFL) $(\forall x, y, z \in A)((z \rightarrow ((y \rightarrow x) \rightsquigarrow x) \in F \wedge z \in F)) \implies (x \rightarrow y) \rightsquigarrow y \in F)$,

(NFR) $(\forall x, y, z \in A)((z \rightsquigarrow ((y \rightsquigarrow x) \rightarrow x) \in F \wedge z \in F)) \implies (x \rightsquigarrow y) \rightarrow y \in F)$.

In the general case, the filter in a pseudo quasi-ordered residuated system does not have to be a normal filter.

On the other hand, from (F2) it immediately follows that (F0) holds if F is a normal filter in $\mathfrak{pA}$.

Example 3.1: Let G be as in Example 2.3. If K is a convex normal lattice ordered subgroup of G. The $F = K \cap nG$ is a normal filter in nG. □

Theorem 3.1. *If $\mathfrak{pA}$ is a strong pseudo quasi-ordered residuated system, then any filter of $\mathfrak{pA}$ is a normal filter of $\mathfrak{pA}$.*

Proof. Let F be a filter of a strong pseudo quasi-ordered residuated system $\mathfrak{pA}$ and suppose that $x, y, z \in A$ are arbitrary elements.

Let it be $z \in F$ and $z \to ((y \to x) \rightsquigarrow x) \in F$. Then $(y \to x) \rightsquigarrow x \in F$ by (F3L). Since $(y \to x) \rightsquigarrow x \equiv_{\preccurlyeq} (x \to y) \rightsquigarrow y$ because $\mathfrak{pA}$ is a strong pseudo quasi-ordered residuated system, we conclude that $(x \to y) \rightsquigarrow y \in F$ is valid. This proves the validity of the formula (NFL). The validity of the formula (NFR) can be proved in a similar way. So, F is a normal filter. $\square$

In what follows, we need the following lemma:

Lemma 3.1. *Let a subset F of a pseudo quasi-ordered residuated system $\mathfrak{pA}$ satisfies the condition (F2). Then the following holds*

$$(\forall u \in A)((u \in F \iff 1 \to u \in F) \wedge (u \in F \iff 1 \rightsquigarrow u \in F)).$$

Now, we have:

Theorem 3.2. *Let F be a filter of a pseudo quasi-ordered residuated system $\mathfrak{pA}$. F is a normal filter of $\mathfrak{pA}$ if and only if the followings holds*

(18) $(\forall x, y \in A)((y \to x) \rightsquigarrow x \in F \implies (x \to y) \rightsquigarrow y \in F)$ *and*

(19) $(\forall x, y \in A)((y \rightsquigarrow x) \to x \in F \implies (x \rightsquigarrow y) \to y \in F)$.

Proof. Let F be a normal filter of a pseudo quasi-ordered residuated system $\mathfrak{pA}$ and let $x, y \in A$ be elements such that $(y \to x) \rightsquigarrow x \in F$. Since $(y \to x) \rightsquigarrow x \in F$ is equivalent with $1 \to ((y \to x) \rightsquigarrow x) \in F$ by Lemma 3.1 and since $1 \in F$, thus $(x \to y) \rightsquigarrow y \in F$ because F is a normal filter in $\mathfrak{pA}$. This proves the implication (18). The implication (19) can be proved in a similar way.

Suppose that the filter F in a pseudo quasi-ordered residuated system $\mathfrak{pA}$ satisfies the conditions (18) and (19). Let $x, y, z \in A$ be such that $z \to ((y \to x) \rightsquigarrow x) \in F$ and $z \in F$. Since F is a filter of $\mathfrak{pA}$, then $(y \to x) \rightsquigarrow x \in F$ by (F3L). Thus we get that $(x \to y) \rightsquigarrow y \in F$ by the hypothesis (18). This shows that (NFL) holds. The validity of the formula (NFR) can be proved in a similar way. Hence, F is a normal filter in $\mathfrak{pA}$. $\square$

3.3. **Relationship between normal and implicative filters.** In proving the following result that connects implicative and normal filters in a pseudo quasi-ordered residuated system we need the following proposition:

Proposition 3.2. *For any implicative filter F in a pseudo quasi-ordered residuated system $\mathfrak{pA}$ the following holds*

(IF5) $(\forall v, z \in A)((v \to z) \rightsquigarrow v \in F \implies v \in F)$ *and*

(IF6) $(\forall v, z \in A)((v \rightsquigarrow z) \to v \in F \implies v \in F)$.

