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ON A TYPE OF IDEAL GENERALIZED SEMIOPEN SETS

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Abstract. In this paper, we obtain two weakly ideal generalized semiopen sets via μ-semiopen sets in ideal τ_μ-topological spaces and study some of the ideal τ_μ-topological characterizations through these sets.

1. Introduction and Preliminaries

A. Csaszar(see[4-11]) introduced the concept of generalized topology and he obtained the concepts of μ-open sets, μ-ξ-open sets (where "ξ" stands for α, semi, pre and β) and defined the operators μ-interior (resp. μ-ξ-interior) and μ-closure (resp. resp. μ-ξ-closure) in generalized topological spaces. Saravanakumar et al.(see[28-33]) initiated a $\tilde{\mu}$-open set in generalized topology and studied $\tilde{\mu}$-T_i, where ($i = 0, \frac{1}{2}, 1, 2$) spaces using through the $\tilde{\mu}$-open and $\tilde{\mu}$-closed sets and created various types of generalized open sets, generalized closed sets, and generalized continuous functions in both generalized topology and operation topology. Kuratowski[17] and Vaidyanathaswamy [36] defined new concept of topologies namely ideal topological spaces and investigated some of their essential properties by some local operators. Jankovic et al.[16], generated the concept of ideal topological spaces and introduced $\mathcal{I}$-open sets by $()^*$ operator. Authors[1, 2, 3, 13, 15, 16, 17], created new topologies by some local operators and generated innovative open sets. In this paper, we created a slightly larger collection than the μ-semiopen collection using the μ^*-semiopen set and described its characterizations in the τ_μ-topological space. Also, we defined the $\mathcal{I}_\mu$-semiopen set in the ideal τ_μ-topological space and proved the finite intersection property through the μ-semi-hyperconnected space. In addition, we generated a new generalized topology in an ideal τ_μ-topological space by weakly $\mathcal{I}_\mu$-semiopen sets and the collection of $\mathcal{I}_\mu$-semiopen sets is slightly larger than the $\mathcal{I}_\mu$-semiopen collection and creates a topology structure in a μ-semi-hyperconnected space.

