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PROXIMATE APPROXIMATIVE RETRACTS IN COMPACT METRIC SPACES

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Abstract. Retracts are fundamental concepts in topology that help in understanding the structure and properties of spaces. A retract of a topological space X is a subspace A with a continuous map (retraction) from X onto A that leaves A fixed. This notion is crucial as it provides insights into simplifying spaces while retaining essential topological features. This paper introduces the concept of proximate approximative retracts (proxAR) in compact metric spaces. It discusses definitions and properties of various retraction types, including proximate retracts (PR), approximative proximate retracts (APR), weak retracts (WR), and approximative weak retracts (AWR). The relationships between these retraction types and proxAR are explored.

Introduction

This paper introduces the concept of proximate approximative retracts (proxAR) and investigates its relationship with existing retraction types. We examine structural connections among PR, APR, WR, AWR, and proxAR in compact metric spaces, providing a framework for understanding their equivalences.

The paper is structured as follows: Section 1 presents key definitions and preliminaries related to retractions, while Section 2 establishes main results, including proofs of relationships between different retraction types.

Definitions

If $\mathcal{U}, \mathcal{V}$ are two coverings of the space X, then $\mathcal{V}$ is refinement of $\mathcal{U}$ if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$. We write $\mathcal{V} \prec \mathcal{U}$.

For the following five definitions, refer to [7] and [6].

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Definition 1. *Let $\mathcal{V}$ be a covering of Y. A function $f : X \to Y$ is $\mathcal{V}$ - continuous at the point $x \in X$ if there exists a neighborhood U_x of x and $V \in \mathcal{V}$ such that $f(U_x) \subseteq V$. A function $f : X \to Y$ is $\mathcal{V}$-continuous on X if it is $\mathcal{V}$-continuous at every point $x \in X$.*

(The family of all such U_x form a covering $\mathcal{U}$ of X. Shortly, we say that $f : X \to Y$ is $\mathcal{V}-$continuous, if there exists $\mathcal{U}$ such that $f(\mathcal{U}) \prec \mathcal{V}$.)

Definition 2. *Let $f : X \to Y$ be a function and let $\mathcal{V}$ be a covering of Y. We say that $g : X \to Y$ is $\mathcal{V}-$close to f if for every $x \in X$, $f(x)$ and $g(x)$ lie in the same member of $\mathcal{V}$. It is denoted by $f =_{\mathcal{V}} g$.*

Definition 3. *Let $f : X \to Y$ be a function between compacta. Let a positive number ϵ be given. We say that f is ϵ-continuous if there exists a $\delta > 0$ such that $d(f(x), f(x')) < \epsilon$ whenever x and x' are points in X with $d(x, x') < \delta$.*

Definition 4. *The functions $f, g : X \to Y$ are ϵ-homotopic, if there exists an ϵ-continuous function $F : X \times I \to Y$ such that for every $x \in X$,*

$$F(x, 0) = f(x) \quad \textit{and} \quad F(x, 1) = g(x).$$

The relation of ϵ-homotopy is an equivalence relation on the set of ϵ-continuous functions.

Definition 5. *The sequence $(f_n) : X \to Y$ is a proximate sequence from X to Y if there exists a cofinal sequence of coverings of Y, $\mathcal{V}_1 > \mathcal{V}_2 > \ldots > \mathcal{V}_n > \ldots$ and if $m \geq n$, f_m and f_n are $\mathcal{V}_n$-homotopic. We say that (f_n) is a proximate sequence over $\mathcal{V}_n$.*

We now define the approximate retraction property through proximate sequences. The following definitions of other retraction types are taken from [4].

Definition 6. *Let X' be a compact metric space that lies in $M \in AR$, and let X be a compact closed subset of X'. For the proximate sequence $(f_n) : X' \to X$ over $\mathcal{V}_n$, we say that it is a proximate approximative retract from X' to X if $f_n|_X$ and i_X are $\mathcal{V}_n$-close for all n.*

For the space X in Definition 6, we say that it is a proxAR (proximate approximative retract) of X'.

