Математички Билтен
49 (LXXV) 2 (2025), 96–106

UDC: ???
DOI: ???

FUZZY SSP-IRRESOLUTE MAPPINGS

EGZON RRUSTOLLI [1] AND BILJANA KRSTESKA [2]

Abstract. In this paper, we introduce the concepts of fuzzy SSP-irresolute continuous mappings, fuzzy SSP-irresolute open (closed) mappings, and SSP homeomorphisms. We also investigate several of their properties and examine their relationships with other forms of fuzzy continuity.

1. Introduction

After the introduction of the concept of fuzzy sets by Zadeh [11], and fuzzy topological spaces and fuzzy continuity by Chang [2], various weaker forms of fuzzy continuity were investigated by many researchers [1, 3, 10, 13, 12, 7] .

Irresolute mappings form another important class of mappings in fuzzy topology. Unlike ordinary fuzzy continuous mappings, which are defined by the inverse images of fuzzy open sets, irresolute mappings are usually defined by using generalized fuzzy open sets. Thus, they provide a natural way to study mappings that preserve weaker forms of fuzzy openness. Mukherjee and Sinha [6] introduced irresolute and almost open functions between fuzzy topological spaces in relation to fuzzy semiopen sets. Later, Krsteska [4] studied fuzzy SP-irresolute mappings associated with fuzzy strongly preopen sets. In recent years, Makolli and Krsteska [5] also studied fuzzy SSPO irresolute mappings, which further shows the relevance of irresolute-type mappings in fuzzy topology.

Recently, Rrustolli and Krsteska [8] introduced the class of fuzzy semi strongly preopen sets and the corresponding notion of fuzzy semi strong precontinuity.

2010 *Mathematics Subject Classification.* 54A40, 03E72.

Key words and phrases. Fuzzy topological spaces, fuzzy semi strongly preopen set, fuzzy semi strong precontinuity, fuzzy SSP-irresolute continuous mappings, fuzzy SSP-irresolute open(closed) mappings, fuzzy SSP-homeomorphism.

This class is defined by using the fuzzy interior, fuzzy closure and fuzzy preclosure operators, and it is closely related to several known classes of fuzzy open-type sets. Therefore, it is natural to study mappings that preserve this class in the sense of irresoluteness.

Motivated by these developments, in this paper we introduce fuzzy SSP-irresolute continuous mappings. The advantage of the present approach is that fuzzy SSP-irresolute continuity is based on the preservation of fuzzy semi strongly preopen sets rather than only fuzzy open sets. Therefore, the proposed mappings allow one to study transformations that may not be covered by ordinary fuzzy continuity, while still preserving a meaningful fuzzy topological structure associated with FSSPOS. We also introduce fuzzy SSP-irresolute open mappings, fuzzy SSP-irresolute closed mappings and fuzzy SSP-homeomorphisms. Several characterizations of these mappings are obtained, and their relationships with other forms of fuzzy continuity and fuzzy semi strong precontinuity are investigated. Furthermore, several results concerning the composition of mappings are established, particularly in relation to some previously studied types of mappings.

2. Preliminaries

In this paper by (X, τ), or simply by X, we will denote a fuzzy topological space (*fts*) due to Chang[2]. The interior, closure, and complement of a fuzzy set A will be denoted by $\operatorname{int} A$, $\operatorname{cl} A$, and A^c, respectively.

Definition 2.1. *A fuzzy set A of a fts X is called:*

1) *fuzzy semiopen (FSOS) if and only if* $A \leq \operatorname{cl}(\operatorname{int} A)$ [1];
2) *fuzzy preopen (FPOS) if and only if* $A \leq \operatorname{int}(\operatorname{cl} A)$ [9];
3) *fuzzy strongly semiopen (FSSOS) if and only if* $A \leq \operatorname{int}(\operatorname{cl}(\operatorname{int} A))$ [13];

Definition 2.2. *A fuzzy set A of X is called:*

1) *fuzzy semiclosed (FSCS) if and only if* $\operatorname{int}(\operatorname{cl} A) \leq A$ [1];
2) *fuzzy preclosed (FPCS) if and only if* $\operatorname{cl}(\operatorname{int} A) \leq A$ [9];
3) *fuzzy strongly semiclosed (FSSCS) if and only if* $\operatorname{cl}(\operatorname{int}(\operatorname{cl} A)) \leq A$ [13];