Proof. Let F be an implicative filter in a pseudo quasi-ordered redisudated system $\mathfrak{pA}$. If we put $x = 1$ and $y = v$ u (IF3), we immediately get (IF5) with respect to Lemma 3.1 and (F0).

The validity of the formula (IF6) can be demonstrated similarly to the previous proof. □

The reverse is also true:

Proposition 3.3. *If a filter F in a pseudo quasi-ordered residuated system $\mathfrak{pA}$ satisfies the (IF5) and (IF6), then F is an implicative filter in $\mathfrak{pA}$.*

Proof. Let F be a filter satisfy the conditions (F2), (IF5) and (IF6).

Assume that $x, y, z \in A$ are such that $x \to ((y \to z) \rightsquigarrow y) \in F$ and $x \in F$. Then $(y \to z) \rightsquigarrow y \in F$ by (F3L). Thus $y \in F$ according to (IF5). This proves (IF3).

Assume that $x, y, z \in A$ are such that $x \rightsquigarrow ((y \rightsquigarrow z) \to y) \in F$ and $x \in F$. Then $(y \rightsquigarrow z) \to y \in F$ by (F3R). Thus $y \in F$ according to (IF6). This proves (IF4). □

The following theorem shows a connection between implicative and normal filters in QRSs.

Theorem 3.3. *Any implicative filter in a pseudo quasi-ordered residuated system $\mathfrak{pA}$ is a normal filter in $\mathfrak{pA}$.*

Proof. Let F be an implicative filter in a pseudo quasi-ordered residuated system $\mathfrak{pA}$. This means that F satisfies the conditions (F2), (IF3) and (IF4). We need to prove that F satisfies the conditions (NFL) and (NFR). In fact, we will prove that F satisfies the conditions (18) and (19). Let $x, y, z \in A$ be such $(x \to y) \rightsquigarrow y \in F$. According to the claim (g) of Theorem 2.1, from $(x \to y) \cdot x \preccurlyeq x$ we get $x \preccurlyeq (y \to x) \rightsquigarrow x$ by (3R). Then $((y \to x) \rightsquigarrow x) \to y \preccurlyeq x \to y$ by (e) and by repeated procedure by (f), we have $(x \to y) \rightsquigarrow y \preccurlyeq (((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow y$. Thus $(((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow y \in F$ by (F2).

On the other hand, since $y \to x \preccurlyeq y \to x$ is equivalent to $(y \to x) \cdot y \preccurlyeq x$ according to (3L), we have $y \preccurlyeq (y \to x) \rightsquigarrow x$ by (3R). If we treat this inequality with $((y \to x) \rightsquigarrow x) \to y$ using procedure (f) on the left, we get

$$(((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow y \preccurlyeq (((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow ((y \to x) \rightsquigarrow x).$$

Since $(((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow y \in F$, we obtain

$$(((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow ((y \to x) \rightsquigarrow x) \in F$$

according to (F2). If we denote $v =: (y \to x) \rightsquigarrow x$, $z =: y$, then we recognize the previous condition as the hypothesis in (IF5). Therefore $(y \to x) \rightsquigarrow x \in F$, by (IF5) since F is an implicative filter in $\mathfrak{A}$. This proves the validity of formula (18).

The validity of the formula (19) can be demonstrated in a similar way. So, F is a normal filter in $\mathfrak{pA}$. □

3.4. **A connection between comparative and normal filters.** In order to establish a connection between a comparative and a normal filter in a pseudo quasi-ordered residuated system, we need an important property of comparative filters.

Proposition 3.4. *Let F be a comparative filter of a pseudo quasi-ordered residuated system $\mathfrak{pA}$. Then the following holds*

(CF5) $(\forall u, v \in A)(u \to (u \to v) \in F \implies u \to v \in F)$ *and*

(CF6) $(\forall u, v \in A)(u \rightsquigarrow (u \rightsquigarrow v) \in F \implies u \rightsquigarrow v \in F)$.

Proof. If we put $u =: x = y$ and $v =: z$ in (CF3), we immediately obtain the claim (CF5) of this proposition, since for every $u \in A$ always $u \rightsquigarrow u \in F$ holds for every non-empty set F satisfying condition (F2). Indeed, $u \rightsquigarrow u \in F$ follows from $u \preccurlyeq u$, whence $1 \preccurlyeq (u \rightsquigarrow u)$ by $1 \in F$ and (F2).