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Let $X \neq \emptyset$ and let $\mathcal{P}(X)$ be the power set of X. Then $\mu \subseteq \mathcal{P}(X)$ is said to be a generalized topology[4] (shortly, GT) on X if and only if $\emptyset \in \mu$, and U_i $(\neq \emptyset) \in \mu$ for $i \in J$ implies that $\bigcup_{i\in J} U_i \in \mu$. The pair (X, μ) is called a generalized topological space[4] (shortly, GTS) and the elements of μ are called μ-open sets[4]. For a subset $A \subseteq X$, $X \setminus A$ is called a μ-closed set if A is μ-open; μ-interior[5] of A is defined by $i_\mu(A) = \cup\{U : U \in \mu$ and $U \subseteq A\}$ (clearly, it is the largest μ-open set contained in A; μ-closure[5] of A is defined by $c_\mu(A) = \cap\{V : X \setminus V \in \mu$ and $A \subseteq V\}$ (clearly, the smallest μ-closed set containing A). A subset A of X is said to be a μ-semi-open[6] (resp. μ-α-open[6], μ-preopen[6], μ-regular open[34], μ-β-open[6]) set if $A \subseteq c_\mu(i_\mu(A))$ (resp. $A \subseteq i_\mu(c_\mu(i_\mu(A)))$, $A \subseteq i_\mu(c_\mu(A))$, $A = i_\mu(c_\mu(A))$, $A \subseteq c_\mu(i_\mu(c_\mu(A)))$). The complement of a μ-semi-open (resp. μ-α-open, μ-preopen, μ-regularopen, μ-β-open) set is said to be μ-semi-closed (resp. μ-α-closed, μ-preclosed, μ-regularclosed, μ-β-closed). The family of μ-semi-open (resp. μ-α-open, μ-preopen, μ-regular open, μ-β-open) sets is denoted by $\mu SO(X)$ (resp. $\mu\alpha O(X)$, $\mu PO(X)$, $\mu RO(X)$, $\mu\beta O(X)$). For a subset $A \subseteq X$, μ-ξ-interior[6] of A is defined by $i_{\xi_\mu}(A) = \cup\{U : U \in \mu\xi O(X)$ and $U \subseteq A\}$ (clearly, it is the largest μ-ξ-open set contained in A; μ-ξ-closure[6] of A is defined by $c_{\xi_\mu}(A) = \cap\{V : X \setminus V \in \mu\xi O(X)$ and $A \subseteq V\}$ (clearly, the smallest μ-ξ-closed set containing A), (where "ξ" stands for α, semi, pre and β). A subset A of X is said to be a μ^*-open[27] if $A \subseteq cl(i_\mu(A))$. A subset A of X is called semiopen[18] if there exists an open set U of X such that $U \subseteq A \subseteq cl(U)$ and its complement $X \setminus A$ is called semiclosed[18]; semi-interior[18] of A is defined by $sint(A) = \cup\{U : U \in \tau$ and $U \subseteq A\}$; semi-closure[18] of A is defined by $scl(A) = \cap\{V : X \setminus V \in \tau$ and $A \subseteq V\}$. A subset A of X is called $\mathcal{I}_\mu$-open[27] if there exists a μ-open set U such that $U \setminus A \in \mathcal{I}$ and $A \setminus cl(U) \in \mathcal{I}$. A subset A of X is called weakly $\mathcal{I}_\mu$-open[27] if $A = \emptyset$ or if $A \neq \emptyset$, there exists a non-empty μ-open set U such that $U \setminus A \in \mathcal{I}$. A GT μ is said to be a quasi topology (briefly QT)[10] if U, $V \in \mu$ implies $U \cap V \in \mu$. The pair (X, μ) is said to be a QTS if μ is a QT on X. An ideal $\mathcal{I}$[17, 36] on X is a nonempty collection $\mathcal{I} \subseteq \mathcal{P}(X)$ satisfying the following conditions: (i) $A \subseteq B$, $B \in \mathcal{I}$ implies $A \in \mathcal{I}$; (ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Let $(X, \tau, \mathcal{I})$ be an ideal generalized topological space (briefly, IGTS).

Definition 1.1. *Let (X, μ) be a generalized topological space and G a subset of X. Then*

- *(i) (X, μ) is said to be μ-irreducible*[34] *if for any two non-empty μ-open sets U and V of X, $U \cap V \neq \emptyset$;*
- *(ii) G is called μ-dense*[12] *if $c_\mu(G) = X$;*
- *(iii) (X, μ) is said to be μ-hyperconnected*[12] *if G is μ-dense for every non-empty μ-open subset G of (X, μ).*

2. μ^*-SEMIOPEN SETS

Definition 2.1. *Let (X, τ, μ) be a τ_μ-topological space. A subset A of X is called μ^*-semiopen if $A \subseteq cl(i_{s_\mu}(A))$.*

Theorem 2.2. *Let (X, τ, μ) be a τ_μ-topological space. Then A is μ^*-semiopen if and only if there exists a μ-semiopen set U such that $U \subseteq A \subseteq cl(U)$.*

Proof. Let A be a μ^*-semiopen set. Then $A \subseteq cl(i_{s_\mu}(A))$. Let $U = i_{s_\mu}(A)$. Then U is μ-semiopen and $U \subseteq A \subseteq cl(i_{s_\mu}(A)) = cl(U)$. Conversely, let there exists a μ-semiopen U such that $U \subseteq A \subseteq cl(U)$. Then $U \subseteq A \Rightarrow U \subseteq i_{s_\mu}(A) \Rightarrow cl(U) \subseteq cl(i_{s_\mu}(A)) \Rightarrow A \subseteq cl(i_{s_\mu}(A))$. Hence A is μ^*-semiopen. □

Remark 2.3: Let (X, τ, μ) be a τ_μ-topological space and A a μ^*-semiopen set in X. If

(i) $\mu = \tau$, then $A \subseteq cl(sint(A))$;
(ii) $\mu = PO(X)$, then $A \subseteq cl(sint(scl(A)))$.

