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QUASI-VALUATION MAPS ON JU-ALGEBRAS INDUCED BY FILTERS

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Abstract. The subject of this report is the observation of the concept of JU-filters in JU-algebras, which are a generalization of KU-algebras. In such algebras, the concept of quasi-valuation maps based on the ideal in them has already been developed. Additionally, the design of a quasi-metric space on a JU-algebra induced by that quasi-valuation map is considered. In this paper, the concept of quasi-valuation maps on JU-algebras based on a filter was designed. Then, some of its fundamental properties are shown. In addition, the concepts of quasi-valuation maps on JU-algebra based on implicative and comparative filters were constructed and analyzed. The interconnections of these three types of quasi-valuation maps on JU-algebras are also registered.

1. INTRODUCTION

The concept of JU-algebra as a generalization of KU-algebra, introduced in [9], was determined for the first time in 2015 under the name 'pseudo KU-algebra' ([6]) and reformulated in 2019 into the name JU-algebra ([1, 2]). Quasi-valuation map as a tool on logical algebras is applied 2019 to this class of logical algebras based on the concept of JU-ideals in this algebra ([1]). Previously, the concepts of quasi-valuation maps in some other logical algebras were considered (see, for example, [3, 4, 5, 7, 15]). This researcher also contributed to the research of JU-algebras in articles [11, 12, 13, 14]. Thus, while in the article [11], the author dealt with general issues of the internal architecture of JU-algebras, in the article [12] the concept of filters in JU-algebras as well as the concepts of implicative and comparative filters in this classes of algebras were considered. Further on, in the article [14], the author considered two types of non-standard qiasi-valuations on JU-algebras based on ideals.

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This article-report refers to designing three new concepts of quasi-valuation maps $w : A \longrightarrow \mathbb{R}$ on a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$. The first of them based on a filter, the second on an implicative filter and the third on a comparative filter. Their mutual connections are explored, too. The similarities and differences of these three classes of quasi-valuation maps on JU-algebra are registered. The specificity of these mappings is reflected, among other things, to the fact that the set $F_w = \{x \in A : w(x) = 0\}$ is an (implicative, comparative, res.) filter in $\mathfrak{A}$ depending on whether w is based on a filter, on an implicative filter or on a comparative filter, respectively.

The paper is designed in the following way: The Preliminaries section, in addition to providing information necessary for a comfortable follow-up of the facts presented in the third section, enables a more precise view of the possibility of the existence of filters in JU-algebras. Section three is the main part of this report. Concepts of three types of quasi-valuation maps on JU-algebra $\mathfrak{A}$ based on the substructures of filters, on implicative filters and comparative filters in $\mathfrak{A}$, respectively, are designed in it.

2. PRELIMINARIES

In this section, we recall the definitions of JU-algebras, JU-filters and other important terminologies and some related results from literature [1, 2, 11, 12, 13, 14]. The numbering of the formulas in this section is the same as the numbering of the formulas in the papers [12, 14]. The use of logical symbols is literal.

2.1. **JU-algerbas.**

Definition 2.1 ([2], Definition 2.1)**.** *An algebra* $\mathfrak{A} =: (A, \cdot, 1)$ *of type* $(2, 0)$ *with a binary operation "$\cdot$" and a fixed element* 1 *is said to be JU-algebra if it satisfies the following conditions*

(JU-1) $(\forall x, y, z \in A)((y \cdot z) \cdot ((z \cdot x) \cdot (y \cdot x)) = 1)$,

(JU-2) $(\forall x \in A)(1 \cdot x = x)$ *and*

(JU-3) $(\forall x, y \in A)((x \cdot y = 1 \wedge y \cdot x = 1) \implies x = y)$.

We denote this axiom system by ***JU***.

Lemma 1 ([2], Lemma 2.4)**.** *In this axioms system, the following formulas*
(J_{11}) $(\forall \in A)(x \cdot x = 1)$.
(J_{12}) $(\forall x, y, z \in A)(z \cdot (y \cdot x) = y \cdot (z \cdot x))$
are hold.

