

SOME COSINE OPERATOR FUNCTIONS IN \mathbb{R}^3

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Abstract. In this paper, we find out the cosine operator functions

$$C(t) = \begin{bmatrix} a(t) & b(t) & c(t) \\ 0 & d(t) & e(t) \\ 0 & 0 & f(t) \end{bmatrix}$$

in a real vector space \mathbb{R}^3 ($t \in \mathbb{R}$), as the solutions of the second order Cauchy problem

$$C''(t) = \mathcal{A} \cdot C(t), \quad C(0) = I, \quad C'(0) = 0.$$

We find the solutions for the various cases of a given real matrix

$$\mathcal{A} = \begin{bmatrix} A & B & C \\ 0 & D & E \\ 0 & 0 & F \end{bmatrix},$$

which is a generator of these cosine operator functions $C(t)$, $t \in \mathbb{R}$.

1. INTRODUCTION

The family of cosine operator functions in a given Banach space X is a family of linear operators $C(t)$ from X to X which satisfies the conditions

$$C(0) = I \quad \text{and} \quad C(t+s) + C(t-s) = 2C(t)C(s) \quad \text{for all } t, s \in \mathbb{R}, \quad (1.1)$$

with I -identity operator on X . Many mathematicians have studied this family and built the theory of cosine operator functions and its applications (see [1, 2, 5, 6, 7]). The notion of a generator \mathcal{A} of this family of operators $C(t)$ is introduced

with $\mathcal{A} = \lim_{h \rightarrow 0} \frac{2(C(h) - I)}{h^2}$ and it holds

$$C''(t) = \mathcal{A}C(t), \quad C(0) = I, \quad C'(0) = 0. \quad (1.2)$$

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(see [1, 5, 8]). So, the family of cosine operator functions $C(t)$ satisfies the second order Cauchy problem (1.2). In this paper [8] we have defined a family of cosine operator functions and its generator in a different way, as a solution of certain functional equations. This operator \mathcal{A} is a linear and closed operator, defined in everywhere dense set in X . It also holds vice versa, for given linear and closed, everywhere dense defined operator \mathcal{A} in X , there exists a cosine operator function $C(t)$ ($t \in \mathbb{R}$) whose generator is just the same \mathcal{A} and which is the solution of the Cauchy problem (1.2).

We may notice that from (1.2), since $C(0) = I$ it follows $C''(0) = \mathcal{A} \cdot C(0) = \mathcal{A}$. It is known that the Cauchy problem

$$y''(t) = a \cdot y(t), y(0) = 1, y'(0) = 0 \quad (a \in \mathbb{R}), \quad (1.3)$$

has only the following solutions:

$$y(t) = \frac{1}{2} \left(e^{\sqrt{at}} + e^{-\sqrt{at}} \right) = \cosh(\sqrt{at}); \quad \text{for } a > 0,$$

$$y(t) = \frac{1}{2} \left(e^{i\sqrt{-at}} + e^{-i\sqrt{-at}} \right) = \cos(\sqrt{-at}); \quad \text{for } a < 0$$

and $y \equiv 1; \quad \text{for } a = 0.$

These solutions satisfy also D'Alambert functional equation

$$y(t+s) + y(t-s) = 2y(t)y(s), \quad (t, s \in \mathbb{R}). \quad (1.4)$$

Now, we are interested in determining of cosine operator functions $C(t)$ in a real vector space \mathbb{R}^3 , which satisfy the second order Cauchy problem. As we see, the solution $C(t)$ ($t \in \mathbb{R}$) of the Cauchy problem (1.2) depends on the operator \mathcal{A} which generates that cosine operator function. We may identify generator \mathcal{A} and family $C(t)$ ($t \in \mathbb{R}$) of cosine operator functions with corresponding matrices from $M_3(\mathbb{R})$ considering the standard basis of the space \mathbb{R}^3 .

2. RESULTS

Let

$$C(t) = \begin{bmatrix} a(t) & b(t) & c(t) \\ 0 & d(t) & e(t) \\ 0 & 0 & f(t) \end{bmatrix} \quad (t \in \mathbb{R})$$

be a cosine operator function and

$$\mathcal{A} = C''(0) = \begin{bmatrix} A & B & C \\ 0 & D & E \\ 0 & 0 & F \end{bmatrix}$$

its generator with $A, B, C, D, E, F \in \mathbb{R}$. We here consider a special case, when $C(t)$ and \mathcal{A} are upper triangular matrices. Our goal is for various selections of the generator \mathcal{A} , to determine their corresponding cosine functions $\{C(t)\}_{t \in \mathbb{R}}$. In that case it holds:

$$C''(t) = \mathcal{A} \cdot C(t), \quad C(0) = I, \quad C'(0) = 0. \quad (2.1)$$

Hence,

$$\begin{bmatrix} a(t) & b(t) & c(t) \\ 0 & d(t) & e(t) \\ 0 & 0 & f(t) \end{bmatrix}'' = \begin{bmatrix} A & B & C \\ 0 & D & E \\ 0 & 0 & F \end{bmatrix} \cdot \begin{bmatrix} a(t) & b(t) & c(t) \\ 0 & d(t) & e(t) \\ 0 & 0 & f(t) \end{bmatrix} \quad (2.2)$$

and we get the system of differential equations

$$\begin{cases} a''(t) = A \cdot a(t) \\ b''(t) = A \cdot b(t) + B \cdot d(t) \\ c''(t) = A \cdot c(t) + B \cdot e(t) + C \cdot f(t) \\ d''(t) = D \cdot d(t) \\ e''(t) = D \cdot e(t) + E \cdot f(t) \\ f''(t) = F \cdot f(t) \end{cases} \quad (2.3)$$

with the following initial conditions:

$$\begin{cases} a(0) = d(0) = f(0) = 1, b(0) = c(0) = 0 = e(0) = 0, \\ a'(0) = b'(0) = c'(0) = d'(0) = e'(0) = f'(0) = 0 \\ a''(0) = A, b''(0) = B, c''(0) = C, d''(0) = D, e''(0) = E, f''(0) = F. \end{cases} \quad (2.4)$$

The solution of the first equation of the system (2.3), $a''(t) = A \cdot a(t)$, with the initial conditions $a(0) = 1, a'(0) = 0, a''(0) = A$, is

$$a(t) = \begin{cases} \cosh(t\sqrt{A}), & \text{for } A > 0 \\ 1, & \text{for } A = 0 \\ \cos(t\sqrt{-A}), & \text{for } A < 0. \end{cases}$$

This holds, analogously, for the functions $d(t)$ and $f(t)$ as the solutions of the fourth and sixth equations of the system (2.3) with their appropriate initial conditions:

$$d(t) = \begin{cases} \cosh(t\sqrt{D}), & \text{for } D > 0 \\ 1, & \text{for } D = 0 \\ \cos(t\sqrt{-D}), & \text{for } D < 0 \end{cases} \quad \text{and} \quad f(t) = \begin{cases} \cosh(t\sqrt{F}), & \text{for } F > 0 \\ 1, & \text{for } F = 0 \\ \cos(t\sqrt{-F}), & \text{for } F < 0. \end{cases}$$

By differentiating the second equation of the system (2.3) twice, we get

$$b^{(4)}(t) = A \cdot b''(t) + B \cdot d''(t).$$

Further, from the fourth equation of the system (2.3) we have $b^{(4)}(t) = A \cdot b''(t) + B \cdot D \cdot d(t)$. Because $B \cdot d(t) = b''(t) - A \cdot b(t)$ we find $b^{(4)}(t) = A \cdot b''(t) + D \cdot (b''(t) - A \cdot b(t))$, that is

$$b^{(4)}(t) - (A + D) \cdot b''(t) + AD \cdot b(t) = 0, \quad (2.5)$$

with initial conditions

$$b(0) = b'(0) = 0, b''(0) = B. \quad (2.6)$$

After the introduction of substitution $b = e^{rt}$ in (2.5) we get the characteristic equation $r^4 - (A + D)r^2 + AD = 0 \iff (r^2 - A)(r^2 - D) = 0$. The solutions are: $r_{1,2} = \pm\sqrt{A}$, $r_{3,4} = \pm\sqrt{D}$.

2.1. Discussion about the solution of the Cauchy problem (2.5)-(2.6).

A) Case $D \neq A$. Then the general solution of (2.5) is:

$$b(t) = C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} + C_3 e^{t\sqrt{D}} + C_4 e^{-t\sqrt{D}}. \quad (2.7)$$

From the initial conditions (2.6) we find

$$\begin{aligned} C_1 + C_2 + C_3 + C_4 &= 0 \\ \sqrt{A}(C_1 - C_2) + \sqrt{D}(C_3 - C_4) &= 0 \\ A(C_1 + C_2) + D(C_3 + C_4) &= B. \end{aligned}$$

From the (2.7) we find

$$b'''(t) = C_1 A \sqrt{A} e^{t\sqrt{A}} - C_2 A \sqrt{A} e^{-t\sqrt{A}} + C_3 D \sqrt{D} e^{t\sqrt{D}} - C_4 D \sqrt{D} e^{-t\sqrt{D}}.$$

After differentiating with respect to t the second equation of the system (2.3), we find $b'''(t) = A \cdot b'(t) + B \cdot d'(t)$ and for $t = 0$

$$b'''(0) = A\sqrt{A}(C_1 - C_2) + D\sqrt{D}(C_3 - C_4) = A \cdot b'(0) + B \cdot d'(0) = A \cdot 0 + B \cdot 0 = 0.$$

Now we have the system

$$\begin{cases} C_1 + C_2 + C_3 + C_4 = 0 \\ \sqrt{A}(C_1 - C_2) + \sqrt{D}(C_3 - C_4) = 0 \\ A(C_1 + C_2) + D(C_3 + C_4) = B \\ A\sqrt{A}(C_1 - C_2) + D\sqrt{D}(C_3 - C_4) = 0. \end{cases} \quad (2.8)$$

The solution of the system (2.8) is:

$$C_1 = C_2 = \frac{B}{2(A - D)}, \quad C_3 = C_4 = -\frac{B}{2(A - D)} \quad (2.9)$$

and after putting it in (2.7), depending on signs of A and D , we get the following results:

- for $A > 0$ and $D > 0$: $b(t) = \frac{B}{A-D}(\cosh(t\sqrt{A}) - \cosh(t\sqrt{D}))$
- for $A > 0$ and $D < 0$: $b(t) = \frac{B}{A-D}(\cosh(t\sqrt{A}) - \cos(t\sqrt{-D}))$
- for $A < 0$ and $D < 0$: $b(t) = \frac{B}{A-D}(\cos(t\sqrt{-A}) - \cos(t\sqrt{-D}))$
- for $A < 0$ and $D > 0$: $b(t) = \frac{B}{A-D}(\cos(t\sqrt{-A}) - \cosh(t\sqrt{D}))$
- for $A = 0$ and $D > 0$: $b(t) = \frac{-B}{D}(1 - \cosh(t\sqrt{D}))$
- for $A = 0$ and $D < 0$: $b(t) = \frac{-B}{D}(1 - \cos(t\sqrt{-D}))$
- for $A > 0$ and $D = 0$: $b(t) = \frac{B}{A}(\cosh(t\sqrt{A}) - 1)$
- for $A < 0$ and $D = 0$: $b(t) = \frac{B}{A}(\cosh(t\sqrt{-A}) - 1)$.