Remark 2.4: Let (X, τ, μ) be a τ_μ-topological space. Then

(i) every μ-open set is μ^*-open;
(ii) every μ-semiopen set is μ^*-semiopen;
(iii) every μ^*-open set is μ^*-semiopen;
(vi) λ is any other GT on X with $\mu \subseteq \lambda$, then every μ^*-semiopen set is λ^*-semiopen.

Theorem 2.5. *Let (X, τ, μ) be a τ_μ-topological space. Then the collection of all μ^*-semiopen sets forms a GT on X.*

Proof. Clearly $\emptyset$ is a μ^*-semiopen set. Let $\{A_k : k \in J\}$ be a family of μ^*-semiopen sets. Then there exists μ^*-semiopen sets U_k such that $U_k \subseteq A_k \subseteq cl(U_k)$ for each $k \in J$. Thus $\bigcup\{U_k : k \in J\} = U$ (say) $\subseteq \bigcup\{A_k : k \in J\} \subseteq cl(U)$ where U is μ-semiopen. Hence union of all μ^*-semiopen sets is μ^*-semiopen. □

Example 2.6. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$ and $\mu = \{\emptyset, \{c\}, \{b, d\}, \{c, d\}, \{b, c, d\}\}$. Then the set $\{b, c\}$ is μ^*-semiopen, but $\{b, c\}$ is not μ-semiopen. Also the set $\{a, c, d\}$ is μ^*-semiopen, but $\{a, c, d\}$ is not μ^*-open. Further the sets $\{b, d\}$ and $\{c, d\}$ are μ^*-semiopen, but their intersection $\{d\}$ is not μ^*-semiopen.*

Theorem 2.7. *Let (X, τ, μ) be a τ_μ-topological space and A a μ^*-semiopen set such that $A \subseteq B \subseteq cl(A)$. Then B is also a μ^*-semiopen set.*

Proof. Suppose that A is μ^*-semiopen. Then there exists a μ-semiopen set U such that $U \subseteq A \subseteq cl(U)$. Thus $U \subseteq B$. Also $cl(A) \subseteq cl(U) \Rightarrow B \subseteq cl(U)$. Thus $U \subseteq B \subseteq cl(U)$. Hence B is μ^*-semiopen. □

3. $\mathcal{I}_\mu$-SEMIOPEN SETS

Definition 3.1. *Let $(X, \tau, \mu, \mathcal{I})$ be an ideal τ_μ-topological space. A subset A of X is called $\mathcal{I}_\mu$-semiopen if there exists a μ-semiopen set U such that $U \setminus A \in \mathcal{I}$ and $A \setminus cl(U) \in \mathcal{I}$.*

If $A \in \mathcal{I}$, then A is an $\mathcal{I}_\mu$-semiopen set and also by Theorem 2.2, every μ^*-semiopen set is $\mathcal{I}_\mu$-semiopen. Also every $\mathcal{I}_\mu$-open set is $\mathcal{I}_\mu$-semiopen.

In Example 2.6, let $\mathcal{I} = \{\emptyset, \{b\}\}$. Then the set $\{c\}$ is $\mathcal{I}_\mu$-semiopen, but $\{c\} \notin \mathcal{I}$. Also the set $\{b\}$ is $\mathcal{I}_\mu$-semiopen, but $\{b\}$ is not μ^*-semiopen. Further, the set $\{a, c, d\}$ is $\mathcal{I}_\mu$-semiopen, but $\{a, c, d\}$ is not $\mathcal{I}_\mu$-open.

Example 3.2. *(i) Let $\Re$ be the set of reals, $\mathcal{Q}$ be the set of rational numbers and $\mathcal{Q}'$ be the set of irrationals. Consider $\mathcal{I} = \{A \subseteq \Re : A \text{ is finite}\}$ and $\mu = \{\emptyset, \mathcal{Q}', \Re\}$. Then $(\Re, \tau_u, \mu, \mathcal{I})$ is an ideal τ_μ-topological space, where τ_u denotes the usual topology on $\Re$. We note that for all $X \in \mathcal{Q}$, $\{x\}$ is an $\mathcal{I}_\mu$-semiopen set as $\{x\} \in \mathcal{I}$, but $\mathcal{Q} = \bigcup\{\{x\} : x \in \mathcal{Q}\}$ is not $\mathcal{I}_\mu$-semiopen. (ii) Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$, $\mu = \{\emptyset, \{b\}, \{d\}, \{b, d\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the sets $\{b, c\}$ and $\{c, d\}$ are $\mathcal{I}_\mu$-semiopen, but their intersection $\{c\}$ is not $\mathcal{I}_\mu$-semiopen.*