Remark 2.1: In [9], a KU-algebra is defined as a system $(A, \cdot, 0)$ satisfying the following axioms

(KU-1) $(\forall x, y, z \in A)((x \cdot y) \cdot ((y \cdot z) \cdot (x \cdot z)) = 0)$,

(KU-2) $(\forall x \in A)(0 \cdot x = x)$,

(KU-3) $(\forall x \in A)(x \cdot 0 = 0)$ and

(KU-4) $(\forall x, y \in A)((x \cdot y = 0 \wedge y \cdot x = 0) \implies x = y)$.

We denote this axiom system by **KU**. With **wKU** we denote the axiomatic system **KU** without axiom (KU-3). If we follow the formation of the concept of 'weak BCC-algebras' from the 'concept of BCC-algebras', then the name *'weak KU-algebra'* could also be used for a JU-algebra by analogy with the previous one. So, **JU** $\equiv$ **wKU** $\equiv$ **PKU**.

Recall ([8]) that in the axiom system **KU**, the formula (J_{12}) is a valid formula also.

If in the definition of KU-algebras we write 1 instead of 0, then we see that any KU-algebra A is a JU-algebra. Therefore, the concept of JU-algebras is a generalization of the concept of KU-algebras ([2], pp. 136).

Since the system of axioms that determine the concept of KU-algebras is mutually independent, there is certainly (at least one) model that satisfies the axioms (KU1), (KU2) and (KU4) but does not satisfy the axiom (KU3).

Example 2.1: The algebraic construction described in Example 1 in article [2], or Example 2.1 in article [8] is an example of JU-algebra. In paper [10], the algebraic construction $(X, \cdot, 0)$ described in Example (or Example 2 in [2]), is a JU-algebra but not a KU-algebra.

(1) Let $A = \{1, 2, 3, 4\}$ and define binary operation $\cdot$ as follows:

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	1	2	2
3	1	2	1	3
4	1	2	1	1

Then $(A, \cdot, 1)$ is a JU-algebra but it is not a KU-algebra.

(2) Let $A = \{1, a, b\}$ and define binary operation $\cdot$ as follows:

$\cdot$	1	a	b
1	1	a	b
a	1	1	b
b	b	b	1

Then $(A, \cdot, 1)$ is a JU-algebra but it is not a KU-algebra.

(3) Let $A = \{1, 2, 3, 4, 5, 6\}$ and define binary operation $\cdot$ as follows:

$\cdot$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	3	3	5	6
3	1	1	1	2	5	6
4	1	1	1	1	5	6
5	5	5	5	5	1	5
6	1	1	2	1	5	1

Then $(A, \cdot, 1)$ is a JU-algebra but it is not a KU-algebra. □

If A is a JU-algebra, let us define $\varphi : A \longrightarrow A$ by $(\forall x \in A)(\varphi(x) = x \cdot 1)$. According to (JU-2), the equality $\varphi(1) = 1$ is valid for mapping φ. It is obvious that the set $An_\varphi(A) =: \{x \in A : \varphi(x) = 1\}$ is a sub-algebra of the JU-algebra A. Indeed, for $x, y \in An_\varphi(A)$ we have $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) = 1$, according to Proposition 2.4(4) in article [12]. So, $An_\varphi(A)$ is a KU-algebra. Therefore, every JU-algebra has at least one sub-KU-algebra.

Definition 2.2 ([2])**.** *Let $\mathfrak{A} =: (A, \cdot, 1)$ be a JU-algebra. We define a relation $\leqslant$ in A*

$$(\forall x, y \in A)(y \leqslant x \iff x \cdot y = 1).$$

According to Lemmas 2.2 and 2.3 in [2], the relation $\leq$ is a partial order in A, left compatible

(a) $(\forall x, y, z \in A)(x \leqslant y \implies z \cdot x \leqslant z \cdot y)$

and right reverse compatible

(b) $(\forall x, y, z \in A)(x \leqslant y \implies y \cdot z \leqslant x \cdot z)$

with the internal operation in $\mathfrak{A}$.