B) Case $D = A$. Then there are three possibilities $A > 0$, $A < 0$ and $A = 0$.

a) Let $A > 0$. Since $D = A$ we get $a(t) = d(t) = \cosh(t\sqrt{A})$. From $b''(t) = A \cdot b(t) + B \cdot d(t)$ follows $b^{(4)}(t) = A \cdot b''(t) + B \cdot d''(t)$. And from $d''(t) = A \cdot d(t)$ and $B \cdot d(t) = b''(t) - A \cdot b(t)$ we find $b^{(4)}(t) - A \cdot b''(t) = A \cdot (b''(t) - A \cdot b(t))$, that is $b^{(4)}(t) - 2A \cdot b''(t) + A^2 \cdot b(t) = 0$. For $b = e^{rt}$ we get $(r^2 - A)^2 = 0$. From $b(t) = C_1 \cdot e^{t\sqrt{A}} + C_2 \cdot te^{t\sqrt{A}} + C_3 e^{-t\sqrt{A}} + C_4 te^{-t\sqrt{A}}$ and initial conditions $b(0) = b'(0) = 0$, $b''(0) = B$ we get

$$\begin{aligned} C_1 + C_3 &= 0 \\ C_1\sqrt{A} + C_2 - C_3\sqrt{A} + C_4 &= 0 \\ A(C_1 + C_3) + 2\sqrt{A}(C_2 - C_4) &= B. \end{aligned}$$

Hence,

$$C_3 = -C_1, \quad C_2 + C_4 = -2\sqrt{A} \cdot C_1, \quad C_2 - C_4 = \frac{B}{2\sqrt{A}}.$$

and so $C_2 = \frac{B}{4\sqrt{A}} - C_1\sqrt{A}$, $C_4 = \frac{-B}{4\sqrt{A}} - C_1\sqrt{A}$. That is why

$$\begin{aligned} b(t) &= C_1(e^{t\sqrt{A}} - e^{-t\sqrt{A}}) - C_1 t\sqrt{A}(e^{t\sqrt{A}} + e^{-t\sqrt{A}}) + \frac{Bt}{4\sqrt{A}}(e^{t\sqrt{A}} - e^{-t\sqrt{A}}) \\ &= 2C_1 \sinh(t\sqrt{A}) - 2C_1 t\sqrt{A} \cosh(t\sqrt{A}) + \frac{Bt}{2\sqrt{A}} \sinh(t\sqrt{A}). \end{aligned}$$

By calculating $b''(t)$ and putting it into $b''(t) - A \cdot b(t) = B \cdot \cosh(t\sqrt{A})$ we find $C_1 = 0$, so

$$b(t) = \frac{Bt}{2\sqrt{A}} \sinh(t\sqrt{A}).$$

b) Let $A < 0$. Then $d(t) = a(t) = \cos(t\sqrt{-A})$. Again as in part a), from $b^{(4)}(t) - 2Ab''(t) + A^2b(t) = 0$ and $b(t) = e^{rt}$ we have characteristic equation $(r^2 - A)^2 = 0$ and its solutions $r = \pm i\sqrt{-A}$. Searching the solution in the form $b(t) = C_1 e^{it\sqrt{-A}} + C_2 te^{it\sqrt{-A}} + C_3 e^{-it\sqrt{-A}} + C_4 te^{-it\sqrt{-A}}$ and by using the initial conditions $b(0) = b'(0) = 0$, $b''(0) = B$, we get

$$C_3 = -C_1, \quad C_2 + C_4 = -2C_1 i\sqrt{-A}, \quad C_2 - C_4 = \frac{B}{2i\sqrt{-A}}.$$

and $b(t) = 2C_1 i \sin(t\sqrt{-A}) - 2C_1 it\sqrt{-A} \cos(t\sqrt{-A}) + \frac{Bt}{2\sqrt{-A}} \sin(t\sqrt{-A})$. We determine number C_1 by calculating $b''(t)$ and putting it into $b''(t) - A \cdot b(t) = B \cdot \cos(t\sqrt{-A})$. In that way, we find $C_1 = 0$ and hence

$$b(t) = \frac{Bt}{2\sqrt{-A}} \sin(t\sqrt{-A}).$$

c) If $D = A = 0$ then $d(t) = a(t) \equiv 1$ and $b''(t) = B$. We are finding $b(t)$ in the form $b(t) = C_1 + C_2 t + C_3 t^2$. From the initial conditions $b(0) = b'(0) = 0$, $b''(0) = B$ it follows $C_1 = C_2 = 0$, $C_3 = \frac{B}{2}$ and $b(t) = \frac{B}{2} t^2$.

2.2. Determining the function $e(t)$. By differentiating two times the equation $e''(t) = D \cdot e(t) + E \cdot f(t)$, and using the equations $f''(t) = F \cdot f(t)$ and $E \cdot f(t) = e''(t) - D \cdot e(t)$ we obtain $e^{(4)}(t) - (D + F)e''(t) + DFe(t) = 0$. It is known that $e(0) = e'(0) = 0$, $e''(0) = E$. Analogously to discussion in section 2.1. A) and B), depending on the values of scalars D and F , we find the following results.

A). Let $F \neq D$. Then:

a) for $F > 0$ and $D > 0$

$$f(t) = \cosh(t\sqrt{F}) \text{ and } e(t) = \frac{E}{D - F}(\cosh(t\sqrt{D}) - \cosh(t\sqrt{F})).$$

b) for $D > 0$ and $F < 0$

$$f(t) = \cos(t\sqrt{-F}) \text{ and } e(t) = \frac{E}{D - F}(\cosh(t\sqrt{D}) - \cos(t\sqrt{-F})).$$

c) for $D < 0$ and $F < 0$

$$f(t) = \cos(t\sqrt{-F}) \text{ and } e(t) = \frac{E}{D - F}(\cos(t\sqrt{-D}) - \cos(t\sqrt{-F})).$$

d) for $D < 0$ and $F > 0$

$$f(t) = \cosh(t\sqrt{F}) \text{ and } e(t) = \frac{E}{D - F}(\cos(t\sqrt{-D}) - \cosh(t\sqrt{F})).$$

e) for $D = 0$ and $F > 0$

$$f(t) = \cosh(t\sqrt{F}) \text{ and } e(t) = \frac{-E}{F}(1 - \cosh(t\sqrt{F})).$$

f) for $D = 0$ and $F < 0$

$$f(t) = \cos(t\sqrt{-F}) \text{ and } e(t) = \frac{-E}{F}(1 - \cos(t\sqrt{-F})).$$

g) for $D > 0$ and $F = 0$

$$f(t) \equiv 1 \text{ and } e(t) = \frac{E}{D}(\cosh(t\sqrt{D}) - 1).$$

h) for $D < 0$ and $F = 0$

$$f(t) \equiv 1 \text{ and } e(t) = \frac{E}{D}(\cos(t\sqrt{-D}) - 1).$$

B). Let $F = D$. Then depending on the sign of the number D , analogously to discussion in section 2.1. B), we find the following cases:

a) for $D > 0$: $e(t) = \frac{Et}{2\sqrt{D}} \sinh(t\sqrt{D})$ and $f(t) = \cosh(t\sqrt{D})$,

b) for $D < 0$: $e(t) = \frac{Et}{2\sqrt{-D}} \sin(t\sqrt{-D})$ and $f(t) = \cos(t\sqrt{-D})$,

c) for $D = 0$: $e(t) = \frac{E}{2}t^2$ and $f(t) \equiv 1$.

2.3. **Determining the function $c(t)$.** From (2.3) and (2.4) we see

$$\begin{cases} c''(t) - A \cdot c(t) = B \cdot e(t) + C \cdot f(t) \\ c(0) = c'(0) = 0, c''(0) = C. \end{cases} \quad (2.10)$$

The solution of the Cauchy problem (2.10) depends on the values of the elements A, B, C of matrix \mathcal{A} . In discussion of the solution $c(t)$ depending on the values of the scalars D and F we use also the results from the section 2.2.

A) Case $F \neq D$.

a) Let $F > 0, D > 0$. Then

$$f(t) = \cosh(t\sqrt{F}) \text{ and } e(t) = \frac{E}{D-F}(\cosh(t\sqrt{D}) - \cosh(t\sqrt{F})),$$

so we get differential equation

$$c''(t) - A \cdot c(t) = \frac{BE}{D-F}(\cosh(t\sqrt{D}) - \cosh(t\sqrt{F})) + C \cdot \cosh(t\sqrt{F}). \quad (2.11)$$

Solution of the homogenous equation $c''(t) - A \cdot c(t) = 0$ is

$$c_h(t) = C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}}. \quad (2.12)$$

We search the particular solution $c_p(t)$ of nonhomogenous equation in the form

$$c_p(t) = \alpha \cdot \cosh(t\sqrt{D}) + \beta \cdot \cosh(t\sqrt{F}). \quad (2.13)$$

Then, $c_p''(t) = \alpha D \cdot \cosh(t\sqrt{D}) + \beta F \cdot \cosh(t\sqrt{F})$, and after putting it in (2.11) we get

$$\alpha(D - A) = \frac{BE}{D - F} \text{ and } \beta(F - A) = C - \frac{BE}{D - F}.$$

1. If $A \neq D, F$ then

$$\alpha = \frac{BE}{(D - A)(D - F)} \text{ and } \beta = \frac{C(D - F) - BE}{(F - A)(D - F)}. \quad (2.14)$$

By putting (2.14) in (2.13) we obtain

$$c_p(t) = \frac{BE}{(D - A)(D - F)} \cosh(t\sqrt{D}) + \frac{C(D - F) - BE}{(F - A)(D - F)} \cosh(t\sqrt{F}). \quad (2.15)$$

From (2.12) and (2.15) we have

$$\begin{aligned} c(t) &= c_h(t) + c_p(t) = \\ &= C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} + \frac{BE}{(D - A)(D - F)} \cosh(t\sqrt{D}) + \\ &\quad + \frac{C(D - F) - BE}{(F - A)(D - F)} \cosh(t\sqrt{F}). \end{aligned} \quad (2.16)$$

After taking into account the initial conditions $c(0) = c'(0) = 0, c''(0) = C$, we find

$$C_1 = C_2 = \frac{AC + BE - CD}{2(D - A)(F - A)}, \text{ (for } D \neq A, D \neq F). \quad (2.17)$$

Including (2.17) into (2.16) we find that the solutions of a Cauchy problem (2.10), depending on the sign of number $A \neq D, F$ are:

1.i) for $A > 0$,

$$c(t) = \frac{AC + BE - CD}{(D - A)(F - A)} \cosh(t\sqrt{A}) + \frac{BE}{(D - A)(D - F)} \cosh(t\sqrt{D}) + \\ + \frac{C(D - F) - BE}{(F - A)(D - F)} \cosh(t\sqrt{F})$$

1.ii) for $A < 0$,

$$c(t) = \frac{AC + BE - CD}{(D - A)(F - A)} \cos(t\sqrt{-A}) + \frac{BE}{(D - A)(D - F)} \cosh(t\sqrt{D}) + \\ + \frac{C(D - F) - BE}{(F - A)(D - F)} \cosh(t\sqrt{F})$$

1.iii) for $A = 0$,

$$c(t) = \frac{BE - CD}{DF} + \frac{BE}{D(D - F)} \cosh(t\sqrt{D}) + \frac{C(D - F) - BE}{(F(D - F))} \cosh(t\sqrt{F}).$$

2. If $A = D \neq F$ then we search the particular solution in a form

$$c_p(t) = at \cdot \cosh(t\sqrt{A}) + bt \sinh(t\sqrt{A}) + \beta \cdot \cosh(t\sqrt{F}).$$

Then,

$$c'_p(t) = a \cosh(t\sqrt{A}) + at\sqrt{A} \sinh(t\sqrt{A}) + b \sinh(t\sqrt{A}) + \\ + bt\sqrt{A} \cosh(t\sqrt{A}) + \beta\sqrt{F} \sinh(t\sqrt{F})$$

and

$$c''_p(t) = 2a\sqrt{A} \sinh(t\sqrt{A}) + aAt \cosh(t\sqrt{A}) + 2b\sqrt{A} \cosh(t\sqrt{A}) + \\ + bAt \sinh(t\sqrt{A}) + \beta F \cdot \cosh(t\sqrt{F}).$$

Now we get:

$$c''_p(t) - A \cdot c_p(t) = 2a\sqrt{A} \sinh(t\sqrt{A}) + aAt \cosh(t\sqrt{A}) + 2b\sqrt{A} \cosh(t\sqrt{A}) + \\ + bAt \sinh(t\sqrt{A}) + \beta F \cdot \cosh(t\sqrt{F}) - aAt \cdot \cosh(t\sqrt{A}) - \\ - bAt \sinh(t\sqrt{A}) - \beta A \cdot \cosh(t\sqrt{F}) = \\ = \frac{BE}{A - F} (\cosh(t\sqrt{A}) - \cosh(t\sqrt{F})) + C \cdot \cosh(t\sqrt{F}).$$

On that way, it follows $a = 0$, $b = \frac{BE}{2\sqrt{A}(A - F)}$, $\beta = \frac{BE - AC + CF}{(A - F)^2}$. So,

$$c_p(t) = \frac{BE}{2\sqrt{A}(A - F)} \cdot t \cdot \sinh(t\sqrt{A}) + \frac{BE - AC + CF}{(A - F)^2} \cdot \cosh(t\sqrt{F})$$

and

$$c(t) = c_h(t) + c_p(t) = C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} + \frac{BE}{2\sqrt{A}(A - F)} \cdot t \cdot \sinh(t\sqrt{A}) +$$

$$+ \frac{BE - AC + CF}{(A - F)^2} \cdot \cosh(t\sqrt{F}).$$

When we apply the initial conditions $c(0) = c'(0) = 0$ we get

$$C_1 = C_2 = -\frac{BE - AC + CF}{2(A - F)^2}.$$

The third condition $c''(0) = C$ is now satisfied. Finally, we find the solution

$$c(t) = \frac{BE - AC + CF}{(A - F)^2} \left(\cosh(t\sqrt{F}) - \cosh(t\sqrt{A}) \right) + \frac{BE}{2\sqrt{A}(A - F)} \cdot t \cdot \sinh(t\sqrt{A}).$$