Definition 2.3. [9] *Let A be a fuzzy set of fts X. Then,*

1) $\operatorname{pint} A = \vee\{B : B \leq A,\ B$ *is a fuzzy preopen set of* $X\}$ *is called a fuzzy preinterior of A;*
2) $\operatorname{pcl} A = \wedge\{B : B \geq A, B$ *is a fuzzy preclosed set of* $X\}$ *is called a fuzzy preclosure of A;*

Lemma 1. [9] *Let A be a fuzzy set of fts X. Then,*

1) $\operatorname{pcl} A^c = (\operatorname{pint} A)^c$;
2) $\operatorname{pint} A^c = (\operatorname{pcl} A)^c$.

Definition 2.4. [8]

(1) *A fuzzy set A of a fts X is called fuzzy semi strongly preopen set (FSSPOS) if and only if* $A \leq \operatorname{cl}(\operatorname{int}(\operatorname{pcl} A))$.
(2) *A fuzzy set A of fts X is called fuzzy semi strongly preclosed set (FSSPCS) if and only if* $\operatorname{int}(\operatorname{cl}(\operatorname{pint} A)) \leq A$.

Theorem 1. [8] *Let (X, τ) be a fts. Every fuzzy open set is a FSSPOS.*

Lemma 2. [2] *Let $f : X \to Y$ be a mapping. For fuzzy sets A and B of X and Y respectively, the following statements hold:*

1) $ff^{-1}(B) \leq B$;
2) $f^{-1}f(A) \geq A$;
3) $f(A^c) \geq f(A)^c$;
4) $f^{-1}(B^c) = f^{-1}(B)^c$;
5) *if f is injective, then* $f^{-1}f(A) = A$;
6) *if f is surjective, then* $ff^{-1}(B) = B$;
7) *if f is bijective, then* $f(A^c) = f(A)^c$.

Definition 2.5. [8] *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts (X, τ_1) into a fts (Y, τ_2). The mapping f is called:*

(1) *fuzzy semi strong precontinuous if $f^{-1}(B) \in FSSPOS(\tau_1)$ for each $B \in \tau_2$.*
(2) *fuzzy semi strongly preopen(preclosed) if $f(A) \in FSSPOS(\tau_2)$ $(f(A) \in FSSPCS(\tau_2))$ for each $A \in \tau_1$ $(A^c \in \tau_1)$.*

Definition 2.6. [8] *Let A be a fuzzy set of fts X.*

1) *The union of all FSSPOSs contained in A is called a fuzzy semi strong preinterior of A, denoted by* sspint A.
2) *The intersection of all FSSPCSs containing A is called a fuzzy semi strong preclosure of A, denoted by* sspcl A.

3. Fuzzy SSP-irresolute continuous mappings

Definition 3.1. *A mapping $f : (X, \tau_1) \to (Y, \tau_2)$ from a fts (X, τ_1) into a fts (Y, τ_2) is called SSP-irresolute continuous mapping if $f^{-1}(B)$ is a FSSPOS of X for each FSSPOS B of Y.*

Example 3.1. *Let (X, τ) be a fuzzy topological space. Then the identity mapping*

$$id_X : (X, \tau) \to (X, \tau)$$

is a fuzzy SSP-irresolute continuous mapping. Indeed, for every FSSPOS B of X, we have $id_X^{-1}(B) = B$. Therefore, $id_X^{-1}(B)$ is a FSSPOS of X for every FSSPOS B of X.

Theorem 2. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts (X, τ_1) into a fts (Y, τ_2). The following statements are equivalent:*

i) *f is fuzzy SSP-irresolute continuous;*
ii) *$f^{-1}(B)$ is FSSPCS of X for each FSSPCS B of Y;*
iii) $f(\text{sspcl}\, A) \leq \text{sspcl}\, f(A)$, *for each fuzzy set A of X;*
iv) $\text{sspcl}\, f^{-1}(B) \leq f^{-1}(\text{sspcl}\, B)$, *for each fuzzy set B of Y;*
v) $f^{-1}(\text{sspint}\, B) \leq \text{sspint}\, f^{-1}(B)$, *for each fuzzy set B of Y.*

Proof. $i) \Rightarrow ii)$ Let B be a FSSPCS of Y, B^c is a FSSPOS of Y. By assumptions it follows that $f^{-1}(B^c)$ is a FSSPOS of X. Thus, $f^{-1}(B) = (f^{-1}(B^c))^c$ is a FSSPCS.