2.2. **JU-filters in JU-algebras.**

In this subsection, we recall the concept of JU-filters in a JU-algebra introduced in the article [12]. We also repeat the concept of two types of filters in this class of algebras: implicative JU-filters and comparative JU-filters. Information about this class of substructures in JU-algebras, taken from the paper [12], is given below for consistency of the presented material.

Definition 2.3 ([12], Definition 3.1)**.** *A non-empty subset F of a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$ is called a JU-*filter *in $\mathfrak{A}$ if*

(F1) $1 \in F$ *and*

(F2) $(\forall x, y \in A)((x \cdot y \in F \wedge y \in F) \implies x \in F)$.

It is easy to conclude that the following applies:

Lemma 2. *Let F be a nonempty subset of a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$ satisfying the condition (F2). Then F also satisfies the condition (F1).*

Proof. Let F be a nonempty subset of a JU-algebra $\mathfrak{A}$ satisfying the condition (F2). Then $F \neq \emptyset$. So, there is an element $x \in F$. Now, from $1 \cdot x = x \in F$ and $x \in F$ it follows $1 \in F$ by (F2). □

Therefore, $\mathbf{JU}, F \neq \emptyset \models (F2) \implies (F1)$.

Lemma 3 ([12], Lemma 3.1)**.** *Let F be a JU-filter in a JU-algebra A. Then*

(8) $(\forall x, y \in A)((x \leqslant y \wedge x \in F) \implies y \in F)$.

Example 2.2: Let $\mathfrak{A}$ be as in Example 2.1(1). The order relation $\leqslant$ in this JU-algebra $\mathfrak{A}$ is given by

$$\leqslant = \{(1,1), (2,2), (3,3), (4,4), (1,3), (1,4), (3,4)\}.$$

The set $F =: \{1, 3, 4\}$ is a JU-filter in $\mathfrak{A}$. □

Example 2.3: Let $\mathfrak{A}$ be as in Example 2.1(2). The order relation $\leqslant$ in this JU-algebra $\mathfrak{A}$ is given by $\leqslant = \{(1,1), (a,a), (b,b), (1,a)\}$. The set $F =: \{1, a\}$ is a JU-filter in $\mathfrak{A}$. □

Example 2.4: Let $\mathfrak{A}$ be as in Example 2.1(3). The order relation $\leqslant$ in this JU-algebra $\mathfrak{A}$ is given by $\leqslant = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(1,2),(1,3),(1,4),(1,6),(2,3),(2,4),(2,6),(3,4),(4,6)\}$. The set $F =: \{1,2,3,4,6\}$ is a JU-filter in $\mathfrak{A}$.

However, the set $G =: \{1,5\}$ is not a filter in $\mathfrak{A}$ because, for example, $5 \in G$ and $2 \cdot 5 = 5 \in G$ but $2 \notin G$. □

Lemma 4. *Let F be a JU-filter in a JU-algebra A. Then*
(9) $(\forall x \in A)(\varphi(x) \in F \implies x \in F)$.

Proof. If we put $y = 1$ in (F2), we get (9). □

An important consequence of the previous lemma enables us to better understand the concept of filters in JU-algebras.

Corollary 0.1. *Let F be a JU-filter in a JU-algebra A. Then holds*
(10) $(\forall x \in A)(x \notin F \implies x \notin An_{\varphi}(A))$.

Proof. The logical contraposition of the statement (9) gives $x \notin F \implies \varphi(x) \notin F$. On the other hand, if for some $x \notin F$ there were $x \in An_{\varphi}(A)$, we would have $\varphi(x) = 1 \in F$, whence, according to Lemma 4, it would follow $x \in F$ which contradicts the assumption $x \notin F$. So, it must be $x \notin An_{\varphi}(A)$. □

From this consideration we conclude $F \supseteq An_{\varphi}(A)$.

Remark 2.2: Let $\mathfrak{A} =: (A, \cdot, 1)$ be a KU-algebra. Then $\mathfrak{A}$ is a unique filter in $\mathfrak{A}$. Indeed, let F be a KU-filter in $\mathfrak{A}$. For $x \in A$ we have $x \cdot 1 = 1 \in F$, according to (KU-3). Now, from this and from $1 \in F$ it follows that $x \in F$ by (F2).