3. If $A = F \neq D$ then we search the particular solution in the form

$$c_p(t) = at \cdot \cosh(t\sqrt{A}) + bt \sinh(t\sqrt{A}) + \beta \cdot \cosh(t\sqrt{D}).$$

We find

$$c'_p(t) = a \cosh(t\sqrt{A}) + at\sqrt{A} \sinh(t\sqrt{A}) + b \sinh(t\sqrt{A}) + \\ + bt\sqrt{A} \cosh(t\sqrt{A}) + \beta\sqrt{D} \sinh(t\sqrt{D})$$

and

$$c''_p(t) = 2a\sqrt{A} \sinh(t\sqrt{A}) + aAt \cosh(t\sqrt{A}) + 2b\sqrt{A} \cosh(t\sqrt{A}) + \\ + bAt \sinh(t\sqrt{A}) + \beta D \cdot \cosh(t\sqrt{D}).$$

Now we get:

$$c''_p(t) - A \cdot c_p(t) = 2a\sqrt{A} \sinh(t\sqrt{A}) + aAt \cosh(t\sqrt{A}) + 2b\sqrt{A} \cosh(t\sqrt{A}) + \\ + bAt \sinh(t\sqrt{A}) + \beta D \cdot \cosh(t\sqrt{D}) - aAt \cdot \cosh(t\sqrt{A}) - \\ - bAt \sinh(t\sqrt{A}) - \beta A \cdot \cosh(t\sqrt{D}) = \\ = \frac{BE}{D - A} \cosh(t\sqrt{D}) + \frac{CD - AC - BE}{D - A} \cdot \cosh(t\sqrt{A}).$$

It follows $a = 0$, $b = \frac{CD - AC - BE}{2\sqrt{A}(D - A)}$, $\beta = \frac{BE}{(D - A)^2}$. So,

$$c_p(t) = \frac{CD - AC - BE}{2\sqrt{A}(D - A)} \cdot t \sinh(t\sqrt{A}) + \frac{BE}{(D - A)^2} \cdot \cosh(t\sqrt{D})$$

and

$$c(t) = C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} + \frac{CD - AC - BE}{2\sqrt{A}(D - A)} \cdot t \sinh(t\sqrt{A}) + \frac{BE}{(D - A)^2} \cdot \cosh(t\sqrt{D}).$$

By applying the initial conditions $c(0) = c'(0) = 0$ we get $C_1 = C_2 = -\frac{BE}{2(D - A)^2}$. The third condition $c''(0) = C$ is also satisfied. Finally, we have the solution

$$c(t) = \frac{BE}{(D - A)^2} \left(\cosh(t\sqrt{D}) - \cosh(t\sqrt{A}) \right) + \frac{CD - AC - BE}{2\sqrt{A}(D - A)} \cdot t \sinh(t\sqrt{A}).$$

b) Let $D > 0$ and $F < 0$. Then $f(t) = \cos(t\sqrt{-F})$ and $e(t) = \frac{E}{D-F}(\cosh(t\sqrt{D}) - \cos(t\sqrt{-F}))$, so the first equation in (2.10) becomes

$$c''(t) - A \cdot c(t) = \frac{BE}{D-F}(\cosh(t\sqrt{D})) + \frac{C(D-F) - BE}{D-F} \cos(t\sqrt{-F}). \quad (2.18)$$

The solution of the corresponding homogeneous equation is given with (2.12) and we are looking for a particular solution of (2.18) in the form

$$c_p(t) = \alpha \cosh(t\sqrt{D}) + \beta \cos(t\sqrt{-F}).$$

Now let introduce $c_p(t)$ and $c_p''(t) = \alpha D \cosh(t\sqrt{D}) + \beta F \cos(t\sqrt{-F})$ into (2.18).

$$\begin{aligned} \alpha D \cosh(t\sqrt{D}) + \beta F \cos(t\sqrt{-F}) - \alpha A \cosh(t\sqrt{D}) - \beta A \cos(t\sqrt{-F}) &= \\ &= \frac{BE}{D-F}(\cosh(t\sqrt{D})) + \frac{C(D-F) - BE}{D-F} \cos(t\sqrt{-F}) \\ \implies \alpha(D-A) &= \frac{BE}{D-F} \text{ and } \beta(F-A) = \frac{C(D-F) - BE}{D-F}. \end{aligned}$$

1. Case $A \neq D$ and $A \neq F$

If $D \neq A$ and $F \neq A$ then $\alpha = \frac{BE}{(D-A)(D-F)}$, $\beta = \frac{C(D-F) - BE}{(F-A)(D-F)}$ and

$$c_p(t) = \frac{BE}{(D-A)(D-F)}(\cosh(t\sqrt{D}) + \frac{C(D-F) - BE}{(F-A)(D-F)} \cos(t\sqrt{-F})).$$

The general solution of (2.18) is $c(t) = c_h(t) + c_p(t)$, i.e.

$$\begin{aligned} c(t) = C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} + \frac{BE}{(D-A)(D-F)} \cosh(t\sqrt{D}) + \\ + \frac{C(D-F) - BE}{(F-A)(D-F)} \cos(t\sqrt{-F}). \quad (2.19) \end{aligned}$$

By using initial conditions we get

$$c(0) = C_1 + C_2 + \frac{BE}{(D-A)(D-F)} + \frac{C(D-F) - BE}{(F-A)(D-F)} = 0. \quad (2.20)$$

$$\begin{aligned} c'(t) = C_1 \sqrt{A} e^{t\sqrt{A}} - C_2 \sqrt{A} e^{-t\sqrt{A}} + \frac{BE\sqrt{D}}{(D-A)(D-F)} \sinh(t\sqrt{D}) - \\ - \frac{(C(D-F) - BE)\sqrt{-F}}{(F-A)(D-F)} \sin(t\sqrt{-F}). \end{aligned}$$

$c'(0) = C_1 \sqrt{A} - C_2 \sqrt{A} = 0 \implies C_1 = C_2$. Now from (2.20) we get

$$C_1 = C_2 = \frac{BE - CD + AC}{2(D-A)(F-A)}. \quad (2.21)$$

$$\begin{aligned} c''(t) = C_1 A e^{t\sqrt{A}} + C_2 A e^{-t\sqrt{A}} + \frac{BED}{(D-A)(D-F)} \cosh(t\sqrt{D}) + \\ + \frac{(C(D-F) - BE)F}{(F-A)(D-F)} \cos(t\sqrt{-F}). \end{aligned}$$

$c''(0) = (C_1 + C_2)A + \frac{BED}{(D-A)(D-F)} + \frac{(C(D-F)-BE)F}{(F-A)(D-F)} = C$. After putting (2.21) into this last equation we see it becomes identity. So, from initial conditions we have found C_1 and C_2 and using (2.21) in (2.19) lead us to the solution of considering Cauchy problem. Depending on the sign of number A here are the solutions.

1.i) for $A > 0$

$$c(t) = \frac{BE - CD + AC}{(D-A)(F-A)} \cosh(t\sqrt{A}) + \frac{BE}{(D-A)(D-F)} \cosh(t\sqrt{D}) + \frac{C(D-F) - BE}{(F-A)(D-F)} \cos(t\sqrt{-F}).$$

1.ii) for $A < 0$

$$c(t) = \frac{BE - CD + AC}{(D-A)(F-A)} \cos(t\sqrt{-A}) + \frac{BE}{(D-A)(D-F)} \cosh(t\sqrt{D}) + \frac{C(D-F) - BE}{(F-A)(D-F)} \cos(t\sqrt{-F}).$$

1.iii) for $A = 0$ the differential equation in (2.10) becomes

$$c''(t) = \frac{BE}{D-F} (\cosh(t\sqrt{D}) - \cos(t\sqrt{-F})) + C \cos(t\sqrt{-F})$$

i.e.

$$c''(t) = \frac{BE}{D-F} \cosh(t\sqrt{D}) + \frac{C(D-F) - BE}{D-F} \cos(t\sqrt{-F}). \quad (2.22)$$

$$c'(t) = \int \left[\frac{BE}{D-F} \cosh(t\sqrt{D}) + \frac{C(D-F) - BE}{D-F} \cos(t\sqrt{-F}) \right] dt$$

$$= \frac{BE}{(D-F)\sqrt{D}} \sinh(t\sqrt{D}) - \frac{C(D-F) - BE}{(D-F)\sqrt{-F}} \sin(t\sqrt{-F}) + C_1.$$

$$c(t) = \int \left[\frac{BE}{(D-F)\sqrt{D}} \sinh(t\sqrt{D}) - \frac{C(D-F) - BE}{(D-F)\sqrt{-F}} \sin(t\sqrt{-F}) + C_1 \right] dt.$$

$$c(t) = \frac{BE}{(D-F)D} \cosh(t\sqrt{D}) + \frac{C(D-F) - BE}{(D-F)F} \cos(t\sqrt{-F}) + C_1 t + C_2. \quad (2.23)$$

From the first condition $c(0) = 0$ we have

$$c(0) = \frac{BE}{(D-F)D} + \frac{C(D-F) - BE}{(D-F)F} + C_2 = 0 \implies (D-F)[CD - BE + C_2 DF] = 0.$$

Since $D \neq F$ it follows $CD - BE + C_2 DF = 0$ and

$$C_2 = \frac{BE - CD}{DF}. \quad (2.24)$$

From the second initial condition we get

$$c'(0) = C_1 = 0. \quad (2.25)$$

From (2.22) we find $c''(0) = C$, so the third initial condition is satisfied. Finally, from (2.25), (2.24) and (2.23) we get

$$c(t) = \frac{BE}{(D-F)D} \cosh(t\sqrt{D}) + \frac{C(D-F) - BE}{(D-F)F} \cos(t\sqrt{-F}) + \frac{BE - CD}{DF}. \quad (2.26)$$

2. If $A = D$, then we put

$$c_p(t) = at \cdot \cosh(t\sqrt{A}) + bt \sinh(t\sqrt{A}) + \beta \cdot \cos(t\sqrt{-F}).$$

We find

$$c'_p(t) = a \cosh(t\sqrt{A}) + at\sqrt{A} \sinh(t\sqrt{A}) + b \sinh(t\sqrt{A}) + bt\sqrt{A} \cosh(t\sqrt{A}) - \beta\sqrt{-F} \sin(t\sqrt{-F})$$

and

$$c''_p(t) = 2a\sqrt{A} \sinh(t\sqrt{A}) + aAt \cosh(t\sqrt{A}) + 2b\sqrt{A} \cosh(t\sqrt{A}) + bAt \sinh(t\sqrt{A}) + \beta F \cdot \cos(t\sqrt{-F}).$$

Now we get:

$$\begin{aligned} c''_p(t) - A \cdot c_p(t) &= 2a\sqrt{A} \sinh(t\sqrt{A}) + aAt \cosh(t\sqrt{A}) + 2b\sqrt{A} \cosh(t\sqrt{A}) + \\ &\quad + bAt \sinh(t\sqrt{A}) + \beta F \cdot \cos(t\sqrt{-F}) - aAt \cdot \cosh(t\sqrt{A}) - \\ &\quad - bAt \sinh(t\sqrt{A}) - \beta A \cdot \cos(t\sqrt{-F}) \\ &= \frac{BE}{A-F} \cosh(t\sqrt{A}) + \frac{AC - CF - BE}{A-F} \cos(t\sqrt{-F}). \end{aligned}$$

Hence, it follows $a = 0$, $b = \frac{BE}{2\sqrt{A}(A-F)}$, $\beta = \frac{BE-AC+CF}{(A-F)^2}$. So,

$$c_p(t) = \frac{BE}{2\sqrt{A}(A-F)} \cdot t \cdot \sinh(t\sqrt{A}) + \frac{BE - AC + CF}{(A-F)^2} \cdot \cos(t\sqrt{-F})$$

and

$$c(t) = C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} + \frac{BE}{2\sqrt{A}(A-F)} t \cdot \sinh(t\sqrt{A}) + \frac{BE - AC + CF}{(A-F)^2} \cos(t\sqrt{-F}).$$

When we apply the initial conditions $c(0) = c'(0) = 0$ we get $C_1 = C_2 = -\frac{BE-AC+CF}{2(A-F)^2}$.