Definition 7. *Let X be a compact subset of the metric space (X', d). If for every $\epsilon > 0$ there exists an ϵ-continuous function $r_\epsilon : X' \to X$ such that $r_\epsilon(x) = x$ for all $x \in X$, then we say that X is a PR (proximate retract) of X'.*

Definition 8. *Let X be a compact closed subset of the metric space (X', d). If for every $\epsilon > 0$ there exists an ϵ-continuous function $r_\epsilon : X' \to X$ such that $d(r_\epsilon(x), x) < \epsilon$ for all $x \in X$, then we say that X is an APR (approximative proximate retract) of X'.*

Definition 9. *Let X be a compact closed subset of the metric space (X', d) and let M be an ANR space such that $X' \subseteq M$. If for every neighborhood U of X in M there exists a continuous function $r : X' \to U$ such that $r(x) = x$ for all $x \in X$, then we say that X is a WR (weak retract) of X'.*

Definition 10. *Let X be a compact closed subset of the metric space (X', d) and let M be an ANR space such that $X' \subseteq M$. If for every neighborhood U of X in M and for every $\epsilon > 0$ there exists a continuous function $r_\epsilon : X' \to U$ such that $d(r_\epsilon(x), x) < \epsilon$ for all $x \in X$, then we say that X is an AWR (approximative weak retract) of X'.*

Lets denote by M the class of compact metric spaces. We say that the compact metric space X is PR(M), APR(M), WR(M), AWR(M), proxAR(M) if for every homeomorphic image $h(X) = Y$ of X in some compactum $Y' \in M$, Y is PR, APR, WR, AWR, proxAR of Y'.

Main Results

What is the relationship between PR, APR, WR, AWR and proxAR? Note that in this paper, we consider only compact metric spaces, all of which can be embedded in the Hilbert cube, which is an AR.

Lemma 1. *Let (X, ρ) and (Y, d) be metric spaces. If $A \underset{closed}{\subseteq} X$ and Y is PR(M), then for any $\epsilon > 0$, there exists $\delta > 0$ such that every δ-continuous function $f : A \to Y$ has an ϵ-continuous extension $f' : X \to Y$ (i.e., there exists a ϵ-continuous function $f' : X \to Y$ such that $f'|_A = f$).*

Proof. According to the Kuratowski-Wojdysławski theorem, we can embed Y as a closed subset of a normed linear space and hence in a Banach space H (the Banach space of all bounded functions with the supremum norm).

Since Y is PR(M), we have that there exists an $\epsilon/2$-continuous retraction $r_{\epsilon/2} : H \to Y$.

Let $\delta < \epsilon$ be such that if $\rho(x, y) < \delta$, then $d(r_{\epsilon/2}(x), r_{\epsilon/2}(y)) < \epsilon/2$.

Let $f : A \to Y$ be a δ-continuous function, then:

Since H, as a Banach space, is locally convex, from the Dugundji extension theorem [2] and from Ho's theorem [3], we can construct the functions as follows:

(1) From $Y \in$ PR(M), there exists an ϵ-continuous function $r_\epsilon : H \to Y$ such that $r_\epsilon(y) = y$ for all $y \in Y$. Since r_ϵ is ϵ-continuous, we have that there exists δ such that if $\rho(x, y) < \delta$, then $d(r_\epsilon(x), r_\epsilon(y)) < \epsilon$. This is possible since we are working with compact metric spaces and from [5].

(2) From Ho's theorem, there exists a continuous δ-approximation $\bar{f} : A \to H$ for f such that $d(\bar{f}(x), f(x)) < \delta$ for all $x \in A$.

(3) From Dugundji's theorem, there exists an extension $\tilde{f} : X \to H$ for the function $\bar{f}$ such that $\tilde{f}(a) = \bar{f}(a)$ for all $a \in A$.

Define the mapping $\hat{f} : X \to Y$ by

$$\hat{f}(x) = \begin{cases} r_{\epsilon/2} \circ \tilde{f}(x), & x \in X \setminus A \\ f(x), & x \in A \end{cases}$$

We will show that $\hat{f} : X \to Y$ is an ϵ-continuous function.

Let $a \in A$, we have:

$$\begin{aligned} d(r_{\epsilon/2}(\tilde{f}(a)), f(a)) &= d(r_{\epsilon/2}(\bar{f}(a)), f(a)) \\ \le d(r_{\epsilon/2}(\bar{f}((a)), r_{\epsilon/2}(f(a))) &+ d(r_{\epsilon/2}(f(a)), f(a)) \\ &< \epsilon/2 + 0 = \epsilon/2 \end{aligned} \qquad (*)$$

Clearly, $\hat{f}$ is ϵ-continuous at points in int(A) ∪ int($X \setminus A$) = int(A) ∪ $X \setminus A$ (by the Pasting lemma). Is it continuous at ∂A?