ii) $\Rightarrow$ *iii*) Let A be a fuzzy set of X. $\operatorname{sspcl} f(A)$ is a FSSPCS, by assumptions, $f^{-1}(\operatorname{sspcl} f(A))$ is a FSSPCS of X. Thus,

$$\begin{aligned} f(\operatorname{sspcl} A) \leq f(\operatorname{sspcl} f^{-1}(f(A))) &\leq f(\operatorname{sspcl} f^{-1}(\operatorname{sspcl} f(A))) \\ &= f(f^{-1}(\operatorname{sspcl} f(A))) \leq \operatorname{sspcl} f(A). \end{aligned}$$

iii) $\Rightarrow$ *iv*) Let B be a fuzzy set of Y. Therefore, $f^{-1}(B)$ is a fuzzy set of X, by assumptions, it follows that

$$f(\operatorname{sspcl} f^{-1}(B)) \leq \operatorname{sspcl} f(f^{-1}(B)) \leq \operatorname{sspcl} B$$

Thus,

$$\operatorname{sspcl} f^{-1}(B) \leq f^{-1}(f(\operatorname{sspcl} f^{-1}(B))) \leq f^{-1}(\operatorname{sspcl} B).$$

iv) $\Rightarrow$ *v*) Let B be a fuzzy set of Y. By assumptions, it follows that

$$\operatorname{sspcl} f^{-1}(B^c) \leq f^{-1}(\operatorname{sspcl} B^c)$$

Thus,

$$f^{-1}(\operatorname{sspint} B) = (f^{-1}(\operatorname{sspcl} B^c))^c \leq (\operatorname{sspcl} f^{-1}(B^c))^c = \operatorname{sspint} f^{-1}(B).$$

v) $\Rightarrow$ *i*) Let B be a FSSPOS of Y, $\operatorname{sspint} B = B$. Thus, by assumptions, it follows that

$$f^{-1}(B) \leq \operatorname{sspint} f^{-1}(B)$$

Therefore, $f^{-1}(B)$ is a FSSPOS of X. Indeed, f is SSP-irresolute continuous mapping. □

Theorem 3. *Let (X, τ_1) and (Y, τ_2) be fts. Every SSP-irresolute continuous mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is also fuzzy semi strong precontinuous.*

Proof. Let B be a fuzzy open set of Y. By Theorem 1, B is a FSSPOS. Therefore, by assumptions, it follows that $f^{-1}(B)$ is a FSSPOS of X. Indeed, f is fuzzy semi strong precontinuous mapping. □

Remark 3.1: Converse of Theorem 3 may not be true.

Example 3.2. *Let $X = \{x, y\}$. Let U, V, W and A fuzzy sets of X defined as it follows:*

$$\begin{aligned} U(x) &= 0.2, & U(y) &= 0.2 \\ V(x) &= 0.7, & V(y) &= 0.7 \\ W(x) &= 0.3, & W(y) &= 0.3 \\ A(x) &= 0.4, & A(y) &= 0.4 \end{aligned}$$

Let $\tau_1 = \{0, U, V, 1\}$, $\tau_2 = \{0, W, 1\}$ and $f = id : (X, \tau_2) \to (X, \tau_1)$. f is fuzzy semi strong precontinuous mapping. Since A is a FSSPOS on (X, τ_2) but not in (X, τ_1) it follows that f is not SSP-irresolute continuous mapping.

Theorem 4. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a bijective mapping from a fts (X, τ_1) into a fts (Y, τ_2). The mapping f is a fuzzy SSP-irresolute continuous if and only if* $\operatorname{sspint} f(A) \leq f(\operatorname{sspint} A)$, *for each fuzzy set A of X.*

Proof. ($\Rightarrow$) Assume that f is a fuzzy SSP-irresolute continuous mapping, and let A be an arbitrary fuzzy set of X. Since sspint $f(A)$ is a FSSPOS, also $f^{-1}(\text{sspint}\, f(A))$ is a FSSPOS of X. Thus, by Theorem 2(v), it follows that

$$f^{-1}(\text{sspint}\, f(A)) \leq \text{sspint}\, f^{-1}(f(A)) = \text{sspint}\, A$$

Therefore,

$$\text{sspint}\, f(A) = f f^{-1}(\text{sspint}\, f(A)) \leq f(\text{sspint}\, A)$$