The third condition $c''(0) = C$ is now satisfied. Finally, the solution is

$$c(t) = \frac{BE - AC + CF}{(A-F)^2} \left(\cos(t\sqrt{-F}) - \cosh(t\sqrt{A}) \right) + \frac{BE}{2\sqrt{A}(A-F)} \cdot t \cdot \sinh(t\sqrt{A}).$$

3. If $A = F$, then $c_p(t) = at \cdot \cos(t\sqrt{-A}) + bt \sin(t\sqrt{-A}) + \beta \cdot \cosh(t\sqrt{D})$. We find

$$c'_p(t) = a \cos(t\sqrt{-A}) - at\sqrt{-A} \sin(t\sqrt{-A}) + b \sin(t\sqrt{-A}) + bt\sqrt{-A} \cos(t\sqrt{-A}) + \beta\sqrt{D} \sinh(t\sqrt{D})$$

and

$$c''_p(t) = -2a\sqrt{-A} \sin(t\sqrt{-A}) + aAt \cos(t\sqrt{-A}) + 2b\sqrt{-A} \cos(t\sqrt{-A}) +$$

$$+ bAt \sin(t\sqrt{-A}) + \beta D \cdot \cosh(t\sqrt{D}).$$

Now we get:

$$\begin{aligned} c_p''(t) - A \cdot c_p(t) &= -2a\sqrt{-A} \sin(t\sqrt{-A}) + aAt \cos(t\sqrt{-A}) + 2b\sqrt{-A} \cos(t\sqrt{-A}) + \\ &\quad + bAt \sin(t\sqrt{-A}) + \beta D \cdot \cosh(t\sqrt{D}) - aAt \cdot \cos(t\sqrt{-A}) - \\ &\quad - bAt \sin(t\sqrt{-A}) - \beta A \cdot \cosh(t\sqrt{D}) = \\ &= \frac{BE}{D-A} \cosh(t\sqrt{D}) + \frac{CD-AC-BE}{D-A} \cdot \cos(t\sqrt{-A}) \end{aligned}$$

It follows $a = 0$, $b = \frac{CD-AC-BE}{2\sqrt{-A}(D-A)}$, $\beta = \frac{BE}{(D-A)^2}$. So,

$$c_p(t) = \frac{CD-AC-BE}{2\sqrt{-A}(D-A)} \cdot t \sin(t\sqrt{-A}) + \frac{BE}{(D-A)^2} \cdot \cosh(t\sqrt{D})$$

and

$$c(t) = C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} + \frac{CD-AC-BE}{2\sqrt{-A}(D-A)} \cdot t \sin(t\sqrt{-A}) + \frac{BE}{(D-A)^2} \cdot \cosh(t\sqrt{D}).$$

By applying the initial conditions $c(0) = c'(0) = 0$ we get $C_1 = C_2 = -\frac{BE}{2(D-A)^2}$. The third condition $c''(0) = C$ is also satisfied. Finally, we have the solution

$$c(t) = \frac{BE}{(D-A)^2} \left(\cosh(t\sqrt{D}) - \cos(t\sqrt{-A}) \right) + \frac{CD-AC-BE}{2\sqrt{-A}(D-A)} \cdot t \sin(t\sqrt{-A}).$$

c) Let $D < 0$, $F < 0$ and $D \neq F$. Then we have an equation:

$$\begin{aligned} c''(t) - A \cdot c(t) &= \frac{BE}{D-F} (\cos(t\sqrt{-D}) - \cos(t\sqrt{-F})) + C \cdot \cos(t\sqrt{-F}) = \\ &= \frac{BE}{D-F} \cos(t\sqrt{-D}) + \left(C - \frac{BE}{D-F} \right) \cos(t\sqrt{-F}). \end{aligned}$$

Let us find the particular solution. $c_p(t) = \alpha \cos(t\sqrt{-D}) + \beta \cos(t\sqrt{-F})$. It follows $c_p''(t) = \alpha D \cos(t\sqrt{-D}) + \beta F \cos(t\sqrt{-F})$. We get

$$\begin{aligned} \alpha D \cos(t\sqrt{-D}) + \beta F \cos(t\sqrt{-F}) - \alpha A \cos(t\sqrt{-D}) - \beta A \cos(t\sqrt{-F}) &= \\ = \frac{BE}{D-F} \cos(t\sqrt{-D}) + \frac{C(D-F) - BE}{D-F} \cos(t\sqrt{-F}), \end{aligned}$$

so

$$\alpha(D-A) = \frac{BE}{D-F} \quad \text{and} \quad \beta(F-A) = \frac{CD-CF-BE}{D-F}.$$

1. If $A \neq D$ and $A \neq F$, then $\alpha = \frac{BE}{(D-A)(D-F)}$ and $\beta = \frac{CD-CF-BE}{(F-A)(D-F)}$. Hence,

$$c_p(t) = \frac{BE}{(D-A)(D-F)} \cdot \cos(t\sqrt{-D}) + \frac{CD-CF-BE}{(F-A)(D-F)} \cdot \cos(t\sqrt{-F})$$

and

$$c(t) = C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} + \frac{BE}{(D-A)(D-F)} \cdot \cos(t\sqrt{-D}) +$$

$$+ \frac{CD - CF - BE}{(F - A)(D - F)} \cdot \cos(t\sqrt{-F}).$$

Further,

$$c(0) = C_1 + C_2 + \frac{BE}{(D - A)(D - F)} + \frac{CD - CF - BE}{(F - A)(D - F)} = 0. \quad (2.27)$$

$$c'(t) = C_1\sqrt{A}e^{t\sqrt{A}} - C_2\sqrt{A}e^{-t\sqrt{A}} - \frac{BE\sqrt{-D}}{(D - A)(D - F)} \cdot \sin(t\sqrt{-D}) - \frac{(CD - CF - BE)\sqrt{-F}}{(F - A)(D - F)} \cdot \sin(t\sqrt{-F}).$$

$$c'(0) = C_1\sqrt{A} - C_2\sqrt{A} = 0 \implies C_1 = C_2.$$

From (2.27) now follows

$$C_1 = C_2 = \frac{BE - CD + AC}{2(F - A)(D - A)} \quad (2.28)$$

and

$$c(t) = \frac{BE - CD + AC}{2(F - A)(D - A)} e^{t\sqrt{A}} + \frac{BE - CD + AC}{2(F - A)(D - A)} e^{-t\sqrt{A}} + \frac{BE}{(D - A)(D - F)} \cdot \cos(t\sqrt{-D}) + \frac{CD - CF - BE}{(F - A)(D - F)} \cdot \cos(t\sqrt{-F}).$$

The condition $c''(0) = C$ is also satisfied. Depending on the sign of number A , the solutions are:

1.i) for $A > 0$

$$c(t) = \frac{BE - CD + AC}{(F - A)(D - A)} \cosh(t\sqrt{A}) + \frac{BE}{(D - A)(D - F)} \cdot \cos(t\sqrt{-D}) + \frac{CD - CF - BE}{(F - A)(D - F)} \cdot \cos(t\sqrt{-F}).$$

1.ii) for $A < 0$

$$c(t) = \frac{BE - CD + AC}{(F - A)(D - A)} \cos(t\sqrt{-A}) + \frac{BE}{(D - A)(D - F)} \cdot \cos(t\sqrt{-D}) + \frac{CD - CF - BE}{(F - A)(D - F)} \cdot \cos(t\sqrt{-F}).$$

1.iii) for $A = 0$

$$c''(t) = \frac{BE}{D - F} \cos(t\sqrt{-D}) + \frac{C(D - F) - BE}{D - F} \cos(t\sqrt{-F}).$$

After integrating this equation two times we get

$$c(t) = \frac{BE}{D(D - F)} \cos(t\sqrt{-D}) + \frac{C(D - F) - BE}{F(D - F)} \cos(t\sqrt{-F}) + C_1t + C_2.$$

We find constants C_1 and C_2 from initial conditions.

$$c(0) = 0 \Rightarrow C_2 = \frac{BE - CD}{DF}, \quad c'(t) = 0 \Rightarrow C_1 = 0.$$

$$c(t) = \frac{BE - CD}{DF} + \frac{BE}{D(D - F)} \cos(t\sqrt{-D}) + \frac{C(D - F) - BE}{F(D - F)} \cos(t\sqrt{-F}).$$

2. If $A = D (\neq F)$, then we put

$$c_p(t) = at \cdot \cos(t\sqrt{-A}) + bt \sin(t\sqrt{-A}) + \beta \cdot \cos(t\sqrt{-F}).$$

We find

$$c'_p(t) = a \cos(t\sqrt{-A}) - at\sqrt{-A} \sin(t\sqrt{-A}) + b \sin(t\sqrt{-A}) +$$

$$+ bt\sqrt{-A} \cos(t\sqrt{-A}) - \beta\sqrt{-F} \sin(t\sqrt{-F})$$

and

$$c''_p(t) = -2a\sqrt{-A} \sin(t\sqrt{-A}) + aAt \cos(t\sqrt{-A}) + 2b\sqrt{-A} \cos(t\sqrt{-A}) +$$

$$+ bAt \sin(t\sqrt{-A}) + \beta F \cdot \cos(t\sqrt{-F}).$$

Now we get:

$$c''_p(t) - A \cdot c_p(t) = -2a\sqrt{-A} \sin(t\sqrt{-A}) + aAt \cos(t\sqrt{-A}) + 2b\sqrt{-A} \cos(t\sqrt{-A}) +$$

$$+ bAt \sin(t\sqrt{-A}) + \beta F \cdot \cos(t\sqrt{-F}) - aAt \cdot \cos(t\sqrt{-A}) -$$

$$- bAt \sin(t\sqrt{-A}) - \beta A \cdot \cos(t\sqrt{-F})$$

$$= \frac{BE}{A - F} \cos(t\sqrt{-A}) + \frac{AC - CF - BE}{A - F} \cos(t\sqrt{-F}).$$

Hence, it follows $a = 0$, $b = \frac{BE}{2\sqrt{-A}(A - F)}$, $\beta = \frac{BE - AC + CF}{(A - F)^2}$. So,

$$c_p(t) = \frac{BE}{2\sqrt{-A}(A - F)} \cdot t \sin(t\sqrt{-A}) + \frac{BE - AC + CF}{(A - F)^2} \cdot \cos(t\sqrt{-F})$$

and

$$c(t) = C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} + \frac{BE}{2\sqrt{-A}(A - F)} \cdot t \sin(t\sqrt{-A}) +$$

$$+ \frac{BE - AC + CF}{(A - F)^2} \cdot \cos(t\sqrt{-F}).$$

When we apply the initial conditions $c(0) = c'(0) = 0$ we get $C_1 = C_2 = -\frac{BE - AC + CF}{2(A - F)^2}$.

The third condition $c''(0) = C$ is now satisfied. Finally, the solution is

$$c(t) = \frac{BE - AC + CF}{(A - F)^2} \left(\cos(t\sqrt{-F}) - \cos(t\sqrt{-A}) \right) + \frac{BE}{2\sqrt{-A}(A - F)} \cdot t \sin(t\sqrt{-A}).$$

3. If $A = F (\neq D)$, then $c_p(t) = at \cdot \cos(t\sqrt{-A}) + bt \sin(t\sqrt{-A}) + \beta \cdot \cos(t\sqrt{-D})$.

We find

$$c'_p(t) = a \cos(t\sqrt{-A}) - at\sqrt{-A} \sin(t\sqrt{-A}) + b \sin(t\sqrt{-A}) +$$

$$+ bt\sqrt{-A} \cos(t\sqrt{-A}) - \beta\sqrt{-D} \sin(t\sqrt{-D})$$

and

$$c_p''(t) = -2a\sqrt{-A} \sin(t\sqrt{-A}) + aAt \cos(t\sqrt{-A}) + 2b\sqrt{-A} \cos(t\sqrt{-A}) + \\ + bAt \sin(t\sqrt{-A}) + \beta D \cdot \cos(t\sqrt{-D}).$$

Now we get:

$$c_p''(t) - A \cdot c_p(t) = -2a\sqrt{-A} \sin(t\sqrt{-A}) + aAt \cos(t\sqrt{-A}) + 2b\sqrt{-A} \cos(t\sqrt{-A}) + \\ + bAt \sin(t\sqrt{-A}) + \beta D \cdot \cos(t\sqrt{-D}) - aAt \cdot \cos(t\sqrt{-A}) - \\ - bAt \sin(t\sqrt{-A}) - \beta A \cdot \cos(t\sqrt{-D}) \\ = \frac{BE}{D-A} \cos(t\sqrt{-D}) + \frac{CD-AC-BE}{D-A} \cdot \cos(t\sqrt{-A}).$$

It follows $a = 0$, $b = \frac{CD-AC-BE}{2\sqrt{-A}(D-A)}$, $\beta = \frac{BE}{(D-A)^2}$. So, we have

$$c_p(t) = \frac{CD-AC-BE}{2\sqrt{-A}(D-A)} \cdot t \sin(t\sqrt{-A}) + \frac{BE}{(D-A)^2} \cdot \cos(t\sqrt{-D})$$

and

$$c(t) = C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} + \frac{CD-AC-BE}{2\sqrt{-A}(D-A)} \cdot t \sin(t\sqrt{-A}) + \frac{BE}{(D-A)^2} \cdot \cos(t\sqrt{-D}).$$