Let $x \in \partial A$ and let $r_{\epsilon/2}(\tilde{f}(x)) = y'$, $f(x) = y$. Since $r_{\epsilon/2}\tilde{f}(x) : X \to Y$ is $\epsilon/2$-continuous, we have that there exists a neighborhood U_x of x such that $r_{\epsilon/2}(U_x) \subseteq B(y', \epsilon/2)$, i.e., $d(r_{\epsilon/2}(\tilde{f}(x')), r_{\epsilon/2}(\tilde{f}(x))) < \epsilon/2$ for $x' \in U_x \cap X$ (1).

If $a \in U_x \cap A$, we have:

$$d(\hat{f}(a), \hat{f}(x)) = d(f(a), f(x)) < \delta < \epsilon$$

If $x' \in U_x \cap X \setminus A$:

$$\begin{aligned} d(\hat{f}(x'), \hat{f}(x)) &= d(r_{\epsilon/2}(\tilde{f}(x')), f(x)) \\ &\leq d(r_{\epsilon/2}(\tilde{f}(x')), r_{\epsilon/2}(\tilde{f}(x))) + d(r_{\epsilon/2}(\tilde{f}(x)), f(x)) \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

The first inequality holds from (1). The second is satisfied from (*).

We get that $f(U_x) \subseteq B(y, \epsilon)$. This is possible for all $x \in \partial A$ and hence we get that $\hat{f}$ is ϵ-continuous on X. □

In this lemma, if instead of PR we take the class APR, then in (*) we have

$$\begin{aligned} d(r_{\epsilon/3}(\tilde{f}(a)), f(a)) &= d(r_{\epsilon/3}(\bar{f}(a)), f(a)) \\ &\leq d(r_{\epsilon/3}(\bar{f}(a)), r_{\epsilon/3}(f(a))) + d(r_{\epsilon/3}(f(a)), f(a)) \\ &< \epsilon/3 + \epsilon/3 = 2\epsilon/3. \qquad (**) \end{aligned}$$

For (**), we have if $a \in U_x \cap X \setminus A$, then $d(f(a), y) < \epsilon$, i.e.,

$$f(U_x \cap X \setminus A) \subseteq B(y, \epsilon).$$

Thus, we obtain the following Corollary:

Corollary 1. *Let (X, ρ) and (Y, d) be metric spaces. If $A \subseteq X$ and Y is from APR, then for any $\epsilon > 0$, there exists $\delta > 0$ such that every δ-continuous function $f : A \to Y$ has an ϵ-continuous extension in X (i.e., there exists a function $f' : X \to Y$ such that $f'|_A = f$).*

Lemma 2. *Let (X, ρ) and (Y, d) be compact metric spaces. If $A \underset{closed}{\subseteq} X$ and Y is PR, then for any covering $\mathcal{W}$ of A, there exists a covering $\mathcal{V}$ of X such that every $\mathcal{W}$-continuous function $f : A \to Y$ has a $\mathcal{V}$-continuous extension in X (i.e., there exists a $\mathcal{V}$-continuous function $f' : X \to Y$ such that $f'|_A = f$).*

Proof. Let $\mathcal{W}$ be a covering of Y. By the Lebesgue covering lemma, there exists $\epsilon > 0$ such that every subset of Y with diameter less than ϵ is contained in some element of $\mathcal{W}$. From Lemma 1, for ϵ there exists $\delta > 0$ such that every δ-continuous function has an ϵ-continuous extension in X. This means there exists an ϵ-continuous extension $f' : X \to Y$ for the function f. Let $\mathcal{V} = \{B(y, \delta) \mid y \in Y\}$, then every $\mathcal{W}$-continuous function has a $\mathcal{V}$-continuous extension in X. □

Definition 11. *Let $\epsilon > 0$. We say that Y is ϵ-contractible if the identity mapping $1_Y : Y \to Y$ is ϵ-homotopic to the constant mapping $C_{y_0} : Y \to \{y_0\}$ for some $y_0 \in Y$.*

Lemma 3. *If $Y \in PR(M)$, then Y is ϵ-contractible for all $\epsilon > 0$.*

Proof. Let $\epsilon > 0$ be arbitrary. The set $A = (Y \times \{0\}) \cup (Y \times \{1\})$ is a closed subset of $Y \times I$. Since $Y \in PR(M)$ from Lemma 1, for ϵ there exists $\delta > 0$ such that every δ-continuous mapping has an ϵ-continuous extension in Y. For the δ-continuous mapping $f : A \to Y$ (where f is continuous by the pasting lemma) defined by $f(y, 0) = y$ and $f(y, 1) = y_0$ for all $y \in Y$, there exists an ϵ-continuous extension $F : Y \times I \to Y$. Thus, we obtain that F is an ϵ-homotopy connecting 1_Y and C_{y_0}. □