The validity of the formula (CF6) can be proved in a similar way. □

Theorem 3.4. *Let F be a comparative and normal filter in a pseudo quasi-ordered residuated system $\mathfrak{pA}$. Then F is an implicative filter in $\mathfrak{pA}$.*

Proof. Let F be a comparative and normal filter in a pseudo quasi-ordered residuated system $\mathfrak{pA}$. We will prove (IF5) and (IF6).

Let $x, y \in A$ be such that $(x \rightarrow y) \rightsquigarrow x \in F$ and $(x \rightsquigarrow y) \rightarrow x \in F$.

We start from a valid formula $x \rightarrow y \preccurlyeq x \rightarrow y$. Then $(x \rightarrow y) \cdot x \preccurlyeq y$ by (3L). Thus $x \preccurlyeq (x \rightarrow) \rightsquigarrow y$ according (3R). If we act on the previous inequality from the left with $(x \rightarrow y)$, according to (f), we get

$$(x \rightarrow y) \rightsquigarrow x \preccurlyeq (x \rightarrow y) \rightsquigarrow ((x \rightarrow y) \rightsquigarrow y).$$

From here we get $(x \rightarrow y) \rightsquigarrow ((x \rightarrow y) \rightsquigarrow y) \in F$, due of (F2). From here, according to (CF6), we get $(x \rightarrow y) \rightsquigarrow y \in F$. Hence $(y \rightarrow x) \rightsquigarrow x \in F$ according (18).

We start from a valid formula $x \rightsquigarrow y \preccurlyeq x \rightsquigarrow y$. Then $x \cdot (x \rightsquigarrow y) \preccurlyeq y$ by (3R). Thus $x \preccurlyeq (x \rightsquigarrow) \rightarrow y$ according (3L). If we act on the previous inequality from the left with $(x \rightsquigarrow y)$, according to (e), we get

$$(x \rightsquigarrow y) \rightarrow x \preccurlyeq (x \rightsquigarrow y) \rightarrow ((x \rightsquigarrow y) \rightarrow y).$$

From here we get $(x \rightsquigarrow y) \rightarrow ((x \rightsquigarrow y) \rightarrow y) \in F$, due of (F2). From here, according to (CF5), we get $(x \rightsquigarrow y) \rightarrow y \in F$. Hence $(y \rightsquigarrow x) \rightarrow x \in F$ according (19).

On the other hand, from $y \cdot x \preccurlyeq y$, according to (g), we get $y \preccurlyeq x \rightarrow y$ due to (3L). If we apply (f) to this inequality acting on it by the right side with x, we get $(x \rightarrow y) \rightsquigarrow x \preccurlyeq y \rightsquigarrow x$. Then $y \rightsquigarrow x \in F$ by (F2). If we start from $x \cdot y \preccurlyeq y$, we have $y \preccurlyeq x \rightsquigarrow y$ according to (3L). If we apply (e) to this inequality acting on it by the right side with x, we get $(x \rightsquigarrow y) \rightarrow x \preccurlyeq y \rightarrow x$. Then $y \rightarrow x \in F$ by (F2).

Finally, from $(y \rightarrow x) \rightsquigarrow x \in F$ and $y \rightarrow x \in F$ it follows $x \in F$ by (F2). Also, from $(y \rightsquigarrow x) \rightarrow x \in F$ and $y \rightsquigarrow x \in F$ it follows $x \in F$ due to (F2). Therefore, the validity of formulas (IF5) and (IF6) is proven. So, F is an implicative filter in $\mathfrak{pA}$. □

4. Conclusions and future works

In articles [24, 26], this author dealt with pseudo quasi-ordered residuated systems as well as their substructures. While implicative filters and comparative filters appear in paper [24], paper [26] is dedicated to fantistic and associative filters in these algebraic structures. The aim of this paper, as a continuation of previous research, is to introduce the notions of normal filters in pseudo quasi-ordered residuated systems and to investigate its properties. The correlation of the normal filter with the implicative and comparative filters in this algebraic structure was established. In the future, there are more directions for further and deeper analysis of the properties of pseudo quasi-ordered residuated systems, such as, for example, establishing the relationship between filters and relations determined by them. This is the domain in which this algebraic structure differs from commutative residuated lattices and from hoop-algebras.

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