Theorem 3.3. *Let $(X, \tau, \mu, \mathcal{I})$ be an ideal τ_μ-topological space. If $\mathcal{I}$ is a minimal ideal on X i.e., $\mathcal{I} = \{\emptyset\}$, then the concept of μ^*-semiopenness and $\mathcal{I}_\mu$-semiopenness are the same.*

Proof. Suppose that $\mathcal{I} = \{\emptyset\}$. It is sufficient to show that whenever A is an $\mathcal{I}_\mu$-semiopen set it is μ^*-semiopen. Indeed, if A is $\mathcal{I}_\mu$-semiopen, then there exists a μ-semiopen set U such that $U \setminus A \in \mathcal{I}$, $A \setminus cl(U) \in \mathcal{I} = \{\emptyset\}$ and so $U \subseteq A \subseteq cl(U)$. Hence by Theorem 2.2, A is μ^*-semiopen. □

The converse of the above theorem is false even if $\mathcal{I}$ is not countable additive as shown by the next example.

Example 3.4. *Let X be any infinite set, $\tau = \mu = P(X)$, the power set of X, $\mathcal{I}$ be the ideal consisting of all finite subsets of X (including empty set). Then $(X, \tau, \mu, \mathcal{I})$ is an ideal τ_μ-topological space. Clearly, $\mathcal{I}$ is not countably additive. Then the collection of all μ^*-semiopen subsets of X as well as the collection of all $\mathcal{I}_\mu$-semiopen subsets of X coincide with $P(X)$. However $\mathcal{I} \neq \{\emptyset\}$.*

Remark 3.5: Let $\mathcal{I}$ and $\mathcal{I}'$ be two ideals on τ_μ-topological space (X, τ, μ). If $\mathcal{I} \subseteq \mathcal{I}'$, then every $\mathcal{I}_\mu$-semiopen sets is $\mathcal{I}'_\mu$-semiopen (see Definition 3.1) and hence if A is $(\mathcal{I} \cap \mathcal{I}')_\mu$-semiopen, then it is $\mathcal{I}_\mu$-semiopen sets as well as $\mathcal{I}'_\mu$-semiopen.

Proposition 3.6. *Let $(X, \tau, \mu, \mathcal{I})$ be an ideal τ_μ-topological space. The union of finite number of $\mathcal{I}_\mu$-semiopen sets is an $\mathcal{I}_\mu$-semiopen set.*

Proof. Let A and B be two $\mathcal{I}_\mu$-semiopen sets. Then there exists two μ-semiopen sets G and H such that $G \setminus A \in \mathcal{I}$, $A \setminus cl(G) \in \mathcal{I}$, $H \setminus B \in \mathcal{I}$, $B \setminus cl(H) \in \mathcal{I}$. Let $U = G \cup H$ be μ-semiopen and observe that $U \setminus (A \cup B) \subseteq ((G \setminus A) \setminus B) \cup ((H \setminus B) \setminus A) \in \mathcal{I}$. Also $A \cup B \setminus cl(G \cup H) = ((A \setminus cl(G)) \setminus cl(H)) \cup ((B \setminus cl(H)) \setminus cl(G)) \in \mathcal{I}$. Thus $A \cup B$ is $\mathcal{I}_\mu$-semiopen. □

Proposition 3.7. *Let $(X, \tau, \mu, \mathcal{I})$ be an ideal τ_μ-topological space and suppose there exists a dense μ-semiopen subset $A \in \mathcal{I}$. Then every subset B of X is $\mathcal{I}_\mu$-semiopen.*