($\Leftarrow$) Let B be a FSSPOS of Y. Thus, by assumptions, it follows that

$$B = \text{sspint}\, B = \text{sspint}\, f(f^{-1}(B)) \leq f(\text{sspint}\, f^{-1}(B))$$

Therefore,

$$f^{-1}(B) \leq \text{sspint}\, f^{-1}(B)$$

Thus, $f^{-1}(B)$ is a FSSPOS. Indeed, f is fuzzy SSP-irresolute continuous mapping. □

Theorem 5. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts (X, τ_1) into a fts (Y, τ_2). The following statements are equivalent:*

i) f is fuzzy SSP-irresolute continuous;
ii) $f^{-1}(\text{sspint}\, B) \leq \text{cl}(\text{int}(\text{pcl}\, f^{-1}(B)))$, for each fuzzy set B of Y;
iii) $\text{int}(\text{cl}(\text{pint}\, f^{-1}(B)))) \leq f^{-1}(\text{sspcl}\, B)$, for each fuzzy set B of Y.

Proof. $i) \Rightarrow ii)$ Let f be a fuzzy SSP-irresolute continuous mapping, and let B be an arbitrary fuzzy set of Y. Since sspint B is a FSSPOS, by assumptions $f^{-1}(\text{sspint}\, B)$ is a FSSPOS on X. Thus,

$$f^{-1}(\text{sspint}\, B) \leq \text{cl}(\text{int}(\text{pcl}\, f^{-1}(\text{sspint}\, B))) \leq \text{cl}(\text{int}(\text{pcl}\, f^{-1}(B)))$$

$ii) \Rightarrow iii)$ Directly by using complement.
$iii) \Rightarrow i)$ Let B be a FSSPC S of Y, sspcl $B = B$. Thus,

$$\text{int}(\text{cl}(\text{pint}\, f^{-1}(B))) \leq f^{-1}(B)$$

it follows that $f^{-1}(B)$ is a FSSPCS of X. Indeed, by Theorem 2, f is fuzzy SSP-irresolute continuous mapping. □

Theorem 6. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts (X, τ_1) into a fts (Y, τ_2). If f is SSP-irresolute continuous mapping then $f^{-1}(B) \leq \text{sspint}\, f^{-1}(\text{cl}(\text{int}(\text{pcl}\, B)))$, for each FSSPOS B of Y.*

Proof. Let B be a FSSPOS of Y, by assumptions, $f^{-1}(B)$ is a FSSPOS of X. Then, $f^{-1}(B) \leq f^{-1}(\text{cl}(\text{int}(\text{pcl}\, B)))$, and

$$f^{-1}(B) = \text{sspint}\, f^{-1}(B) \leq \text{sspint}\, f^{-1}(\text{cl}(\text{int}(\text{pcl}\, B))).$$

□

Theorem 7. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts (X, τ_1) into a fts (Y, τ_2). If f is SSP-irresolute continuous mapping then $\text{sspcl}\, f^{-1}(\text{int}(\text{cl}(\text{pint}\, B))) \leq f^{-1}(B)$, for each FSSPCS B of Y.*

Proof. In similar manner as the proof of Theorem 6. □

Theorem 8. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ and $g : (Y, \tau_2) \to (Z, \tau_3)$, where (X, τ_1), (Y, τ_2) and (Z, τ_3) are fts.*

(1) if f and g are fuzzy SSP-irresolute continuous mappings, then gf is also SSP-irresolute continuous mapping;
(2) if f is fuzzy SSP-irresolute continuous mapping and g is fuzzy semi strong precontinuous mapping, then gf is fuzzy semi strong precontinuous mapping.

Proof. The proof is straightforward. □

4. Fuzzy SSP-irresolute open and fuzzy SSP-irresolute closed mappings

Definition 4.1. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts (X, τ_1) into a fts (Y, τ_2). The mapping f is called a fuzzy SSP-irresolute open(closed) mapping if $f(A)$ is a FSSPOS(FSSPCS) of Y, for each FSSPOS(FSSPCS) A of X.*

Example 4.1. *Let (X, τ) be a fuzzy topological space. Then the identity mapping*

$$id_X : (X, \tau) \to (X, \tau)$$

is both a fuzzy SSP-irresolute open mapping and a fuzzy SSP-irresolute closed mapping. Indeed, for every FSSPOS A of X, we have $id_X(A) = A$, and for every FSSPCS B of X, we also have $id_X(B) = B$.