Lemma 4. *If X is a metric space and Y is ϵ-contractible for every $\epsilon > 0$, then there exists $\delta > 0$ such that any two δ-continuous functions $f_\delta, g_\delta : X \to Y$ are ϵ-homotopic.*

Proof. Since Y is ϵ-contractible, we have that $1_Y \approx_\epsilon C_{y_0}$. On the other hand, it is straightforward to verify the following property of ϵ-homotopy: there exists $\delta > 0$ such that for every δ-continuous function c, if $a \approx_\epsilon b$, then $ac \approx_\epsilon bc$. Hence $f_\delta = 1_Y f_\delta \approx_\epsilon C_{y_0} f_\delta$ and $g_\delta = 1_Y g_\delta \approx_\epsilon C_{y_0} g_\delta$. From the fact that $C_{y_0} f_\delta = C_{y_0} g_\delta = C_{y_0}|_X$ it follows $f \approx_\epsilon g$. □

From Corollary 1, we have that Lemma 3 and Lemma 4 also hold for the class APR.

Using these lemmas, we will prove that if A is APR, then it is also proxAR.

Lemma 5. *If (X', d) is a compact metric space and $X \subseteq X'$, then if X is APR on X', it follows that X is PR on X'.*

Proof. Assume X is APR, i.e., for every $\epsilon > 0$, there exists an $\epsilon/2$-continuous function $f : X' \to X$ such that for all $x \in X$, we have $d(x, f(x)) < \epsilon/2$.

Define the mapping $f_1 : X' \to X$ as follows:

$$f_1(x) = \begin{cases} f(x), & x \in X' \setminus X \\ x, & x \in X \end{cases}$$

We will show that f_1 is ϵ-continuous. It is clear that f_1 is ϵ-continuous at points $x \in \text{int}(X) \cup X' \setminus X$.

Let $x \in X \setminus \text{int}(X)$. From the ϵ-continuity of f, we have that there exists $\delta > 0$ such that if $d(x, y) < \delta$ then $d(f(x), f(y)) < \epsilon$. We can assume that $\delta < \epsilon$.

Let $y \in B(x, \delta)$, the following cases are possible:

1) If $y \in X' \setminus X$, then:

$$d(f_1(x), f_1(y)) = d(x, f(y)) \leq d(x, f(x)) + d(f(x), f(y)) < \epsilon/2 + \epsilon/2 = \epsilon$$

2) If $y \in X$, we have:

$$d(f_1(x), f_1(y)) = d(f(x), f(y)) < \delta < \epsilon$$

Thus, we obtain that f_1 is an ϵ-continuous mapping and further $f_1|_X = 1_X$. Hence, X is PR. □

Lemma 6. *If (X', d) is a compact metric space and $X \underset{closed}{\subseteq} X'$, then if X is AWR on X', it follows that X is APR on X'.*

Proof. Let X be AWR on X', $\epsilon > 0$, and let X be embedded in the Hilbert cube (Q, ρ). Let $U = \{x \in Q \mid \rho(x, X) < \epsilon/5\}$. Since U is a neighborhood of X, there exists a continuous mapping $f : X' \to U$ such that $f|_X$ and 1_X are $\epsilon/5$-close.

Define the mapping $g : U \to X$ as follows:

$$g(x) = \begin{cases} y \in X \text{ such that } \rho(y, x) < 2\rho(x, X) & \text{if } x \in U \setminus X \text{ (choose one)} \\ x & \text{if } x \in X \end{cases}$$

Consider $x, x' \in U$ such that $\rho(x, x') < \epsilon/5$. The following cases are possible:

1) If $x \in U \setminus X$, $x' \in X$, then:

$$\rho(g(x), g(x')) = \rho(g(x), x') \le \rho(g(x), x) + \rho(x, x') < \frac{2\epsilon}{5} + \frac{\epsilon}{5} = \epsilon$$

2) If $x \in X$, $x' \in X$, then:

$$\rho(g(x), g(x')) = \rho(x, x') < \frac{\epsilon}{5} < \epsilon$$

3) If $x \in U \setminus X$, $x' \in U \setminus X$, then:

$$\rho(g(x), g(x')) \le \rho(g(x), x) + \rho(x, x') + \rho(x', g(x')) < \frac{2\epsilon}{5} + \frac{\epsilon}{5} + \frac{2\epsilon}{5} = \epsilon$$

Now, define $r_\epsilon : X' \to X$ as $r_\epsilon = g \circ f$. We have that r_ϵ is ϵ-continuous as a composition of a continuous function with an ϵ-continuous function.