Proof. Let B be any subset of X. Note that (as $\mathcal{I}$ is an ideal, $A \in \mathcal{I}$, $A \setminus B \subseteq A$), we shall have $A \setminus B \in \mathcal{I}$. Furthermore, $B \setminus cl(A) = B \setminus X = \emptyset \in \mathcal{I}$. Put $U = A$ is μ-semiopen. Then $U \setminus B = A \setminus B \in \mathcal{I}$ and $B \setminus cl(U) = B \setminus cl(A) = B \setminus X = \emptyset \in \mathcal{I}$. Hence B is $\mathcal{I}_\mu$-semiopen. □

Example 3.8. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$, $\mu = \{\emptyset, \{b\}, \{a, b\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $cl(\{b\}) = X$ where $\{b\} \notin \mathcal{I}$. Also, we obtained that $\{c\}$ is not $\mathcal{I}_\mu$-semiopen.*

Proposition 3.9. *Let $(X, \tau, \mu, \mathcal{I})$ be an ideal τ_μ-topological space and A a μ-semiopen set such that $A \subseteq B \subseteq cl(A)$. Then B is an $\mathcal{I}_\mu$-semiopen set.*

Proof. Obvious. □

Definition 3.10. *Let (X, μ) be a generalized topological space and G a subset of X. Then*

(i) (X, μ) is said to be μ-semi-irreducible if for any two non-empty μ-semiopen sets U and V of X, $U \cap V \neq \emptyset$;

(ii) G is called μ-semi-dense if $c_{s_\mu}(G) = X$;

(iii) (X, μ) is said to be μ-semi-hyperconnected if G is μ-semi-dense for every non-empty μ-semiopen subset G of (X, μ).

Remark 3.11: A generalized topological space (X, μ) is μ-semi-hyperconnected if and only if (X, μ) is μ-semi-irreducible.

Proposition 3.12. *Let $(X, \tau, \mu, \mathcal{I})$ be an ideal τ_μ-topological space where every non-empty μ-semiopen subset is dense in (X, τ). Then for any subset A of X,*

(a) if A is $\mathcal{I}_\mu$-semiopen with $A \notin \mathcal{I}$, (i) $A \subseteq B$, then B is $\mathcal{I}_\mu$-semiopen; (ii) $A \cup B$ is $\mathcal{I}_\mu$-semiopen for any subset B of X;

(b) Moreover, if μ is closed under finite intersection and (X, μ) is μ-semi-hyperconnected, then intersection of two $\mathcal{I}_\mu$-semiopen sets is $\mathcal{I}_\mu$-semiopen.

Proof. (a) (i) Suppose that A is $\mathcal{I}_\mu$-semiopen and $A \subseteq B$. Then there is a μ-semiopen set G such that $G \setminus A \in \mathcal{I}$ and $A \setminus cl(G) \in \mathcal{I}$. We first observe that $G \neq \emptyset$ for otherwise $cl(G) = \emptyset$ (and $A \in \mathcal{I}$). Since $A \subseteq B$, we have $G \setminus B \subseteq G \setminus A \in \mathcal{I}$ and $B \setminus cl(G) = B \setminus X = \emptyset \in \mathcal{I}$. Hence B is $\mathcal{I}_\mu$-semiopen. (ii) As $A \subseteq A \cup B$, (ii) follows directly from (i).

(b) Let A and B be two $\mathcal{I}_\mu$-semiopen sets. If $A \cap B = \emptyset$, then the proof is trivial. We assume that $A \cap B \neq \emptyset$. By assumption there exists two μ-semiopen sets G and H such that $G \setminus A \in \mathcal{I}$, $A \setminus cl(G) \in \mathcal{I}$, $H \setminus B \in \mathcal{I}$, $B \setminus cl(H) \in \mathcal{I}$. Consider the μ-semiopen set $G \cap H$ which is non-empty (by μ-semi-hyperconnected of (X, μ)). Since $(G \cap H) \setminus (A \cap B) = ((G \setminus A) \cap H) \cup ((H \setminus B) \cap G) \in \mathcal{I}$, $(A \cap B) \setminus cl(G \cap H) = (A \cap B) \setminus X = \emptyset \in \mathcal{I}$. Hence $A \cap B$ is $\mathcal{I}_\mu$-semiopen. □

Note that from Example 3.8, we have every non-empty μ-semiopen set is not dense in (X, τ). Also, we obtained that $\{a\}$ is an $\mathcal{I}_\mu$-semiopen set, but $\{a, c\}$ is not $\mathcal{I}_\mu$-semiopen.