Theorem 9. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts (X, τ_1) into a fts (Y, τ_2). The following statements are equivalent:*

i) f is SSP-irresolute open mapping;
ii) $f(\text{sspint}\, A) \leq \text{sspint}\, f(A)$, for each fuzzy set A of X;
iii) $\text{sspint}\, f^{-1}(B) \leq f^{-1}(\text{sspint}\, B)$, for each fuzzy set B of Y;
iv) $f^{-1}(\text{sspcl}\, B) \leq \text{sspcl}\, f^{-1}(B)$, for each fuzzy set B of Y.

Proof. *i)* $\Rightarrow$ *ii)* Let f be a SSP-irresolute open mapping, and A be a fuzzy set of X. By assumptions, it follows that $f(\text{sspint}\, A)$ is a FSSPOS of Y. Thus,

$$f(\text{sspint}\, A) = \text{sspint}\, f(\text{sspint}\, A) \leq \text{sspint}\, f(A)$$

ii) $\Rightarrow$ *iii)* Let B be a fuzzy set of Y, $f^{-1}(B)$ is a fuzzy set of X. By assumptions and Lemma 2, it follows that

$$f(\text{sspint}\, f^{-1}(B)) \leq \text{sspint}\, f(f^{-1}(B) \leq \text{sspint}\, B$$

Thus,

$$\text{sspint}\, f^{-1}(B) \leq f^{-1}(f(\text{sspint}\, f^{-1}(B))) \leq f^{-1}(\text{sspint}\, B).$$

iii) $\Leftrightarrow$ *iv)* Directly by using complement. *iii)* $\Rightarrow$ *ii)* Let A be a fuzzy set of X, by assumptions, it follows that

$$\text{sspint}\, f^{-1}(f(A)) \leq f^{-1}(\text{sspint}\, f(A))$$
$$\text{sspint}\, A \leq f^{-1}(\text{sspint}\, f(A))$$
$$f(\text{sspint}\, A) \leq f f^{-1}(\text{sspint}\, f(A)) \leq \text{sspint}\, f(A).$$

ii) $\Rightarrow$ *i*) Let A be a FSSPOS of X, sspint $A = A$. By assumptions, it follows that

$$f(A) \leq \operatorname{sspint} f(A)$$

Therefore, $f(A)$ is a FSSPOS. Indeed, f is a SSP-irresolute open mapping. $\square$

Theorem 10. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts (X, τ_1) into a fts (Y, τ_2). f is a fuzzy SSP-irresolute open mapping if and only if $f(\operatorname{sspint} A) \leq \operatorname{cl}(\operatorname{int}(\operatorname{pcl} f(A)))$ for each fuzzy set A of X.*

Proof. ($\Rightarrow$) Let A be a fuzzy set of X. Since sspint A is a FSSPOS, by assumptions, it follows that

$$f(\operatorname{sspint} A) \leq \operatorname{cl}(\operatorname{int}(\operatorname{pcl} f(\operatorname{sspint} A))) \leq \operatorname{cl}(\operatorname{int}(\operatorname{pcl} f(A))).$$

($\Leftarrow$) Let A be a FSSPOS of X. By assumptions, it follows that

$$f(A) = f(\operatorname{sspint} A) \leq \operatorname{cl}(\operatorname{int}(\operatorname{pcl} f(A)))$$

Thus, $f(A)$ is a FSSPOS; it follows that f is fuzzy SSP-irresolute open mapping. $\square$

Theorem 11. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts (X, τ_1) into a fts (Y, τ_2). The following statements are equivalent:*

i) f is a fuzzy SSP-irresolute closed mapping;
ii) $\operatorname{sspcl} f(A) \leq f(\operatorname{sspcl} A)$, for each fuzzy set A of X.