The aforementioned conclusion can be demonstrated in the following way: Let A be a KU-algebra. Then A is a unique filter in A. Indeed, let F be a KU-filter in A. For $x \in A$ we have $x \cdot 1 = 1 \in F$, according to (KU-3). So $(x \cdot 1) \cdot 1 = 1 \in F$ by (KU-3). On the other hand, from $(x \cdot 1) \cdot 1 \leqslant x$ it follows $x \in F$ according to (8). This proves that $A = F$ holds.

The condition (F2) can be replaced by the following condition:

Proposition 2.1 ([12], Proposition 3.1)**.** *Let F be a subset of a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$ such that $1 \in F$. Then the condition* (F2) *is equivalent to the condition*

(F3) $(\forall x, y, z \in A)(((x \cdot y) \cdot z \in F \wedge z \in F) \implies x \cdot y \in F)$.

Proposition 2.2 ([12], Proposition 3.2)**.** *Let F be a subset of a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$ such that $1 \in F$. Then the condition* (F2) *is equivalent the condition*

(F4) $(\forall x, y, z \in A)((y \cdot (x \cdot z) \in F \wedge y \cdot z \in F) \implies x \in F)$.

In following definition we introduce the concept of implicative filters of a JU-algebra.

Definition 2.4 ([12], Definition 3.2)**.** *A subset F of a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$ is an* implicative *filter in $\mathfrak{A}$ if the following hold*

(F1) $1 \in F$ *and*

(F5) $(\forall x, y, z \in A)((x \cdot (y \cdot z) \in F \wedge x \cdot y \in F) \implies x \cdot z \in F)$

In what follows, we introduce comparative filters of a JU-algebra.

Definition 2.5 ([12], Definition 3.3)**.** *A subset F of a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$ is a* comparative filter *in $\mathfrak{A}$ if the following hold*

(F1) $1 \in F$ *and*

(F6) $(\forall x, y, z \in A)(((y \cdot z) \cdot (z \cdot x) \in F \wedge x \in F) \implies y \in F)$.

3. THE MAIN RESULTS

This section is the central part of this report. It consists of three subsections. The first one introduces the concept of standard quasi-valuation map w on a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$ based on a JU-filter in $\mathfrak{A}$ so that the set $F_w =: \{x \in A : w(x) = 0\}$ is a JU-filter in $\mathfrak{A}$. In the second and third subsections, the concepts of quasi-valuation map w on a JU-algebra $\mathfrak{A}$ induced an implicative filter and induced a comparative filter in $\mathfrak{A}$ respectively are introduced and analyzed. The specificity of these last quasi-valuation maps is that they are a standard quasi-valuation map, on the one hand, and that the induced set F_w is an implicative filter in $\mathfrak{A}$ (that is, it is a comparative filter in $\mathfrak{A}$, res.).

3.1. **Standard quasi-valuation map based on a filter.**

Definition 3.1. *A real-valued function $w : A \longrightarrow \mathbb{R}$ on a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$ is called a* standard quasi-valuation map *on $\mathfrak{A}$ if it satisfies the following two conditions:*

(V1) $w(1) = 0$,

(V2) $(\forall x, y \in A)(w(y) \geqslant w(y \cdot x) + w(x))$.

A quasi-valuation map w on a JU-algebra $\mathfrak{A}$ satisfying the following condition:

(V3) $(\forall u \in A)(w(u) = 0 \implies u = 1)$.

is called a valuation on $\mathfrak{A}$.