By applying the initial conditions $c(0) = c'(0) = 0$ we get $C_1 = C_2 = -\frac{BE}{2(D-A)^2}$. The third condition $c''(0) = C$ is also satisfied. Finally, we have the solution

$$c(t) = \frac{BE}{(D-A)^2} \left(\cos(t\sqrt{-D}) - \cos(t\sqrt{-A}) \right) + \frac{CD-AC-BE}{2\sqrt{-A}(D-A)} \cdot t \sin(t\sqrt{-A}).$$

d) Let $D < 0$ and $F > 0$. Now we have an equation

$$c''(t) - A \cdot c(t) = \frac{BE}{D-F} \cos(t\sqrt{-D}) + \frac{CD-CF-BE}{D-F} \cosh(t\sqrt{F}). \quad (2.29)$$

We put

$$c_p(t) = \alpha \cos(t\sqrt{-D}) + \beta \cosh(t\sqrt{F}) \text{ and } c_p''(t) = \alpha D \cos(t\sqrt{-D}) + \beta F \cosh(t\sqrt{F}).$$

When we imply these in (2.29) we get

$$\alpha D \cos(t\sqrt{-D}) + \beta F \cosh(t\sqrt{F}) - \alpha A \cos(t\sqrt{-D}) - \beta A \cosh(t\sqrt{F}) = \\ = \frac{BE}{D-F} \cos(t\sqrt{-D}) + \frac{C(D-F)-BE}{D-F} \cosh(t\sqrt{F}).$$

$$\implies \alpha(D-A) = \frac{BE}{D-F} \text{ and } \beta(F-A) = \frac{CD-CF-BE}{D-F}.$$

1. If $A \neq D, F$ then $\alpha = \frac{BE}{(D-A)(D-F)}$ and $\beta = \frac{CD-CF-BE}{(F-A)(D-F)}$. Therefore

$$c_p(t) = \frac{BE}{(D-A)(D-F)} \cdot \cos(t\sqrt{-D}) + \frac{CD-CF-BE}{(F-A)(D-F)} \cdot \cosh(t\sqrt{F})$$

and

$$\begin{aligned}
c(t) = c_h(t) + c_p(t) &= C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} + \frac{BE}{(D-A)(D-F)} \cdot \cos(t\sqrt{-D}) + \\
&\quad + \frac{CD - CF - BE}{(F-A)(D-F)} \cdot \cosh(t\sqrt{F}). \\
c(0) &= C_1 + C_2 + \frac{BE}{(D-A)(D-F)} + \frac{CD - CF - BE}{(F-A)(D-F)} = 0. \quad (2.30)
\end{aligned}$$

$$\begin{aligned}
c'(t) &= C_1 \sqrt{A} e^{t\sqrt{A}} - C_2 \sqrt{A} e^{-t\sqrt{A}} - \\
&\quad - \frac{BE\sqrt{-D}}{(D-A)(D-F)} \cdot \sin(t\sqrt{-D}) + \frac{(CD - CF - BE)\sqrt{F}}{(F-A)(D-F)} \cdot \sinh(t\sqrt{F}). \\
c'(0) &= C_1 \sqrt{A} - C_2 \sqrt{A} = 0 \Rightarrow C_1 = C_2. \text{ From (2.30) it follows} \\
C_1 = C_2 &= \frac{BE - CD + AC}{2(F-A)(D-A)}. \quad (2.31)
\end{aligned}$$

We get

$$\begin{aligned}
c(t) &= \frac{BE - CD + AC}{(F-A)(D-A)} \left(\frac{e^{t\sqrt{A}} + e^{-t\sqrt{A}}}{2} \right) + \\
&\quad + \frac{BE}{(D-A)(D-F)} \cdot \cos(t\sqrt{-D}) + \frac{CD - CF - BE}{(F-A)(D-F)} \cdot \cosh(t\sqrt{F}).
\end{aligned}$$

1.i) for $A > 0$

$$\begin{aligned}
c(t) &= \frac{BE - CD + AC}{(F-A)(D-A)} \cosh(t\sqrt{A}) + \\
&\quad + \frac{BE}{(D-A)(D-F)} \cdot \cos(t\sqrt{-D}) + \frac{CD - CF - BE}{(F-A)(D-F)} \cdot \cosh(t\sqrt{F}).
\end{aligned}$$

1.ii) for $A < 0$

$$\begin{aligned}
c(t) &= \frac{BE - CD + AC}{(F-A)(D-A)} \cos(t\sqrt{-A}) + \frac{BE}{(D-A)(D-F)} \cdot \cos(t\sqrt{-D}) + \\
&\quad + \frac{CD - CF - BE}{(F-A)(D-F)} \cdot \cosh(t\sqrt{F}).
\end{aligned}$$

1.iii) for $A = 0$, the equation (2.29) becomes

$$c''(t) = \frac{BE}{D-F} \cos(t\sqrt{-D}) + \frac{CD - CF - BE}{D-F} \cosh(t\sqrt{F}).$$

Then

$$\begin{aligned}
c'(t) &= \frac{BE}{(D-F)\sqrt{-D}} \sin(t\sqrt{-D}) + \frac{CD - CF - BE}{(D-F)\sqrt{F}} \sinh(t\sqrt{F}) + C_1. \\
c(t) &= \frac{BE}{(D-F)D} \cos(t\sqrt{-D}) + \frac{CD - CF - BE}{(D-F)F} \cosh(t\sqrt{F}) + C_1 t + C_2. \\
c(0) &= \frac{BE}{(D-F)D} + \frac{CD - CF - BE}{(D-F)F} + C_2 = 0 \Rightarrow C_2 = \frac{BE - CD}{DF}.
\end{aligned}$$

$$c'(0) = C_1 = 0.$$

Hence,

$$c(t) = \frac{BE - CD}{DF} + \frac{BE}{D(D - F)} \cos(t\sqrt{-D}) + \frac{CD - CF - BE}{F(D - F)} \cosh(t\sqrt{F}).$$

2. If $A = D$, then

$$c''(t) - A \cdot c(t) = \frac{BE}{A - F} \cos(t\sqrt{-A}) + \frac{CA - CF - BE}{A - F} \cosh(t\sqrt{F}). \quad (2.32)$$

$$c_p(t) = at \cos(t\sqrt{-A}) + bt \sin(t\sqrt{-A}) + \beta \cosh(t\sqrt{F})$$

and

$$c_p''(t) = -2a\sqrt{-A} \sin(t\sqrt{-A}) + 2b\sqrt{-A} \cos(t\sqrt{-A}) + aAt \cos(t\sqrt{-A}) + bAt \sin(t\sqrt{-A}) + \beta F \cosh(t\sqrt{F}).$$

When we put these in (2.32) we get

$$\begin{aligned} & -2a\sqrt{-A} \sin(t\sqrt{-A}) + 2b\sqrt{-A} \cos(t\sqrt{-A}) + aAt \cos(t\sqrt{-A}) + \\ & \quad + bAt \sin(t\sqrt{-A}) + \beta F \cosh(t\sqrt{F}) - aAt \cos(t\sqrt{-A}) - \\ & \quad - bAt \sin(t\sqrt{-A}) - \beta A \cosh(t\sqrt{F}) = \\ & = \frac{BE}{A - F} \cos(t\sqrt{-A}) + \frac{CA - CF - BE}{A - F} \cosh(t\sqrt{F}). \end{aligned}$$

$$\implies a = 0, b = \frac{BE}{2\sqrt{-A}(A - F)} \text{ and } \beta = \frac{BE + CF - AC}{(A - F)^2}. \text{ So,}$$

$$c_p(t) = \frac{BE}{2\sqrt{-A}(A - F)} t \sin(t\sqrt{-A}) + \frac{BE + CF - AC}{(A - F)^2} \cosh(t\sqrt{F})$$

and

$$\begin{aligned} c(t) &= C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} + \frac{BE}{2\sqrt{-A}(A - F)} t \sin(t\sqrt{-A}) + \frac{BE + CF - AC}{(A - F)^2} \cosh(t\sqrt{F}). \\ c(0) &= C_1 + C_2 + \frac{BE + CF - AC}{(A - F)^2} = 0. \end{aligned} \quad (2.33)$$

$$\begin{aligned} c'(t) &= C_1 \sqrt{A} e^{t\sqrt{A}} - C_2 \sqrt{A} e^{-t\sqrt{A}} + \frac{BE}{2\sqrt{-A}(A - F)} \cdot \sin(t\sqrt{-A}) + \\ & \quad + \frac{BE}{2(A - F)} \cdot t \cos(t\sqrt{-A}) + \frac{BE + CF - AC}{(A - F)^2} \sqrt{F} \sinh(t\sqrt{F}). \end{aligned}$$

$c'(0) = C_1 \sqrt{A} - C_2 \sqrt{A} = 0 \implies C_1 = C_2$. From (2.33) it follows

$$C_1 = C_2 = -\frac{BE + CF - AC}{2(A - F)^2}. \quad (2.34)$$

It also values $c''(0) = C$. Thus, we find

$$c(t) = -\frac{BE + CF - AC}{(A - F)^2} \cos(t\sqrt{-A}) +$$

$$\begin{aligned}
& + \frac{BE}{2\sqrt{-A}(A-F)} t \sin(t\sqrt{-A}) + \frac{BE+CF-AC}{(A-F)^2} \cosh(t\sqrt{F}) \\
\Rightarrow c(t) & = \frac{BE+CF-AC}{(A-F)^2} \left(\cosh(t\sqrt{F}) - \cos(t\sqrt{-A}) \right) + \\
& + \frac{BE}{2\sqrt{-A}(A-F)} t \sin(t\sqrt{-A}).
\end{aligned}$$

3. If $A = F$, then

$$c''(t) - A \cdot c(t) = \frac{BE}{D-A} \cos(t\sqrt{-D}) + \frac{CD-CA-BE}{D-A} \cosh(t\sqrt{A}). \quad (2.35)$$

$$c_p(t) = at \cosh(t\sqrt{A}) + bt \sinh(t\sqrt{A}) + \beta \cos(t\sqrt{-D})$$

and

$$\begin{aligned}
c_p''(t) & = 2a\sqrt{A} \sinh(t\sqrt{A}) + 2b\sqrt{A} \cosh(t\sqrt{A}) + aAt \cosh(t\sqrt{A}) + \\
& + bAt \sinh(t\sqrt{A}) + \beta D \cos(t\sqrt{-D}).
\end{aligned}$$

When we put these in (2.35) we get

$$\begin{aligned}
& 2a\sqrt{A} \sinh(t\sqrt{A}) + 2b\sqrt{A} \cosh(t\sqrt{A}) + aAt \cosh(t\sqrt{A}) + bAt \sinh(t\sqrt{A}) + \\
& + \beta D \cos(t\sqrt{-D}) - aAt \cosh(t\sqrt{A}) - bAt \sinh(t\sqrt{A}) - \beta A \cos(t\sqrt{-D}) = \\
& = \frac{BE}{D-A} \cos(t\sqrt{-D}) + \frac{CD-CA-BE}{D-A} \cosh(t\sqrt{A}) \\
\Rightarrow a = 0, \beta(D-A) & = \frac{BE}{D-A} \text{ and } 2b\sqrt{A} = \frac{CD-CA-BE}{D-A}. \text{ It follows } \beta = \frac{BE}{(D-A)^2}, \\
b & = \frac{CD-AC-BE}{2\sqrt{A}(D-A)} \text{ and}
\end{aligned}$$

$$c_p(t) = \frac{CD-AC-BE}{2\sqrt{A}(D-A)} \cdot t \sinh(t\sqrt{A}) + \frac{BE}{(D-A)^2} \cos(t\sqrt{-D}).$$

We get

$$\begin{aligned}
c(t) & = C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} + \frac{CD-AC-BE}{2\sqrt{A}(D-A)} \cdot t \sinh(t\sqrt{A}) + \frac{BE}{(D-A)^2} \cos(t\sqrt{-D}). \\
c(0) & = C_1 + C_2 + \frac{BE}{(D-A)^2} = 0. \quad (2.36)
\end{aligned}$$

$$\begin{aligned}
c'(t) & = C_1 \sqrt{A} e^{t\sqrt{A}} - C_2 \sqrt{A} e^{-t\sqrt{A}} + \frac{CD-AC-BE}{2\sqrt{A}(D-A)} \cdot \sinh(t\sqrt{A}) + \\
& + \frac{CD-AC-BE}{2(D-A)} \cdot t \cosh(t\sqrt{A}) - \frac{BE}{(D-A)^2} \sqrt{-D} \sin(t\sqrt{-D}).
\end{aligned}$$