Proof. *i*) $\Rightarrow$ *ii*) Let A be a fuzzy set of X. By assumptions, $f(\operatorname{sspcl} A)$ is FSSPCS. Thus, by definition of the operator sspcl, it follows that

$$\operatorname{sspcl} f(A) \leq f(\operatorname{sspcl} A).$$

ii) $\Rightarrow$ *i*) Let A be a FSSPCS of X. By assumptions, it follows that

$$\operatorname{sspcl} f(A) \leq f(\operatorname{sspcl} A) = f(A).$$

$f(A)$ is a FSSPCS, indeed, f is fuzzy SSP-irresolute closed mapping. $\square$

Theorem 12. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts (X, τ_1) into a fts (Y, τ_2).*

(1) if $f(\operatorname{cl}(\operatorname{int}(\operatorname{pcl} A))) \leq \operatorname{cl}(\operatorname{int}(\operatorname{pcl} f(A)))$, for each FSSPOS A of X, then f is a fuzzy SSP-irresolute open mapping;
(2) if $\operatorname{int}(\operatorname{cl}(\operatorname{pint} f(A))) \leq f(\operatorname{int}(\operatorname{cl}(\operatorname{pint} A))$, for each FSSPCS A of X, then f is a fuzzy SSP-irresolute closed mapping.

Proof. 1) Let A be a FSSPOS of X. Thus,

$$f(A) \leq f(\operatorname{cl}(\operatorname{int}(\operatorname{pcl} A))) \leq \operatorname{cl}(\operatorname{int}(\operatorname{pcl} f(A))).$$

$f(A)$ is a FSSPOS, indeed, f is SSP-irresolute open mapping.
2) Similar. $\square$

Theorem 13. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a bijective mapping from a fts (X, τ_1) into a fts (Y, τ_2).*

(1) f is a fuzzy SSP-irresolute open if and only if f is fuzzy SSP-irresolute closed
(2) f is a fuzzy SSP-irresolute open(closed) if and only if f^{-1} is fuzzy SSP-irresolute continuous

Proof. The proof is straightforward. $\square$

Corollary 13.1. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a bijective mapping from a fts (X, τ_1) into a fts (Y, τ_2). The following statements are equivalent:*

i) *f is a fuzzy SSP-irresolute closed mapping;*
ii) $\text{sspcl}\, f(A) \leq f(\text{sspcl}\, A)$, *for each fuzzy set A of X;*
iii) $f^{-1}(\text{sspcl}\, B) \leq (\text{sspcl}\, f^{-1}(B))$, *for each fuzzy set B of Y;*
iv) $\text{sspint}\, f^{-1}(B) \leq f^{-1}(\text{sspint}\, B)$, *for each fuzzy set B of Y.*

Theorem 14. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts (X, τ_1) into a fts (Y, τ_2). f is fuzzy SSP-irresolute closed mapping if and only if* $\text{int}(\text{cl}(\text{pint}\, f(A))) \leq f(\text{sspcl}\, A)$ *for each fuzzy set A of X.*

Proof. $(\Rightarrow)$ Let A be a fuzzy set of X. Since $\text{sspcl}\, A$ is a FSSPCS of X, by assumptions, it follows that

$$\text{int}(\text{cl}(\text{pint}\, f(A))) \leq \text{int}(\text{cl}(\text{pint}\, f(\text{sspcl}\, A))) \leq f(\text{sspcl}\, A).$$

$(\Leftarrow)$ Let A be an arbitrary FSSPCS of X. By assumptions, it follows that

$$\text{int}(\text{cl}(\text{pint}\, f(A))) \leq f(\text{sspcl}\, A) = f(A)$$

Thus, $f(A)$ is a FSSPCS of Y. Indeed, f is fuzzy SSP-irresolute closed mapping. $\square$

Theorem 15. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts (X, τ_1) into a fts (Y, τ_2). The mapping f is fuzzy SSP-irresolute open if and only if for every fuzzy set B of Y and every FSSPCS A of X, such that $f^{-1}(B) \leq A$, there exists a FSSPCS C of Y, such that $B \leq C$ and $f^{-1}(C) \leq A$.*

Proof. $(\Rightarrow)$ Let A be an arbitrary FSSPCS of X, and B be a fuzzy set of Y such that $f^{-1}(B) \leq A$. By assumptions, $f(A^c)$ is a FSSPOS of Y, and $A^c \leq f^{-1}(B^c)$. Thus, $f(A^c) \leq ff^{-1}(B^c) \leq B^c$, it follows that $B \leq f(A^c)^c$. Define $C = f(A^c)^c$, which is a FSSPCS of Y. Then,

$$\begin{aligned} f(A^c) &= C^c \\ A^c \leq f^{-1}f(A^c) &= f^{-1}(C^c) \\ f^{-1}(C) &\leq A \end{aligned}$$