Proposition 3.13. *Let $(X, \tau, \mu, \mathcal{I})$ be an ideal τ_μ-topological space with $\tau \subseteq \mu$ and every non-empty μ-semiopen subset is dense in (X, τ). A subset A is $\mathcal{I}_\mu$-semiopen if and only if $cl(A)$ is $\mathcal{I}_\mu$-semiopen.*

Proof. Let A be $\mathcal{I}_\mu$-semiopen. Then as $A \subseteq cl(A)$, by Proposition 3.12(i) $cl(A)$ is also $\mathcal{I}_\mu$-semiopen (if $A \notin \mathcal{I}$). For $A \in \mathcal{I}$, we proceed as follows: As A is $\mathcal{I}_\mu$-semiopen, there exists μ-semiopen set U such that $U \setminus A$ and $cl(A) \setminus U$ are both in $\mathcal{I}$ which implies $(U \setminus A) \cup A = U \cup A \in \mathcal{I}$. Thus $U \setminus cl(A) \in \mathcal{I}$ (as $U \setminus cl(A) \subseteq U \subseteq U \cup A$). Also, $cl(A) \setminus cl(U) = cl(A) \setminus X = \emptyset \in \mathcal{I}$. Conversely, suppose that $cl(A)$ is $\mathcal{I}_\mu$-semiopen. Then there exists a μ-semiopen set G such that $G \setminus cl(A) \in \mathcal{I}$ and $cl(A) \setminus cl(G) \in \mathcal{I}$. If $G = \emptyset$, then $cl(A) \setminus cl(G) = cl(A) \in \mathcal{I} \Rightarrow A \in \mathcal{I}$. Thus A

is $\mathcal{I}_\mu$-semiopen. If G is non-empty, consider the μ-semiopen set $H = G \setminus cl(A) = G \cap (X \setminus cl(A)) \in \mathcal{I}$. Again $H \setminus A = G \cap (X \setminus cl(A)) \cap (X \setminus A) \subseteq G \cap (X \setminus cl(A)) \in \mathcal{I}$. Thus $H \setminus A \in \mathcal{I}$ and $A \setminus cl(H) = A \setminus cl(G \cap (X \setminus cl(A))) = A \setminus X = \emptyset$. Hence A is $\mathcal{I}_\mu$-semiopen. □

Theorem 3.14. *Let $(X, \tau, \mu, \mathcal{I})$ be an ideal τ_μ-topological space. Then $X \setminus A$ is $\mathcal{I}_\mu$-semiopen if and only if there exists a μ-semiclosed set F such that $int(F) \setminus A \in \mathcal{I}$ and $A \setminus F \in \mathcal{I}$.*

Proof. First suppose that $X \setminus A$ is $\mathcal{I}_\mu$-semiopen. Then there exists a μ-semiopen set G such that $G \setminus (X \setminus A)) = A \setminus (X \setminus G)) \in \mathcal{I}$ and $(X \setminus A) \setminus cl(G) = int(X \setminus G) \setminus A \in \mathcal{I}$. Let $F = X \setminus G$. Then F is μ-semiclosed and the rest follows. The converse part can be done similarly by taking $G = X \setminus F$. □

4. Weakly $\mathcal{I}_\mu$-semiopen sets

Definition 4.1. *Let $(X, \mu, \mathcal{I})$ be an ideal generalized topological space. A subset A of X is called weakly $\mathcal{I}_\mu$-semiopen if $A = \emptyset$ or if there exists a non-empty μ-semiopen set U such that $U \setminus A \in \mathcal{I}$, for $A \neq \emptyset$. The complement of a weakly $\mathcal{I}_\mu$-semiopen set is termed as weakly $\mathcal{I}_\mu$-semiclosed set.*

It follows that for an ideal τ_μ-topological space $(X, \tau, \mu, \mathcal{I})$, any $\mathcal{I}_\mu$-semiopen set is weakly $\mathcal{I}_\mu$-semiopen but the converse is false. This follows from the Example 3.8, the set $\{c\}$ is a weakly $\mathcal{I}_\mu$-semiopen set, but $\{c\}$ is not $\mathcal{I}_\mu$-semiopen.