Similarly, for $x \in X$, assuming $\rho(f(x), x) < \epsilon/5$, we have:

$$\rho(r_\epsilon(x), x) = \rho(g(f(x)), x) \le \rho(g(f(x)), f(x)) + \rho(f(x), x) < \frac{2\epsilon}{5} + 0 = \epsilon/2$$

Thus, we obtain that X is an APR on X'. □

The following diagram holds:

$$\text{AWR} \xrightarrow{\text{Lemma 5}} \text{APR} \xrightarrow{\text{Lemma 6}} \text{PR} \xleftrightarrow{[1]} \text{WR}$$

Therefore, we have that the equivalences hold (since PR → APR and WR → AWR hold from definitions):

$$\text{APR} \leftrightarrow \text{AWR} \leftrightarrow \text{PR} \leftrightarrow \text{WR} \leftrightarrow \text{proxAR}$$

Theorem 1. *For compact metric spaces that are subsets of a metric space M, we have:*

$$proxAR = PR = APR = WR = AWR.$$

(i.e., $X \subseteq M$ is proxAR with respect to M if X is APR (or AWR) with respect to M).

Proof. $\Leftarrow$: We will prove that PR $\subseteq$ proxAR. Lets assume that X is PR.

From the compactness of X, we have that there exists a finite covering

$$\mathcal{V}_1 = \left\{ B_{\frac{1}{2}}\left(x_i^1\right) \mid x_i^1 \in X, i = 1, 2, 3, \ldots, m_1 \right\} \text{ for } X.$$

Let us take $\delta_1 = 1$. From the Lebesgue lemma for the covering $\mathcal{V}_1$, there exists $\delta_2 > 0$ such that every subset of X with diameter less than δ_2 is contained in some member of $\mathcal{V}_1$. We can choose $\delta_2 < \frac{1}{2}$.

Then it certainly holds that for the choice $\delta_2 < \frac{1}{2}$, every subset of X with diameter less than δ_2 is contained in some member of $\mathcal{V}_1$. This means that the covering

$$\mathcal{V}_2 = \left\{ B_{\frac{\delta_2}{2}}\left(x_i^2\right) \mid x_i^2 \in X, i = 1, 2, 3, \ldots, m_2 \right\}$$

is a finite subcovering of $\mathcal{V}_1$. Clearly, $\mathcal{V}_2 \succ \mathcal{V}_1$. By this process, we obtain a sequence of coverings

$$\mathcal{V}_1 \succ \mathcal{V}_2 \succ \mathcal{V}_3 \succ \ldots \succ \mathcal{V}_n \succ \ldots,$$

and a sequence of positive real numbers (δ_n) such that $\delta_1 > \delta_2 > \delta_3 > \ldots$ and $\delta_1 = 1, \delta_2 < \frac{1}{2}, \delta_3 < \frac{1}{3}, \ldots$. From the Lebesgue lemma, we have that the sequence $(\mathcal{V}_n)$ of finite coverings of X is cofinal.

From Lemma 2, we have that there exist coverings $\mathcal{W}_1, \mathcal{W}_2, \ldots$ such that each $\mathcal{W}_n$-continuous function defined on $(M \times \{0\}) \cup (M \times \{1\})$ has a $\mathcal{W}_n$-continuous extension on the set $M \times I$.

Since X is PR, we have that for every covering $\mathcal{W}_n$ of X, there exists a $\mathcal{W}_n$-continuous function $f_{\mathcal{W}_n} : M \to X$ such that $f_{\mathcal{W}_n}|_X = 1_X$.