$c'(0) = C_1\sqrt{A} - C_2\sqrt{A} = 0 \Rightarrow C_1 = C_2$. From (2.36) it follows $C_1 = C_2 = -\frac{BE}{2(D-A)^2}$. Further, $c''(0) = C_1A + C_2A + \frac{CD-AC-BE}{D-A} + \frac{BE}{(D-A)^2} \cdot D = C$. Finally,

$$\begin{aligned} c(t) &= -\frac{BE}{(D-A)^2} \left(\frac{e^{t\sqrt{A}} + e^{-t\sqrt{A}}}{2} \right) + \\ &\quad + \frac{CD-AC-BE}{2\sqrt{A}(D-A)} t \sinh(t\sqrt{A}) + \frac{BE}{(D-A)^2} \cos(t\sqrt{-D}) \Rightarrow \\ c(t) &= \frac{BE}{(D-A)^2} \left(\cos(t\sqrt{-D}) - \cosh(t\sqrt{A}) \right) + \frac{CD-AC-BE}{2\sqrt{A}(D-A)} \cdot t \sinh(t\sqrt{A}). \end{aligned}$$

e) Let $D = 0$ and $F > 0$. The equation (2.10) becomes

$$\begin{aligned} c''(t) - A \cdot c(t) &= \frac{-BE}{F} \left(1 - \cosh(t\sqrt{F}) \right) + C \cosh(t\sqrt{F}) = \\ &= \frac{-BE}{F} + \frac{BE+CF}{F} \cosh(t\sqrt{F}). \end{aligned}$$

We search for the particular solution $c_p(t) = \alpha + \beta \cosh(t\sqrt{F})$ and then $c_p''(t) = \beta F \cosh(t\sqrt{F})$. We get

$$\begin{aligned} c_p''(t) - A \cdot c_p(t) &= \beta F \cosh(t\sqrt{F}) - A\alpha - A\beta \cosh(t\sqrt{F}) = \\ &= \frac{-BE}{F} + \frac{BE+CF}{F} \cosh(t\sqrt{F}) \\ \Rightarrow -A\alpha &= \frac{-BE}{F} \quad \text{and} \quad \beta(F-A) = \frac{BE+CF}{F}. \end{aligned}$$

1. If $A \neq 0$ and $A \neq F$ then $\alpha = \frac{BE}{AF}$ and $\beta = \frac{BE+CF}{F(F-A)}$. Hence $c_p(t) = \frac{BE}{AF} + \frac{BE+CF}{F(F-A)} \cosh(t\sqrt{F})$ and

$$c(t) = C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} + \frac{BE}{AF} + \frac{BE+CF}{F(F-A)} \cosh(t\sqrt{F}). \quad (2.37)$$

$$c'(t) = C_1 \sqrt{A} e^{t\sqrt{A}} - C_2 \sqrt{A} e^{-t\sqrt{A}} + \frac{(BE+CF)\sqrt{F}}{F(F-A)} \sinh(t\sqrt{F}).$$

$$c''(t) = C_1 A e^{t\sqrt{A}} + C_2 A e^{-t\sqrt{A}} + \frac{BE+CF}{F-A} \cosh(t\sqrt{F}).$$

$$c(0) = C_1 + C_2 + \frac{BE}{AF} + \frac{BE+CF}{F(F-A)} = 0$$

$$c'(0) = C_1 \sqrt{A} - C_2 \sqrt{A} = 0 \Rightarrow C_1 = C_2.$$

$$C_1 = C_2 = -\frac{BE+AC}{2A(F-A)}.$$

From (2.37) it follows

$$c(t) = -\frac{BE+AC}{A(F-A)} \left(\frac{e^{t\sqrt{A}} + e^{-t\sqrt{A}}}{2} \right) + \frac{BE}{AF} + \frac{BE+CF}{F(F-A)} \cosh(t\sqrt{F}).$$

1.i) If $A > 0$: $c(t) = -\frac{BE+AC}{A(F-A)} \cosh(t\sqrt{A}) + \frac{BE}{AF} + \frac{BE+CF}{F(F-A)} \cosh(t\sqrt{F})$.

1.ii) If $A < 0$: $c(t) = -\frac{BE+AC}{A(F-A)} \cos(t\sqrt{-A}) + \frac{BE}{AF} + \frac{BE+CF}{F(F-A)} \cosh(t\sqrt{F})$.

2. If $A = 0$:

$$c''(t) = \frac{-BE}{F} + \frac{BE+CF}{F} \cosh(t\sqrt{F}) \implies$$

$$c'(t) = \frac{-BE}{F}t + \frac{BE+CF}{F\sqrt{F}} \sinh(t\sqrt{F}) + C_1 \implies$$

$$c(t) = \frac{-BE}{2F}t^2 + \frac{BE+CF}{F^2} \cosh(t\sqrt{F}) + C_1t + C_2.$$

$$c(0) = \frac{BE+CF}{F^2} + C_2 = 0 \implies C_2 = -\frac{BE+CF}{F^2}; c'(0) = C_1 = 0 \implies C_1 = 0.$$

$$c''(0) = \frac{-BE}{F} + \frac{BE+CF}{F} = C.$$

Hence,

$$\begin{aligned} c(t) &= \frac{-BE}{2F}t^2 - \frac{BE+CF}{F^2} + \frac{BE+CF}{F^2} \cosh(t\sqrt{F}) = \\ &= \frac{BE+CF}{F^2} (\cosh(t\sqrt{F}) - 1) - \frac{BE}{2F}t^2. \end{aligned}$$

3. If $A = F$, then $c''(t) - Ac(t) = -\frac{BE}{A} + \frac{BE+AC}{A} \cosh(t\sqrt{A})$. We search for a particular solution $c_p(t) = \alpha + at \cosh(t\sqrt{A}) + bt \sinh(t\sqrt{A})$ and then

$$c_p''(t) = 2a\sqrt{A} \sinh(t\sqrt{A}) + aAt \cosh(t\sqrt{A}) + 2b\sqrt{A} \cosh(t\sqrt{A}) + bAt \sinh(t\sqrt{A}).$$

It follows

$$\begin{aligned} c_p''(t) - Ac_p(t) &= 2a\sqrt{A} \sinh(t\sqrt{A}) + aAt \cosh(t\sqrt{A}) + 2b\sqrt{A} \cosh(t\sqrt{A}) + \\ &\quad + bAt \sinh(t\sqrt{A}) - A\alpha - aAt \cosh(t\sqrt{A}) - bAt \sinh(t\sqrt{A}) \\ &= -\frac{BE}{A} + \frac{BE+AC}{A} \cosh(t\sqrt{A}). \end{aligned}$$

We get $a = 0$, $\alpha = \frac{BE}{A^2}$ and $b = \frac{BE+AC}{2\sqrt{AA}}$. Hence, $c_p(t) = \frac{BE}{A^2} + \frac{BE+AC}{2\sqrt{AA}}t \sinh(t\sqrt{A})$ and

$$c(t) = C_1e^{t\sqrt{A}} + C_2e^{-t\sqrt{A}} + \frac{BE}{A^2} + \frac{BE+AC}{2\sqrt{AA}}t \sinh(t\sqrt{A}).$$

$$\begin{aligned} c(0) = C_1 + C_2 + \frac{BE}{A^2} = 0, c'(t) &= C_1\sqrt{A}e^{t\sqrt{A}} - C_2\sqrt{A}e^{-t\sqrt{A}} + \\ &\quad + \frac{BE+AC}{2\sqrt{AA}} \sinh(t\sqrt{A}) + \frac{BE+AC}{2A}t \cosh(t\sqrt{A}) \end{aligned}$$

$$c'(0) = C_1\sqrt{A} - C_2\sqrt{A} = 0 \implies C_1 = C_2.$$

$$c''(0) = 2C_1A + \frac{BE+AC}{A} = C.$$

Thus

$$c(t) = -\frac{BE}{A^2} \left(\frac{e^{t\sqrt{A}} + e^{-t\sqrt{A}}}{2} \right) + \frac{BE}{A^2} + \frac{BE+AC}{2\sqrt{AA}} t \sinh(t\sqrt{A}) \implies$$

$$c(t) = \frac{BE}{A^2} \left(1 - \cosh(t\sqrt{A}) \right) + \frac{BE+AC}{2\sqrt{AA}} t \sinh(t\sqrt{A}).$$

f) Let $D = 0$ and $F < 0$. Then

$$c''(t) - Ac(t) = \frac{-BE}{F} \left(1 - \cos(t\sqrt{-F}) \right) + C \cos(t\sqrt{-F}) =$$

$$= \frac{-BE}{F} + \frac{BE+CF}{F} \cos(t\sqrt{-F}).$$

We search for the particular solution $c_p(t) = \alpha + \beta \cos(t\sqrt{-F})$ and hence $c''_p(t) = \beta F \cos(t\sqrt{-F})$. It follows

$$\beta F \cos(t\sqrt{-F}) - A\alpha - A\beta \cos(t\sqrt{-F}) = \frac{-BE}{F} + \frac{BE+CF}{F} \cos(t\sqrt{-F}) \implies$$

$$A\alpha = \frac{BE}{F} \text{ and } \beta(F-A) = \frac{BE+CF}{F}.$$

1. If $A \neq 0$ then $\alpha = \frac{BE}{AF}$, $\beta = \frac{BE+CF}{F(F-A)}$ and $c(t) = C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} + \frac{BE}{AF} + \frac{BE+CF}{F(F-A)} \cos(t\sqrt{-F})$. Further,

$$c'(t) = C_1 \sqrt{A} e^{t\sqrt{A}} - C_2 \sqrt{A} e^{-t\sqrt{A}} - \frac{(BE+CF)\sqrt{-F}}{F(F-A)} \sin(t\sqrt{-F}).$$

$$c''(t) = C_1 A e^{t\sqrt{A}} + C_2 A e^{-t\sqrt{A}} + \frac{BE+CF}{F-A} \cos(t\sqrt{-F}).$$

$$c(0) = C_1 + C_2 + \frac{BE}{AF} + \frac{BE+CF}{F(F-A)} = 0.$$

$$c'(0) = C_1 \sqrt{A} - C_2 \sqrt{A} = 0 \implies C_1 = C_2.$$

$$C_1 = C_2 = -\frac{BE+AC}{2A(F-A)}.$$

Therefore, $c(t) = -\frac{BE+AC}{A(F-A)} \left(\frac{e^{t\sqrt{A}} + e^{-t\sqrt{A}}}{2} \right) + \frac{BE}{AF} + \frac{BE+CF}{F(F-A)} \cos(t\sqrt{-F})$.

1.i) If $A > 0$: $c(t) = -\frac{BE+AC}{A(F-A)} \cosh(t\sqrt{A}) + \frac{BE}{AF} + \frac{BE+CF}{F(F-A)} \cos(t\sqrt{-F})$.

1.ii) If $A < 0$: $c(t) = -\frac{BE+AC}{A(F-A)} \cos(t\sqrt{-A}) + \frac{BE}{AF} + \frac{BE+CF}{F(F-A)} \cos(t\sqrt{-F})$.