Example 4.2. *Let $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{d\}\}$. It is easy to see that $\{a, b\}$ and $\{a, c\}$ are two weakly $\mathcal{I}_\mu$-semiopen sets, but their intersection $\{a\}$ is not weakly $\mathcal{I}_\mu$-semiopen.*

Proposition 4.3. *Let $(X, \mu, \mathcal{I})$ be an ideal generalized topological space. Then the collection of all weakly $\mathcal{I}_\mu$-semiopen sets form a GT on X.*

Proof. $\emptyset$ is clearly a weakly $\mathcal{I}_\mu$-semiopen. Let $\{A_k : k \in J\}$ be a collection of weakly $\mathcal{I}_\mu$-semiopen sets. Then for each $k \in A$ there exists a non-empty μ-semiopen set U_k such that $U_k \setminus A_k \in \mathcal{I}$. But $U_k \setminus \bigcup\{A_k : k \in J\} \subseteq U_k \setminus A_k$. Thus $U_k \setminus \bigcup\{A_k : k \in J\} \in \mathcal{I}$. Hence $\bigcup\{A_k : k \in J\}$ is a weakly $\mathcal{I}_\mu$-semiopen set. □

Theorem 4.4. *Let $(X, \mu, \mathcal{I})$ be an ideal generalized topological space. Then a non-empty subset A of X is weakly $\mathcal{I}_\mu$-semiopen if and only if there exists a non-empty μ-semiopen set U and a set $C \in \mathcal{I}$ such that $U \setminus C \subseteq A$.*

Proof. Let A be a non-empty weakly $\mathcal{I}_\mu$-semiopen subset of X. Then there exists a non-empty μ-semiopen set such that $U \setminus A \in \mathcal{I}$. Let $C = U \setminus A = U \cap (X \setminus A)$. Then $U \setminus C \subseteq A$. Conversely, let there exists a μ-semiopen set U and $C \in \mathcal{I}$ such that $U \setminus C \subseteq A$. Then $U \setminus A \subseteq U \cap C \in \mathcal{I}$ (as $C \in \mathcal{I}$). Thus $U \setminus A \in \mathcal{I}$. □

Theorem 4.5. *Let $(X, \mu, \mathcal{I})$ be an ideal generalized topological space. Then if a subset A of X is weakly $\mathcal{I}_\mu$-semiclosed, then $A \subseteq M \cup B$ for some μ-semiclosed set M of X and $B \in \mathcal{I}$.*

Proof. Let A be a weakly $\mathcal{I}_\mu$-semiclosed set. Then $X \setminus A$ is a weakly $\mathcal{I}_\mu$-semiopen. If $X \setminus A = \emptyset$, then $A = X$. Thus $A = X \cup \emptyset$. If $A \neq X$, then there is a non-empty μ-semiopen set U and $B \in \mathcal{I}$ such that $U \setminus B \subseteq X \setminus A$. So $A \subseteq X \setminus (U \setminus B) = (X \setminus U) \cup B = M \cup B$ where $M = X \setminus U$ which is a μ-semiclosed set and $B \in \mathcal{I}$. □

Example 4.6. *Let $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{a, b\}, \{a, c\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{d\}\}$. Then $\{a, d\} \subseteq X \cup \{d\}$ where X is μ-semiclosed and $\{d\} \in \mathcal{I}$, but $\{a, d\}$ is not weakly $\mathcal{I}_\mu$-semiclosed.*

Theorem 4.7. *Let $(X, \mu, \mathcal{I})$ be an ideal generalized topological space and $\{a\} \in \mu \cap \mathcal{I}$ for some $a \in X$. Then every subset of X is weakly $\mathcal{I}_\mu$-semiopen.*