Example 3.1: Let $\mathfrak{A}$ be as in examples 2.1(1) and 2.2. If we determine the function $w : A \longrightarrow \mathbb{R}$ in the following way $w(1) = w(3) = w(4) = 0$ and $w(2) = -1$, then w is a standard quasi-valuation map on $\mathfrak{A}$. □

Example 3.2: Let $\mathfrak{A}$ be as in examples 2.1(2) and 2.3. If we determine the function $w : A \longrightarrow \mathbb{R}$ in the following way $w(1) = w(a) = 0$ and $w(b) = -1$, then w is a standard quasi-valuation map on $\mathfrak{A}$. □

Example 3.3: Let $\mathfrak{A}$ be as in examples 2.1(3) and 2.4. If we determine the function $w : A \longrightarrow \mathbb{R}$ in the following way $w(1) = w(2) = w(3) = w(4) = w(6) = 0$ and $w(5) = -1$, then w is a standard quasi-valuation map on $\mathfrak{A}$. □

Theorem 1. *The condition* (V2) *in the previous determination can be replaced with the condition*

(V4) $(\forall x, y, z \in A)(w(x \cdot y) \geqslant w((x \cdot y) \cdot z) + w(z))$,

or with the condition

(V5) $(\forall x, y, z \in A)(w(x) \geqslant w(y \cdot (x \cdot z)) + w(y \cdot z))$.

Proof. (V2) $\Longleftrightarrow$ (V4). Putting $x = 1$ and $z = x$ in (V4) we get (V2). In order to prove the reverse implication we have to put $y = x \cdot y$ and $x = z$ in (V2).

(V2) $\Longleftrightarrow$ (V5). If we put $x = y$, $y = 1$ and $z = x$ in (V5), we obtain (V2). (V5) is obtained from (V2) if we take the product $y \cdot z$ instead of the variable x and $y = x$ in (V2) and apply the equation $x \cdot (y \cdot z) = y \cdot (x \cdot z)$. ☐

The following theorem gives some important properties of the thus determined quasi-valuation map on a JU-algebra based on a filter in it.

Theorem 2. *If w is a standard quasi-valuation map on a JU-algebras $\mathfrak{A} =: (A, \cdot, 1)$, then we have*

(23) $(\forall x, y \in A)(x \leqslant y \implies w(x) \leqslant w(y))$,
(24) $(\forall x, y \in A)(w(y \cdot x) \leqslant w(y) - w(x))$,
(25) $(\forall x, y \in A)(w(x \cdot y) + w(y \cdot x) \leqslant 0)$,
(26) $(\forall x \in A)(w(x) \leqslant 0)$,
(27) $(\forall x, y \in A)(w((x \cdot y) \cdot y) \leqslant w(x))$,
(28) $(\forall x \in A)(w(\varphi(x)) \leqslant w(x))$,
(29) $(\forall x, y, z \in A)(w(x) \geqslant w((y \cdot z)(x \cdot z)) + w(y))$,
(30) $(\forall x, z \in A)(w(x) \geqslant w(z \cdot (x \cdot z)))$,
(31) $(\forall x, y, z \in A)(w(x) \geqslant w(x \cdot ((y \cdot z) \cdot z)) + w(y))$,
(32) $(\forall x, y, z \in A)(w((z \cdot x) \cdot (y \cdot x)) \leqslant w(y \cdot z))$,
(33) $(\forall x, z \in a)(w(x) \geqslant w(\varphi(x) \cdot z) + w(z))$,
(34) $(\forall x, z \in A)(w(x) \leqslant w((z \cdot x) \cdot z))$.

Proof. Let $x, y \in A$ be such that $x \leqslant y$. Then $y \cdot x = 1$. Thus

$$w(y) \geqslant w(y \cdot x) + w(x) = w(1) + w(x) = w(x)$$

by (V2) and (V1).

(24) is obtained directly from (V2).

If we replace the variables in (24), we get $w(x \cdot x) \leqslant w(x) - w(y)$. Adding these two inequalities we get (25).

Since $1 \cdot x = x$, according to (JU-2), we have $0 = w(1) \geqslant w(1 \cdot x) + w(x) = 2w(x)$ by (V2). So, $w(x) \leqslant 0$. This prove (26).

Since $(x \cdot y) \cdot (x \cdot y) = 1$ according to (J_{11}), we have $x \cdot ((x \cdot y) \cdot y) = 1$ using (J_{12}). From here it follows $(x \cdot y) \cdot y \leqslant x$ and $w((x \cdot y) \cdot y) \leqslant w(x)$ according to (23). Therefore, (27) is a valid formula.