$(\Leftarrow)$ Let A be a FSSPOS of X. By Lemma 2, $A \leq f^{-1}f(A)$, it follows that $f^{-1}(f(A))^c \leq A^c$, where A^c is a FSSPCS. Define $B = f(A)^c$. By assumptions, there exists a FSSPCS C of Y, such that $B \leq C$ and $f^{-1}(C) \leq A^c$. Thus,

$$\begin{aligned} A &\leq f^{-1}(C^c) \\ f(A) \leq ff^{-1}(C^c) &\leq C^c \\ C &\leq f(A)^c = B \end{aligned}$$

Conclusively, $f(A)^c = B = C$ is a FSSPCS, it follows that $f(A)$ is a FSSPOS. Indeed, f is fuzzy SSP-irresolute open mapping. $\square$

Theorem 16. *Let $f:(X,\tau_1)\to(Y,\tau_2)$ be a mapping from a fts (X,τ_1) into a fts (Y,τ_2). The mapping f is fuzzy SSP-irresolute closed if and only if for each fuzzy set A and each FSSPOS B of Y, such that $f^{-1}(B)\leq A$, there exists a FSSPOS C of Y, such that $B\leq C$ and $f^{-1}(C)\leq A$.*

Proof. In similar manner as the proof of Theorem 15. □

Theorem 17. *Let $f:(X,\tau_1)\to(Y,\tau_2)$ and $g:(Y,\tau_2)\to(Z,\tau_3)$, where (X,τ_1), (Y,τ_2) and (Z,τ_3) are fts.*

(1) if f and g are fuzzy SSP-irresolute open(closed), then gf is also fuzzy SSP-irresolute open(closed);
(2) if f is a fuzzy semi strong preopen(preclosed) and g is fuzzy SSP-irresolute open(closed) then gf is fuzzy semi strong preopen (preclosed) mapping.

Proof. The proof is straightforward. □

Theorem 18. *Let $f:(X,\tau_1)\to(Y,\tau_2)$ and $g:(Y,\tau_2)\to(Z,\tau_3)$, where (X,τ_1), (Y,τ_2) and (Z,τ_3) are fts and gf is a fuzzy SSP-irresolute open(closed) mapping.*

(1) if g is fuzzy SSP-irresolute continuous and injective, then f is fuzzy SSP-irresolute open(closed);
(2) if f is fuzzy SSP-irresolute continuous and surjective, then g is fuzzy SSP-irresolute open(closed).
(3) if f is fuzzy semi strong precontinuous and surjective, then g is fuzzy semi strong preopen(preclosed).

Proof. 1) Let A be a FSSPOS(FSSPCS) of X, by assumptions, $g(f(A))$ is a FSSPOS(FSSPCS). Since g is SSP-irresolute continuous, then $f(A)=g^{-1}g(f(A))$ is a FSSPOS(FSSPCS). Thus, f is SSP-irresolute open(closed).
2) Let B be a FSSPOS(FSSPCS) of Y, by assumptions, $f^{-1}(B)$ is FSSPOS(FSSPCS) of X. Thus, $g(B)=g(f(f^{-1}(B)))$ is a FSSPOS(FSSPCS) of Z. Indeed, g is SSP-irresolute open(closed) mapping.
3) Let B be a fuzzy open(closed) set of Y. By assumptions, $f^{-1}(B)$ is a FSSPOS(FSSPCS) of X. Thus, $g(B)=g(f(f^{-1}(B)))$ is a FSSPOS(FSSPCS) of Z. Indeed, g is a fuzzy semi strong preopen(preclosed) mapping. □

Theorem 19. *Let $f:(X,\tau_1)\to(Y,\tau_2)$ and $g:(Y,\tau_2)\to(Z,\tau_3)$, where (X,τ_1), (Y,τ_2) and (Z,τ_3) are fts and gf is a fuzzy SSP-irresolute continuous mapping.*

(1) if g is fuzzy SSP-irresolute open(closed) and injective, then f is fuzzy SSP-irresolute continuous;
(2) if f is fuzzy SSP-irresolute open(closed) and surjective, then g is fuzzy SSP-irresolute continuous.