2. If $A = 0$:

$$c''(t) = \frac{-BE}{F} + \frac{BE+CF}{F} \cos(t\sqrt{-F}) \implies$$

$$c'(t) = \frac{-BE}{F} t - \frac{BE+CF}{F\sqrt{-F}} \sin(t\sqrt{-F}) + C_1 \implies$$

$$c(t) = \frac{-BE}{2F} t^2 + \frac{BE+CF}{F^2} \cos(t\sqrt{-F}) + C_1 t + C_2.$$

$$c(0) = \frac{BE + CF}{F^2} + C_2 = 0 \implies C_2 = -\frac{BE + CF}{F^2}; c'(0) = C_1 = 0 \implies C_1 = 0.$$

$$c''(0) = \frac{-BE}{F} + \frac{BE + CF}{F} = C.$$

Hence,

$$c(t) = \frac{-BE}{2F}t^2 - \frac{BE + CF}{F^2} + \frac{BE + CF}{F^2} \cos(t\sqrt{-F}) =$$

$$= \frac{BE + CF}{F^2}(\cos(t\sqrt{-F}) - 1) - \frac{BE}{2F}t^2.$$

3. If $A = F$, then $c''(t) - Ac(t) = \frac{-BE}{A} + \frac{BE+AC}{A} \cos(t\sqrt{-A})$. We put $c_p(t) = \alpha + at \cos(t\sqrt{-A}) + bt \sin(t\sqrt{-A})$ and

$$c_p''(t) = -2a\sqrt{-A} \sin(t\sqrt{-A}) + 2b\sqrt{-A} \cos(t\sqrt{-A}) +$$

$$+ aAt \cos(t\sqrt{-A}) + bAt \sin(t\sqrt{-A}).$$

It follows

$$c_p''(t) - Ac_p(t) = -2a\sqrt{-A} \sin(t\sqrt{-A}) + 2b\sqrt{-A} \cos(t\sqrt{-A}) + aAt \cos(t\sqrt{-A}) +$$

$$+ bAt \sin(t\sqrt{-A}) - A\alpha - Aat \cos(t\sqrt{-A}) - Abt \sin(t\sqrt{-A})$$

$$= \frac{-BE}{A} + \frac{BE + AC}{A} \cos(t\sqrt{-A})$$

$$\implies a = 0, \alpha = \frac{BE}{A^2}, b = \frac{BE + AC}{2\sqrt{-AA}}.$$

So, $c_p(t) = \frac{BE}{A^2} + \frac{BE+AC}{2\sqrt{-AA}} \cdot t \sin(t\sqrt{-A})$ and

$$c(t) = C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} + \frac{BE}{A^2} + \frac{BE + AC}{2\sqrt{-AA}} \cdot t \sin(t\sqrt{-A}).$$

$$c(0) = C_1 + C_2 + \frac{BE}{A^2} = 0.$$

$$c'(t) = C_1 \sqrt{A} e^{t\sqrt{A}} - C_2 \sqrt{A} e^{-t\sqrt{A}} +$$

$$+ \frac{BE + AC}{2\sqrt{-AA}} \sin(t\sqrt{-A}) + \frac{BE + AC}{2A} \cdot t \cos(t\sqrt{-A})$$

$$c'(0) = C_1 \sqrt{A} - C_2 \sqrt{A} = 0 \implies C_1 = C_2.$$

$$2C_1 + \frac{BE}{A^2} = 0 \implies C_1 = -\frac{BE}{2A^2}.$$

The third condition is now satisfied, i.e. $c''(0) = 2C_1 A + \frac{BE+AC}{A} = C$. It follows

$$c(t) = -\frac{BE}{A^2} \left(\frac{e^{t\sqrt{A}} + e^{-t\sqrt{A}}}{2} \right) + \frac{BE}{A^2} + \frac{BE + AC}{2\sqrt{-AA}} \cdot t \sin(t\sqrt{-A}).$$

Since $A = F < 0$, we have

$$c(t) = \frac{BE}{A^2} \left(1 - \cos(t\sqrt{-A})\right) + \frac{BE + AC}{2\sqrt{-AA}} \cdot t \sin(t\sqrt{-A}).$$

g) Let $D > 0$ and $F = 0$. We have an equation

$$c''(t) - Ac(t) = \frac{BE}{D} \cosh(t\sqrt{D}) + \frac{CD - BE}{D}.$$

The particular solution $c_p(t) = \alpha + \beta \cosh(t\sqrt{D})$ and then $c_p''(t) = \beta D \cosh(t\sqrt{D})$.

$$\begin{aligned} \beta D \cosh(t\sqrt{D}) - A\alpha - A\beta \cosh(t\sqrt{D}) &= \frac{CD - BE}{D} + \frac{BE}{D} \cosh(t\sqrt{D}) \implies \\ -A\alpha &= \frac{CD - BE}{D} \text{ and } \beta(D - A) = \frac{BE}{D}. \end{aligned}$$

1. If $A \neq 0$ and $D \neq A$, then $\alpha = \frac{BE - CD}{AD}$ and $\beta = \frac{BE}{D(D - A)}$. So

$$c(t) = C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} + \frac{BE - CD}{AD} + \frac{BE}{D(D - A)} \cosh(t\sqrt{D}).$$

$$c'(t) = C_1 \sqrt{A} e^{t\sqrt{A}} - C_2 \sqrt{A} e^{-t\sqrt{A}} + \frac{BE\sqrt{D}}{D(D - A)} \sinh(t\sqrt{D}).$$

$$c''(t) = C_1 A e^{t\sqrt{A}} + C_2 A e^{-t\sqrt{A}} + \frac{BE}{D - A} \cosh(t\sqrt{D}).$$

$$c(0) = C_1 + C_2 + \frac{BE - CD}{AD} + \frac{BE}{D(D - A)} = 0$$

$$c'(0) = C_1 \sqrt{A} - C_2 \sqrt{A} = 0 \implies C_1 = C_2.$$

So $C_1 = C_2 = \frac{CD - BE - AC}{2A(D - A)}$. It follows

$$c(t) = \frac{CD - BE - AC}{A(D - A)} \left(\frac{e^{t\sqrt{A}} + e^{-t\sqrt{A}}}{2} \right) + \frac{BE - CD}{AD} + \frac{BE}{D(D - A)} \cosh(t\sqrt{D}).$$

1.i) If $A > 0$: $c(t) = \frac{CD - BE - AC}{A(D - A)} \cosh(t\sqrt{A}) + \frac{BE - CD}{AD} + \frac{BE}{D(D - A)} \cosh(t\sqrt{D})$.

1.ii) If $A < 0$: $c(t) = \frac{CD - BE - AC}{A(D - A)} \cos(t\sqrt{-A}) + \frac{BE - CD}{AD} + \frac{BE}{D(D - A)} \cosh(t\sqrt{D})$.

2. If $A = 0$:

$$c''(t) = \frac{CD - BE}{D} + \frac{BE}{D} \cosh(t\sqrt{D}) \implies$$

$$c'(t) = \frac{CD - BE}{D} t + \frac{BE}{D\sqrt{D}} \sinh(t\sqrt{D}) + C_1 \implies$$

$$c(t) = \frac{CD - BE}{2D} t^2 + \frac{BE}{D^2} \cosh(t\sqrt{D}) + C_1 t + C_2.$$

$$c(0) = \frac{BE}{D^2} + C_2 = 0 \implies C_2 = -\frac{BE}{D^2}; \quad c'(0) = C_1 = 0 \implies C_1 = 0.$$

$$c''(0) = \frac{BE}{D} + \frac{CD - BE}{D} = C.$$

Hence, $c(t) = \frac{BE}{D^2}(\cosh(t\sqrt{D}) - 1) + \frac{CD-BE}{2D}t^2$.

3. If $A = D$, then $c''(t) - Ac(t) = \frac{AC-BE}{A} + \frac{BE}{A} \cosh(t\sqrt{A})$. We search the particular solution $c_p(t) = \alpha + at \cosh(t\sqrt{A}) + bt \sinh(t\sqrt{A})$.

$$c_p''(t) = 2a\sqrt{A} \sinh(t\sqrt{A}) + 2b\sqrt{A} \cosh(t\sqrt{A}) + aAt \cosh(t\sqrt{A}) + bAt \sinh(t\sqrt{A}).$$

Then we have

$$\begin{aligned} & 2a\sqrt{A} \sinh(t\sqrt{A}) + 2b\sqrt{A} \cosh(t\sqrt{A}) + aAt \cosh(t\sqrt{A}) + bAt \sinh(t\sqrt{A}) - \\ & \quad - A\alpha - aAt \cosh(t\sqrt{A}) - bAt \sinh(t\sqrt{A}) = \\ & = \frac{AC - BE}{A} + \frac{BE}{A} \cosh(t\sqrt{A}). \end{aligned}$$

$\implies a = 0$, $\alpha = \frac{BE-AC}{A^2}$, $b = \frac{BE}{2\sqrt{AA}}$. Hence, $c_p(t) = \frac{BE-AC}{A^2} + \frac{BE}{2\sqrt{AA}} \cdot t \sinh(t\sqrt{A})$ and $c(t) = C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} + \frac{BE-AC}{A^2} + \frac{BE}{2\sqrt{AA}} \cdot t \sinh(t\sqrt{A})$. Further we find

$$c(0) = C_1 + C_2 + \frac{BE - AC}{A^2} = 0$$

$$c'(t) = C_1 \sqrt{A} e^{t\sqrt{A}} - C_2 \sqrt{A} e^{-t\sqrt{A}} + \frac{BE}{2\sqrt{AA}} \sinh(t\sqrt{A}) + \frac{BE}{2A} \cdot t \cosh(t\sqrt{A})$$

$$c'(0) = C_1 \sqrt{A} - C_2 \sqrt{A} = 0 \implies C_1 = C_2 = -\frac{BE - AC}{2A^2}.$$

$$c''(t) = C_1 A e^{t\sqrt{A}} + C_2 A e^{-t\sqrt{A}} + \frac{BE}{A} \cosh(t\sqrt{A}) + \frac{BE}{2A} \cdot t \sqrt{A} \sinh(t\sqrt{A}).$$

$$c''(0) = C_1 A + C_2 A + \frac{BE}{A} = -\frac{BE - AC}{A^2} \cdot A + \frac{BE}{A} = C.$$

Thus, $c(t) = -\frac{BE-AC}{A^2} \left(\frac{e^{t\sqrt{A}} + e^{-t\sqrt{A}}}{2} \right) + \frac{BE-AC}{A^2} + \frac{BE}{2\sqrt{AA}} \cdot t \sinh(t\sqrt{A})$. Since $A = D > 0$, we have

$$c(t) = \frac{BE - AC}{A^2} \left(1 - \cosh(t\sqrt{A}) \right) + \frac{BE}{2\sqrt{AA}} \cdot t \sinh(t\sqrt{A}).$$

h) Let $D < 0$ and $F = 0$. Now we consider the equation $c''(t) - Ac(t) = \frac{BE}{D} \cos(t\sqrt{-D}) + \frac{CD-BE}{D}$. The particular solution $c_p(t) = \alpha + \beta \cos(t\sqrt{-D})$ and $c_p''(t) = \beta D \cos(t\sqrt{-D})$. We get

$$\begin{aligned} \beta D \cos(t\sqrt{-D}) - A\alpha - A\beta \cos(t\sqrt{-D}) &= \frac{CD - BE}{D} + \frac{BE}{D} \cos(t\sqrt{-D}) \implies \\ -A\alpha &= \frac{CD - BE}{D} \text{ and } \beta(D - A) = \frac{BE}{D}. \end{aligned}$$

1. If $A \neq 0$ and $D \neq A$, then $\alpha = \frac{BE-CD}{AD}$ and $\beta = \frac{BE}{D(D-A)}$. It follows

$$c(t) = C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} + \frac{BE - CD}{AD} + \frac{BE}{D(D - A)} \cos(t\sqrt{-D}).$$

$$c'(t) = C_1 \sqrt{A} e^{t\sqrt{A}} - C_2 \sqrt{A} e^{-t\sqrt{A}} - \frac{BE\sqrt{-D}}{D(D - A)} \sin(t\sqrt{-D}).$$

$$c''(t) = C_1 A e^{t\sqrt{A}} + C_2 A e^{-t\sqrt{A}} + \frac{BE}{D-A} \cos(t\sqrt{-D}).$$

$$c(0) = C_1 + C_2 + \frac{BE-CD}{AD} + \frac{BE}{D(D-A)} = 0$$

$$c'(0) = C_1\sqrt{A} - C_2\sqrt{A} = 0 \implies C_1 = C_2.$$

$$c''(0) = C_1 A + C_2 A + \frac{BE}{D-A}$$

So $C_1 = C_2 = \frac{CD-BE-AC}{2A(D-A)}$. Therefore

$$c(t) = \frac{CD-BE-AC}{A(D-A)} \left(\frac{e^{t\sqrt{A}} + e^{-t\sqrt{A}}}{2} \right) + \frac{BE-CD}{AD} + \frac{BE}{D(D-A)} \cos(t\sqrt{-D}).$$

1.i) If $A > 0$: $c(t) = \frac{CD-BE-AC}{A(D-A)} \cosh(t\sqrt{A}) + \frac{BE-CD}{AD} + \frac{BE}{D(D-A)} \cos(t\sqrt{-D})$.

1.ii) If $A < 0$: $c(t) = \frac{CD-BE-AC}{A(D-A)} \cos(t\sqrt{-A}) + \frac{BE-CD}{AD} + \frac{BE}{D(D-A)} \cos(t\sqrt{-D})$.

2. If $A = 0$

$$c''(t) = \frac{CD-BE}{D} + \frac{BE}{D} \cos(t\sqrt{-D}) \implies$$

$$c'(t) = \frac{CD-BE}{D} t + \frac{BE}{D\sqrt{-D}} \sin(t\sqrt{-D}) + C_1 \implies$$

$$c(t) = \frac{CD-BE}{2D} t^2 + \frac{BE}{D^2} \cos(t\sqrt{-D}) + C_1 t + C_2.$$

$$c(0) = \frac{BE}{D^2} + C_2 = 0 \implies C_2 = -\frac{BE}{D^2}; c'(0) = C_1 = 0 \implies C_1 = 0.$$

$$c''(0) = \frac{BE}{D} + \frac{CD-BE}{D} = C.$$

Hence, $c(t) = \frac{BE}{D^2} (\cosh(t\sqrt{-D}) - 1) - \frac{CD-BE}{2D} t^2$.