For arbitrary element $x \in A$ we have $w(x) \geqslant w(x \cdot 1) + w(1) = w(\varphi(x))$ by definition of function φ and (V2).

Let $x, y, z \in A$ be arbitrary elements. Since $x \cdot y \geqslant (y \cdot z) \cdot (x \cdot z)$ according to (JU-1), we conclude that $w(x \cdot y) \geqslant w((y \cdot z) \cdot (x \cdot z))$ according to (23). So, we have $w(x) \geqslant w(x \cdot y) + w(y) \geqslant w((y \cdot z) \cdot (x \cdot z)) + w(y)$ according to (V2). This proves the inequality (29).

(30) can be obtained from (29), if we put $y = 1$ with respect to (JU-2).

(31) is equivalent to (29) since (J_{12}) is a valid formula in any JU-algebra.

If we put $y = 1$ in (V4), we get $w(\varphi(x)) \geqslant w(\varphi(x) \cdot z) + w(z)$. From here we get (33) due to (28).

First, we can transform (JU-1) into the form *aa* due to (J_{12}). If we put $y = 1$ in this form of (JU-1), we get $((z \cdot x) \cdot z) \cdot x = 1$. Thus means $x \leqslant (z \cdot x) \cdot z$. From here we get (34) by (23). ☐

Theorem 3. *If w is a standard quasi-valuation map on a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$, then the set $F_w =: \{x \in A : w(x) = 0\}$ is a JU-filter in $\mathfrak{A}$.*

Proof. We have $w(1) = 0$, according to (V1) and hence $1 \in F_w$. Let $x, y \in A$ be such that $x \cdot y \in F_w$ and $y \in F_w$. This means $w(x \cdot y) = 0$ and $w(y) = 0$. Since, according to (V2), we have $0 \geqslant w(x) \geqslant w(x \cdot y) + w(y) = 0 + 0 = 0$, we conclude that $w(x) = 0$ which gives $x \in F_w$. Therefore, formula (F2) is valid for the set F_w. So, the set F_w is a JU-filter in $\mathfrak{A}$. ☐

Theorem 4. *Let w be a standard quasi-valuation map on a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$. Then holds*

(35) $(\forall x, y \in A)(w(x \cdot y) \geqslant w(x) + w(y))$.

Proof. If we put $z = y$ in (V4), we get $w(x \cdot y) \geqslant w((x \cdot y) \cdot y) + w(y)$. Applying (V4) to $w((x \cdot y) \cdot y)$, we get $w((x \cdot y) \cdot y) \geqslant w(((x \cdot y) \cdot y) \cdot x) + w(x)$. Thus

$$w(x \cdot y) \geqslant w(((x \cdot y) \cdot y) \cdot x) + w(x) + w(y).$$

On the other side, from $(x \cdot y) \cdot (x \cdot y) = 1$ it follows $x \cdot ((x \cdot y) \cdot y) = 1$, according to ($J_{12}$). This gives $(x \cdot y) \cdot y \leqslant x$. From here, acting on right with x on the previous inequality, we get $1 = x \cdot x \leqslant ((x \cdot y) \cdot y) \cdot x$ according to (b). From here it follows $0 = w(1) = w(x \cdot x) \leqslant w(((x \cdot y) \cdot y) \cdot x) \leqslant 0$ in accordance with (23) and (26). Therefore, $w(((x \cdot y) \cdot y) \cdot x) = 0$ so we have $w(x \cdot y) \geqslant w(x) + w(y)$. ☐

Corollary 4.1. *Let w be a standard quasi-valuation map on a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$. Then holds*

(36) $(\forall x, y, z \in A)(w(x \cdot z) \geqslant w(z \cdot y) + w(x \cdot y))$.

Proof. If we apply the previous theorem to (32), we obtain the required inequality. ☐

3.2. **Quasi-valuation map based on an implicative filter.**

In this subsection, we will observe a quasi-valuation map $w : A \longrightarrow \mathbb{R}$ on a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$ such that the set $F_w = \{x \in A : w(x) = 0\}$ is an implicative filter in $\mathfrak{A}$.