Proof. In similar manner as the proof of Theorem 18. □

5. Fuzzy SSP-homeomorphism

Definition 5.1. *A bijective mapping $f:(X,\tau_1)\to(Y,\tau_2)$ from a fts (X,τ_1) into a fts (Y,τ_2) is called a fuzzy SSP-homeomorphism if both f and f^{-1} are fuzzy SSP-irresolute continuous.*

Theorem 20. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a mapping from a fts (X, τ_1) into a fts (Y, τ_2). The following statements are equivalent:*

i) *f is fuzzy SSP-homeomorphism;*
ii) *f^{-1} is fuzzy SSP-homeomorphism;*
iii) *f is fuzzy SSP-irresolute continuous and fuzzy SSP-irresolute open;*
iv) *f is fuzzy SSP-irresolute continuous and fuzzy SSP-irresolute closed;*
v) $f(\mathrm{sspcl}\, A) = \mathrm{sspcl}\, f(A)$, *for each fuzzy set A of X.*
vi) $\mathrm{sspcl}\, f^{-1}(B) = f^{-1}(\mathrm{sspcl}\, B)$, *for each fuzzy set B of Y;*
vii) $f^{-1}(\mathrm{sspint}\, B) = \mathrm{sspint}\, f^{-1}(B)$, *for each fuzzy set B of Y;*
viii) $\mathrm{sspint}\, f(A) = f(\mathrm{sspint}\, A)$, *for each fuzzy set A of X.*

Proof. Directly by Theorem 2, Theorem 4, Theorem 9 and Theorem 11. □

6. CONCLUSION

In this paper, we introduced the notions of fuzzy SSP-irresolute continuous mappings, fuzzy SSP-irresolute open mappings, fuzzy SSP-irresolute closed mappings, and fuzzy SSP-homeomorphisms. Several characterizations of these mappings were established by using fuzzy semi strong preinterior and fuzzy semi strong preclosure operators. The present study is restricted to fuzzy topological spaces in the sense of Chang. Future research may extend these notions to other generalized fuzzy topological structures.

REFERENCES

[1] K. K. Azad, *On Fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity*, Journal fo Mathematical Analysis and Applications, **82** (1981), 14-32.
[2] C. L. Chang, *Fuzzy topological spaces*, Journal of Mathematical Analysis and Applications, **24** (1968), 182-190.
[3] B. Ghosh, *Semi continuous and semiclosed mappings and semi connectedness in fuzzy settings* , Fuzzy Sets and Systems, **35** (1990), 345-355.
[4] B. Krsteska, *Fuzzy SP-irresolute mappings*, Mathematicki Vesnik, **21** (1997), 67-80.
[5] S. Makolli and B. Krsteska, *A note regarding some fuzzy SSPO mappings*, Matematichki Bilten, **46** (2022), 83-96.
[6] M. N. Mukherjee and S. P. Sinha, *Irresolute and almost open functions between fuzzy topological spaces*, Fuzzy Sets and Systems, **29** (1989), 381-388.
[7] J.H. Park and B.Y. Lee, *Fuzzy semipreopen sets and fuzzy semiprecontinuous mappings*, Fuzzy Sets and Systems **67** (1994), 359-364.
[8] E. Rrustolli and B. Krsteska, *Fuzzy Semi Strongly Preopen Sets and Fuzzy Semi Strong Precontinuity*, Submitted.
[9] M. K. Singal and N. Prakash, *Fuzzy preopen sets and fuzzy preseparation axioms*, Fuzzy Sets and Systems, **44** (1991), 273-281.
[10] T.H. Yalvale, *Semi interior and semiclosure of fuzzy sets*, Journal of Mathematical Analysis and applications, **132** (1988), 356-364.
[11] L. A. Zadeh, *Fuzzy sets*, Inf. Control, vol. 8, no. 3, June 1965, 338-353.
[12] B.Sh. Zhong, *Fuzzy weakly semicontinuity*, Fuzzy Sets and Systems, **47** (1992), 93-98.
[13] B.Sh. Zhong, *Fuzzy strongly semiopen sets and fuzzy strong semicontinuity*, Fuzzy Sets and Systems, **52** (1992), 345-354.

Egzon Rrustolli
University of Prishtina,
Faculty of Economics,
st. "Agim Ramadani", n.n. 10 000 Prishtina, Republic of Kosovo
Email address: egzon.rrustolli@uni-pr.edu

Biljana Krsteska
Ss. Cyril and Methodius University in Skopje,
Faculty of Natural Sciences and Mathematics,
Arhimedova 3, 1000 Skopje, Republic of North Macedonia
Email address: madob2006@gmail.com

Received 09.03.2026
Revised 16.06.2026
Accepted 23.07.2026