For each $n \geq 1$, we define the mapping $F_{n,n+1} : M \times \{0, 1\} \to X$ with $F_{n,n+1}(y, 0) = f_{\mathcal{W}_n}, \forall y \in Y$ and $F_{n,n+1}(y, 1) = 1_X$ for all $y \in Y$. Clearly, $F_{n,n+1}$ is $\mathcal{W}_n$-continuous. From the previous discussion, each $\mathcal{W}_n$-continuous function

$$F_{i,i+1} : (M \times \{0\}) \cup (M \times \{1\}) \to X$$

has a $\mathcal{V}_i$-continuous extension

$$F_{i,i+1} : M \times I \to X$$

on the set $M \times I$.
We obtain that for all $m \geq n$, $f_{\mathcal{W}_n}, f_{\mathcal{W}_m}$ are $\mathcal{V}_n$-homotopic, i.e., $(f_{\mathcal{W}_n}) : M \to X$ is a proximate retract of M with X, i.e., X is proxAR.

$\Rightarrow$: We will prove that X is AWR. Let $X \in$ proxAR, U be an open neighborhood of X, and let $\epsilon > 0$. Since X is compact, we have that there exists $\delta > 0$ such that $B(X, \delta) \subseteq U$. We can choose $\delta < \epsilon$. There exists a cofinal sequence $(\mathcal{V}_n)$ of coverings of X and there exists a proximate sequence $r_n : M \to X$ over $\mathcal{V}_n$ such that $r_n|_X$ and 1_X are $\mathcal{V}_n$-close.

Since $(\mathcal{V}_n)$ is a cofinal sequence, for every $m \in \mathbb{N}$, there exists $n_m \in \mathbb{N}$ such that

$$\mathcal{W}_m = \left\{ B_{\frac{1}{m}}(x) \mid x \in X \right\} \succ \mathcal{V}_{n_m}.$$

From the Archimedean axiom, there exists $m_0 \in \mathbb{N}$ such that $\frac{2}{m_0} < \frac{\delta}{2}$. For the function $r_{m_0} : M \to X$, it holds that r_{m_0} and 1_X are $\mathcal{V}_{n_{m_0}}$-close, i.e.,

$$(\forall x \in X), r_{m_0}(x) \in V \text{ for some } V \in \mathcal{V}_{n_{m_0}}.$$

From $\mathcal{W}_{m_0} > \mathcal{V}_{n_{m_0}}$, $r_{m_0}(x), x \in B_{\frac{1}{m_0}}(y)$ for some $y \in X$. Since $\frac{2}{m_0} < \frac{\delta}{2}$, we have

$$d(r_{m_0}(x), x) < \frac{\delta}{2}, \text{ i.e., } r_{m_0}|_X \text{ and } 1_X \text{ are } \delta/2\text{-close.}$$

According to [6, p. 72, Theorem 1], there exist $n' \in \mathbb{N}$ and an approximate sequence

$$(\hat{r}_n) : M \to X$$

of continuous mappings such that $d(\hat{r}_n, r_n) < \frac{\delta}{2}$ for all $n \geq n'$. ($(\hat{r}_n)$ is obtained from the approximate sequence (r_n) by infinitesimal translation)
For $n_0 = \max\{n', n_{m_0}\}$ it holds:

$$(\forall x \in X), \quad d(\hat{r}_{n_0}(x), x) \leq d(\hat{r}_{n_0}(x), r_{n_0}(x)) + d(r_{n_0}(x), x) < \frac{\delta}{2} + \frac{\delta}{2} = \delta < \epsilon.$$

From this, we obtain that there exists a continuous mapping $\hat{r}_{n_0} : M \to U$ such that $\hat{r}_{n_0}|_X$ and 1_X are ϵ-close, i.e., X is AWR. □

References

[1] S. A. Bogaty, *Approximate and fundamental retracts*, Mat. Sbornik **93** (135) (1974), 90–102 (in Russian). (Math. USSR Sbornik **22** (1974), 91–103).
[2] J. Dugundji, *An extension of Tietze's theorem*, Pacific J. Math. **1**.3 (1951), 353–367.
[3] C. Ho, *On a stability theorem for the fixed point property*, Fund. Math. **111** (1981), 169–177.
[4] J. Klisowski, *A survey of various modifications of the notions of absolute retracts and absolute neighborhood retracts*, Colloquium Mathematicum **46**.1 (1982), 23–35.
[5] J. M R. Sanjurjo, *Stability of the fixed-point property and universal maps*, Proc. Amer. Math. Soc. **105**.1 (1989), 221–230.
[6] N. Shekutkovski and Z. Misajleski, *Equivalence of intrinsic shape and shape*, Annuaire de l'Institut des Mathématiques **42** (2013), 69–79, Skopje.
[7] N. Shekutkovski, *Intrinsic definition of strong shape for compact metric spaces*, Topology Proc. **39** (2012), 27–39.

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