Definition 3.2. *A real-valued function $w : A \longrightarrow \mathbb{R}$ on a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$ is called a* Im-quasi-valuation map *on $\mathfrak{A}$ if it satisfies the following two conditions:*

(V1) $w(1) = 0$,

(V6) $(\forall x, y, z \in A)(w(x \cdot z) \geqslant w(x \cdot (y \cdot z)) + w(x \cdot y))$.

Theorem 5. *Any Im-quasi-valuation map on a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$ is a standard quasi-valuation map on $\mathfrak{A}$.*

Proof. Putting $x = 1$, $y = x$ and $z = y$ in (V6), we get (V2). ☐

The importance of the previous theorem is reflected in the fact that the properties described in Theorem 3.2 also apply to this class of quasi-valuation maps.

Theorem 6. *If w is an Im-quasi-valuation map on a JU-algebra $\mathfrak{A}$, then the set $F_w =: \{x \in A : w(x) = 0\}$ is an implicative filter in $\mathfrak{A}$.*

Proof. Let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in F_w$ and $x \cdot y \in F_w$. This means $w(x \cdot (y \cdot z)) = 0$ and $w(x \cdot y) = 0$. On the other hand, we have

$$0 \geqslant w(x \cdot z) \geqslant w(x \cdot (y \cdot z)) + w(x \cdot y) = 0 + 0 = 0.$$

Thus, $w(x \cdot z) = 0$ and hence $x \cdot z \in F_w$. This shows that F_w is an implicative filter in the JU-algebra $\mathfrak{A}$. ☐

Theorem 7. *Let w be an Im-quasi-valuation map on a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$. Then the following holds*

(37) $(\forall x, z \in A)(w(x \cdot z) \geqslant w(x \cdot (x \cdot z)))$,

(38) $(\forall x \in A)(w(x) \geqslant w(x \cdot \varphi(x)))$,

Proof. If we put $y = x$ in (V6), we get $w(x \cdot z) \geqslant w(x \cdot (x \cdot z)) + w(x \cdot x) = w(x \cdot (x \cdot z))$.

If we put $z = 1$ in (37) we get $w(x) \geqslant w(x \cdot \varphi(x))$. From here we get (38) by (28). ☐

Theorem 8. *Let w be a standard quasi-valuation map on a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$. If w satisfies the additional condition $w((x \cdot z) \cdot (x \cdot y)) \geqslant w(x \cdot (y \cdot z))$, then w is an Im-quasi-valuation map on $\mathfrak{A}$.*

Proof. Let w be a standard quasi-valuation map in a JU-algebra $\mathfrak{A}$. If we put $x \cdot y$ instead of x and $x \cdot z$ instead of y in (V2), we get

$$w(x \cdot z) \geqslant w((x \cdot z) \cdot (x \cdot y)) + w(x \cdot y).$$

Applying an additional assumed condition to this inequality, we obtain (V6). ☐

3.3. **Quasi-valuation map based on a comparative filter.** This subsection is dedicated to a quasi-valuation map w on a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$ such that the set $F_w = \{x \in A : w(x) = 0\}$ is a comparative filter in $\mathfrak{A}$.

Definition 3.3. *A real-valued function $w : A \longrightarrow \mathbb{R}$ on a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$ is called a* C-quasi-valuation map *on $\mathfrak{A}$ if it satisfies the following two conditions:*

(V1) $w(1) = 0$,

(V7) $(\forall x, y, z \in A)(w(y) \geqslant w(((y \cdot z) \cdot (z \cdot x)) + w(x))$.