3. If $A = D$, then $c''(t) - Ac(t) = \frac{AC-BE}{A} + \frac{BE}{A} \cos(t\sqrt{-A})$. We put $c_p(t) = \alpha + at \cos(t\sqrt{-A}) + bt \sin(t\sqrt{-A})$ and then find

$$c'_p(t) = (a + bt\sqrt{-A}) \cos(t\sqrt{-A}) + (b - at\sqrt{-A}) \sin(t\sqrt{-A})$$

$$c''_p(t) = -2a\sqrt{-A} \sin(t\sqrt{-A}) + 2b\sqrt{-A} \cos(t\sqrt{-A}) + aAt \cos(t\sqrt{-A}) + bAt \sin(t\sqrt{-A}).$$

Now we get

$$\begin{aligned} c''_p(t) - Ac_p(t) &= -2a\sqrt{-A} \sin(t\sqrt{-A}) + 2b\sqrt{-A} \cos(t\sqrt{-A}) + aAt \cos(t\sqrt{-A}) + \\ &\quad + bAt \sin(t\sqrt{-A}) - A\alpha - aAt \cos(t\sqrt{-A}) - bAt \sin(t\sqrt{-A}) \\ &= \frac{AC-BE}{A} + \frac{BE}{A} \cos(t\sqrt{-A}). \end{aligned}$$

So, $a = 0$, $\alpha = \frac{BE-AC}{A^2}$ and $b = \frac{BE}{2\sqrt{-AA}}$. Hence we get $c_p(t) = \frac{BE-AC}{A^2} + \frac{BE}{2\sqrt{-AA}} \cdot t \sin(t\sqrt{-A})$ and $c(t) = C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} + \frac{BE-AC}{A^2} + \frac{BE}{2\sqrt{-AA}} \cdot t \sin(t\sqrt{-A})$. Further,

$$c(0) = C_1 + C_2 - \frac{BE-AC}{A^2} = 0$$

$$c'(t) = C_1 \sqrt{A} e^{t\sqrt{A}} - C_2 \sqrt{A} e^{-t\sqrt{A}} + \frac{BE}{2\sqrt{-AA}} \sin(t\sqrt{-A}) + \frac{BE}{2A} \cdot t \cos(t\sqrt{-A})$$

$$c'(0) = C_1 \sqrt{A} - C_2 \sqrt{A} = 0.$$

It follows $C_1 = C_2 = -\frac{BE-AC}{2A^2}$. $c''(t) = C_1 A e^{t\sqrt{A}} + C_2 A e^{-t\sqrt{A}} + \frac{BE}{A} \cos(t\sqrt{-A}) - \frac{BE}{2A} \cdot t\sqrt{-A} \sin(t\sqrt{-A})$. $c''(0) = C_1 A + C_2 A + \frac{BE}{A} = C$. Hence,

$$c(t) = -\frac{BE-AC}{A^2} \left(\frac{e^{t\sqrt{A}} + e^{-t\sqrt{A}}}{2} \right) + \frac{BE-AC}{A^2} + \frac{BE}{2\sqrt{-AA}} \cdot t \sin(t\sqrt{-A}).$$

Since $A = D < 0$, we have

$$c(t) = \frac{BE-AC}{A^2} \left(1 - \cos(t\sqrt{-A}) \right) + \frac{BE}{2\sqrt{-AA}} \cdot t \sin(t\sqrt{-A}).$$

B) Case $F = D$.

a) Let $D > 0$. Then $e(t) = \frac{E}{2\sqrt{D}} \cdot t \sinh(t\sqrt{D})$ and $f(t) = \cosh(t\sqrt{D})$. The equation is $c''(t) - Ac(t) = B \cdot \frac{E}{2\sqrt{D}} t \sinh(t\sqrt{D}) + C \cosh(t\sqrt{D})$. We search the particular solution in a form $c_p(t) = \alpha t \sinh(t\sqrt{D}) + \beta \cosh(t\sqrt{D}) \Rightarrow c_p''(t) = (2\alpha\sqrt{D} + \beta D) \cosh(t\sqrt{D}) + \alpha D t \sinh(t\sqrt{D})$. We get

$$\begin{aligned} \alpha t(D-A) \sinh(t\sqrt{D}) + (2\alpha\sqrt{D} + \beta D - \beta A) \cosh(t\sqrt{D}) &= \\ &= B \cdot \frac{E}{2\sqrt{D}} t \sinh(t\sqrt{D}) + C \cosh(t\sqrt{D}). \end{aligned}$$

It follows $\alpha(D-A) = \frac{BE}{2\sqrt{D}}$ and $\beta(D-A) = C - 2\alpha\sqrt{D}$.

1) If $D \neq A$ then $\alpha = \frac{BE}{2\sqrt{D}(D-A)}$ and $\beta = \frac{C(D-A)-BE}{(D-A)^2}$. Hence,

$$c_p(t) = \frac{BE}{2\sqrt{D}(D-A)} t \sinh(t\sqrt{D}) + \frac{C(D-A)-BE}{(D-A)^2} \cosh(t\sqrt{D}), \text{ and}$$

$$\begin{aligned} c(t) = C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} + \frac{BE}{2\sqrt{D}(D-A)} t \sinh(t\sqrt{D}) + \\ + \frac{C(D-A)-BE}{(D-A)^2} \cosh(t\sqrt{D}) \end{aligned}$$

From the initial conditions we get $C_1 = C_2 = \frac{BE-C(D-A)}{2(D-A)^2}$.

i) for $A > 0$ we get

$$c(t) = \frac{BE}{2\sqrt{D}(D-A)} t \sinh(t\sqrt{D}) + \frac{C(D-A)-BE}{(D-A)^2} (\cosh(t\sqrt{D}) - \cosh(t\sqrt{A})).$$

ii) for $A < 0$ we get

$$c(t) = \frac{BE}{2\sqrt{D}(D-A)} t \sinh(t\sqrt{D}) + \frac{C(D-A) - BE}{(D-A)^2} (\cosh(t\sqrt{D}) - \cos(t\sqrt{-A})).$$

iii) for $A = 0$ we have the equation $c''(t) = \frac{BE}{2\sqrt{D}} t \sinh(t\sqrt{D}) + C \cosh(t\sqrt{D})$. After two times integration we get

$$c(t) = \frac{CD - BE}{D^2} \cosh(t\sqrt{D}) + \frac{BE}{2D\sqrt{D}} t \sinh(t\sqrt{D}) + C_1 t + C_2.$$

From the initial conditions we find $C_1 = 0$ and $C_2 = \frac{BE - CD}{D^2}$, therefore we have

$$c(t) = \frac{BE - CD}{D^2} (1 - \cosh(t\sqrt{D})) + \frac{BE}{2D\sqrt{D}} t \sinh(t\sqrt{D})$$

2) If $D = A$ then we find $BE = 0$ and $c(t) = \frac{C}{2\sqrt{A}} t \sinh(t\sqrt{A})$.

b) Let $D < 0$. Then $e(t) = \frac{E}{2\sqrt{-D}} \cdot t \sinh(t\sqrt{-D})$ and $f(t) = \cos(t\sqrt{-D})$. The equation is $c''(t) - Ac(t) = B \cdot \frac{E}{2\sqrt{-D}} t \sin(t\sqrt{-D}) + C \cos(t\sqrt{-D})$. The particular solution and its second derivative are

$$\begin{aligned} c_p(t) &= \alpha t \sin(t\sqrt{-D}) + \beta \cos(t\sqrt{-D}) \implies \\ c_p''(t) &= (2\alpha\sqrt{-D} + \beta D) \cos(t\sqrt{-D}) + \alpha D t \sin(t\sqrt{-D}). \end{aligned}$$

Hence we get $\alpha(D - A) = \frac{BE}{2\sqrt{-D}}$ and $\beta(D - A) = C - 2\alpha\sqrt{-D}$.

1) If $D \neq A$, then $\alpha = \frac{BE}{2\sqrt{-D}(D-A)}$ and $\beta = \frac{C(D-A) - BE}{(D-A)^2}$. Hence,

$$\begin{aligned} c_p(t) &= \frac{BE}{2\sqrt{-D}(D-A)} t \sin(t\sqrt{-D}) + \frac{C(D-A) - BE}{(D-A)^2} \cos(t\sqrt{-D}) \text{ and} \\ c(t) &= C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} + \frac{BE}{2\sqrt{-D}(D-A)} t \sin(t\sqrt{-D}) + \\ &\quad + \frac{C(D-A) - BE}{(D-A)^2} \cos(t\sqrt{-D}) \end{aligned}$$

From the initial conditions we get $C_1 = C_2 = \frac{BE - C(D-A)}{2(D-A)^2}$.

i) for $A > 0$ we get

$$c(t) = \frac{BE}{2\sqrt{-D}(D-A)} t \sin(t\sqrt{-D}) + \frac{C(D-A) - BE}{(D-A)^2} (\cos(t\sqrt{-D}) - \cosh(t\sqrt{A})).$$

ii) for $A < 0$ we get

$$c(t) = \frac{BE}{2\sqrt{-D}(D-A)} t \sin(t\sqrt{-D}) + \frac{C(D-A) - BE}{(D-A)^2} (\cos(t\sqrt{-D}) - \cos(t\sqrt{-A})).$$

iii) for $A = 0$ we have the equation $c''(t) = \frac{BE}{2\sqrt{-D}} t \sin(t\sqrt{-D}) + C \cos(t\sqrt{-D})$. After two times integration we get

$$c(t) = \frac{CD - BE}{D^2} \cos(t\sqrt{-D}) + \frac{BE}{2D\sqrt{-D}} t \sin(t\sqrt{-D}) + C_1 t + C_2.$$

From the initial conditions we find $C_1 = 0$ and $C_2 = \frac{BE-CD}{D^2}$, therefore we have

$$c(t) = \frac{BE - CD}{D^2}(1 - \cos(t\sqrt{-D})) + \frac{BE}{2D\sqrt{-D}}t \sin(t\sqrt{-D}).$$

2) If $D = A$ then we find $BE = 0$ and $c(t) = \frac{C}{2\sqrt{-A}}t \sin(t\sqrt{-A})$.

c) If $D = 0$ then we have the equation $c''(t) - Ac(t) = \frac{BE}{2}t^2 + C$. We search the particular solution and its second derivative: $c_p(t) = \alpha t^2 + \beta t + \gamma$ and $c_p''(t) = 2\alpha$. Now we get

$$2\alpha - A\alpha t^2 - A\beta t - A\gamma = \frac{BE}{2}t^2 + C \implies -A\alpha = \frac{BE}{2}, \beta = 0, 2\alpha - A\gamma = C.$$

1) If $A \neq 0$ then $\alpha = -\frac{BE}{2A}$ and $\gamma = \frac{-AC-BE}{A^2}$. Hence, $c_p(t) = -\frac{BE}{2A}t^2 - \frac{AC+BE}{A^2}$ and $c(t) = C_1 e^{t\sqrt{A}} + C_2 e^{-t\sqrt{A}} - \frac{BE}{2A}t^2 - \frac{AC+BE}{A^2}$. From the initial conditions we get $C_1 = C_2 = \frac{AC+BE}{2A^2}$.

i) For $A > 0$ the solution is $c(t) = \frac{AC+BE}{A^2}(\cosh(t\sqrt{A}) - 1) - \frac{BE}{2A}t^2$

ii) For $A < 0$ the solution is $c(t) = \frac{AC+BE}{A^2}(\cos(t\sqrt{-A}) - 1) - \frac{BE}{2A}t^2$.

2) For $A = 0$ we have the equation $c''(t) = \frac{BE}{2}t^2 + C$. Then

$$c'(t) = \frac{BE}{6}t^3 + Ct + C_1 \text{ and } c(t) = \frac{BE}{24}t^4 + \frac{C}{2}t^2 + C_1t + C_2.$$

When we apply the initial conditions, we find $c(0) = C_2 = 0$, $c'(0) = C_1 = 0$, $c''(0) = C$. Hence, the solution is

$$c(t) = \frac{BE}{24}t^4 + \frac{C}{2}t^2.$$

3. CONCLUSION

In this paper, we have found the cosine operator functions $C(t)$ as the solutions of the appropriate second order Cauchy problem, for the various cases of generator $\mathcal{A}_{3 \times 3}$. This study is a natural continuation of our previous paper [9] where \mathcal{A} was 2-by-2 matrix. Analogous research could be done for a lower triangle matrix $\mathcal{A}_{3 \times 3}$. These results may be useful in solving various second order differential equations with some boundary conditions in \mathbb{R}^3 . Cosine and sine operator functions defined on a Banach space are useful tools in the study of wide classes of evolution equations ([4]). There are also applications to some linear second order control systems ([3]).

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