Proof. Suppose that $\{b\} \subseteq X$. Then either $\{a\} \setminus \{b\} = \emptyset \in \mathcal{I}$ (if $b = a$) or $\{a\} \setminus \{b\} = \{a\} \in \mathcal{I}$ (if $a \neq b$) where $\{a\} \in \mu$. Thus $\{b\}$ is weakly $\mathcal{I}_\mu$-semiopen. Thus by Proposition 4.3, any subset of X is weakly $\mathcal{I}_\mu$-semiopen. □

Theorem 4.8. *Let $(X, \mu, \mathcal{I})$ be an ideal quasi generalized topological space. If (X, μ) is a μ-semi-hyperconnected, then the collection of all weakly $\mathcal{I}_\mu$-semiopen sets form a topology on X.*

Proof. Due to proposition 4.3, we have only to show that X is weakly $\mathcal{I}_\mu$-semiopen and that the intersection of two weakly $\mathcal{I}_\mu$-semiopen sets is so. If $X \in \mu$, then X is weakly $\mathcal{I}_\mu$-semiopen. If $X \notin \mu$, then there exists a non-empty μ-semiopen set U such that $U \neq X$. Clearly $U \setminus X = \emptyset$. Thus X is weakly $\mathcal{I}_\mu$-semiopen. Let A and B be two weakly $\mathcal{I}_\mu$-semiopen sets. Then there exists non-empty μ-semiopen sets U and V such that $U \setminus A \in \mathcal{I}$ and $V \setminus B \in \mathcal{I}$. Then $(U \cap V) \setminus (A \cap B) = [(U \setminus A) \cap V] \cup [(U \cap (V \setminus B)] \in \mathcal{I}$. Hence $A \cap B$ is a weakly $\mathcal{I}_\mu$-semiopen set. □

Remark 4.9: If $\mu = \{\emptyset\}$ in the last theorem, then the set X (if $\neq \emptyset$) is not weakly $\mathcal{I}_\mu$-semiopen. Consequently, the collection of all weakly $\mathcal{I}_\mu$-semiopen sets does not form a topology on X.

Theorem 4.10. *Let $(X, \tau, \mu, \mathcal{I})$ be an ideal τ_μ-topological space. If μ is a QT and (X, μ) a μ-semi-hyperconnected with $\tau \subseteq \mu$, then any non-dense subset A of (X, τ) is weakly $\mathcal{I}_\mu$-semiopen if and only if $cl(A)$ is weakly $\mathcal{I}_\mu$-semiopen.*

Proof. We first observe that if $A \neq \emptyset$ is weakly $\mathcal{I}_\mu$-semiopen and $A \subseteq B$, then B is weakly $\mathcal{I}_\mu$-semiopen and thus $cl(A)$ is weakly $\mathcal{I}_\mu$-semiopen (as $A \subseteq cl(A)$). Conversely, suppose that $cl(A)$ is a weakly $\mathcal{I}_\mu$-semiopen set. If $cl(A) = \emptyset$, then $A = \emptyset$. Thus A is a weakly $\mathcal{I}_\mu$-semiopen set. If $cl(A) \neq \emptyset$, then $U \setminus cl(A) \in \mathcal{I}$ for some non-empty μ-semiopen set U. Let $V = U \setminus cl(A)$. Then $V \setminus A = U \setminus cl(A) \in \mathcal{I}$. It is now sufficient to show that $V \neq \emptyset$. We note that $X \setminus cl(A) \in \tau \subseteq \mu$ and $U \in \mu \setminus \{\emptyset\}$. Since (X, μ) is a μ-semi-hyperconnected, $(X \setminus cl(A)) \cap U) = V \neq \emptyset$. Hence A is weakly $\mathcal{I}_\mu$-semiopen. □

Example 4.11. *Let $X = \{a, b\}$, $\tau = \mu = \{\emptyset, X\}$ and $\mathcal{I} = \emptyset$. Clearly $\tau \subseteq \mu$. Put $A = \emptyset$ and $B = \{b\}$. Then A is a weakly $\mathcal{I}_\mu$-semiopen subset of X which is not dense in X and $A \subseteq B$. However, B is not weakly $\mathcal{I}_\mu$-semiopen as the only non-empty μ-semiopen subset of X is X itself and $X \setminus B = X \setminus \{b\} = \{a\} \notin \mathcal{I}$.*

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