Theorem 9. *Any C-quasi-valuation map on a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$ is a standard quasi-valuation map on $\mathfrak{A}$.*

Proof. Let w be a C-quasi-valuation map on a JU-algebra $\mathfrak{A}$. If we put z = y in (V7), we get (V2) with respect to (J_{11}) and (JU-2). ☐

Theorem 10. *If w is an C-quasi-valuation map on a JU-algebra $\mathfrak{A}$, then the set $F_w =: \{x \in A : w(x) = 0\}$ is a comparative filter in $\mathfrak{A}$.*

Proof. Let $x, y, z \in A$ be arbitrary elements such that $(y \cdot z)(z \cdot x) \in F_w$ and $x \in F_w$. This means $((y \cdot z)(z \cdot x)) = 0$ and $w(x) = 0$. On the other hand, if we include this in (V7), we get

$$w(y) \geqslant w(((y \cdot z) \cdot (z \cdot x)) + w(x) = 0 + 0 = 0.$$

Hence $w(y) = 0$ and, therefore, $y \in F_w$ showing that F_w is a comparative filter in $\mathfrak{A}$. □

Theorem 11. *Let w be an C-quasi-valuation map on a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$. Then the following holds*

(39) $(\forall y, z \in A)(w(y) \geqslant w((y \cdot z) \cdot \varphi(z)))$,

(40) $(\forall x, y \in A)(w(y) \geqslant w(\varphi(y) \cdot x) + w(x))$,

(41) $(\forall y, z \in A)(w(y) \geqslant w(z \cdot \varphi(y \cdot z)))$.

Proof. Let w be a C-quasi-valuation map on a JU-algebra $\mathfrak{A}$.

Putting $x = 1$ in (V7), we get (39).

If we put $z = 1$ in (V7), we get (40).

(41) is equivalent to (39) since $(\forall u, v \in A)(u \cdot \varphi(v) = v \cdot \varphi(u))$ is a valid formula in any JU-algebra ([12], Proposition 2.1(1)). □

Theorem 12. *Let w be a standard quasi-valuation map on a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$. If w satisfies the additional condition $w(y \cdot x) \geqslant w((y \cdot z) \cdot (z \cdot x))$, then w is an C-quasi-valuation map on $\mathfrak{A}$.*

Proof. If we apply the additional condition to (V2), we get (V7). □

4. FINAL COMMENTS

The concept of JU-algebras, as a generalization of KU-algebras, was introduced and analyzed 2019-2020 in articles [1, 2] by U. Ali, M. A. Ansari, A. Haider, A. N. Koam and M. U. Rehman. However, this algebraic concept was introduced in 2015 in the article [6] by U. Leerawat and C. Prabpayak under the name ′pseudo KU- algebra and in doing so, they used the PKU designation for this class of algebra. Since then, this type of generalization of KU-algebra has been in the focus of interest of the academic community (for example, [10, 11, 12, 13, 14]). The article [12] introduces and analyzes the concept of filters in JU-algebras as well as two special types of filters in these algebras.

In [1], the authors consider pseudo-valuation maps on JU-algebras based on the concept of ideals in them as well as pseudo-metric on JU-algebras induced with these pseudo-valuation maps. The specificity of this determination of the quasi-valuation map $v : \mathfrak{A} \longrightarrow \mathbb{R}$ on a JU-algebra $\mathfrak{A} =: (A, \cdot, 1)$ is reflected to the fact that the set $\{x \in A : v(x) = 0\}$ is an ideal in $\mathfrak{A}$. In [14] some determinations of non-standard quasi-valuation maps on JU-algebras based on ideals are reviewed. In this paper, the determination of quasi-valuation maps on a JU-algebras $\mathfrak{A} =: (A, \cdot, 1)$ based on the concept of filters in its is considered. In addition, some important properties of the quasi-valuation map w determined in this way were registered, the most important of which is that the set $F_w = \{x \in A : w(x) = 0\}$

is a JU-filter in $\mathfrak{A}$. As annexes to these considerations, determinations of quasi-valuation maps on a JU-algebra $\mathfrak{A}$ based on implicative and comparative filters in it were added. A distinctive feature of these additional quasi-valuation maps is that the set F_w is an implicative filter (res. a comparative filter) in $\mathfrak{A}$.

How to design a quasi-metric space on a JU-algebra induced by one of the previously treated quasi-valuation maps on it, could be a question to which attention could be paid in one of the following investigations.

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