Математички Билтен
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ON (NON-STANDARD) PSEUDO-VALUATIONS ON JU-ALGEBRAS

DANIEL ABRAHAM ROMANO

Abstract. JU-algebra, as a generalization of KU-algebra, is the subject of study by several researchers in the second decade of this century. In addition to the previous one, the concept of pseudo-valuations on such algebras, generated by the ideal was introduced in 2019. This paper, as a continuation of previous research, refers to the design of the congruence relation on such an algebra and the analysis of its properties induced by a pseudo-valuation on it. Also, three important properties of the JU-algebra quotient structure created in this way are proved.

1. Introduction

Designing of the congruence on some logical algebraic structure induced by a (pseudo - or quasi-)valuation on it is the subject of study in this century. Thus, Mehrshad and Kouhestani in [6] constructed a congruence relation based on a corresponding pseudo-valuation on a BCK-algebra. Independently of them, Song, Bordbar and Jun in [15] studied the construction of the quotient structure of BCK/BCI-algebras induced by quasi-valuation maps. This construction was also written on KU-algebras ([5]) and on a UP-algebra ([12]). The previous reports are a justification for the analogous possibilities of creating (and registering properties) of congruence on JU-algebras based on quasi-valuation maps on them.

KU-algebra (introduced in [7]) is close to BE-algebra ([9]). In [10], it was shown that a KU-algebra is equivalent to a commutative self-distributive BE-algebra. One of a generalizations of the KU-algebra is the JU-algebra ([1]). This last algebra was also the focus of interest of many authors (see, for example, [2, 3, 4, 8, 11, 13, 14]). In [1], the concept of pseudo-variation on JU-algebras, design by the ideal in it, was introduced, and properties of this mapping were discussed.

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In this article, using the notion of a pseudo-valuation map of a JU-algebra, determined by an ideal, a congruence relation on JU-algebra is designed (Theorem 5), and then the quotient structure of JU-algebra is created with the help of such designed congruence (Theorem 7). In addition to the previous one, three important properties of the JU-algebra quotient structure created in this way are proved (Theorem 8, Theorem 9 and Theorem 10). The preliminaries section in this report is designed to enable the reader to comfortably view the material presented in Section 3, which is the main part of this work, but also the consistency of the content in Section 3. Since this author believes that some of the properties of a quasi-valuation map on JU-algebras, determined in article [1], could be demonstrated in a different way, he expressed his beliefs in the Preliminaries section, which is why this section is now slightly longer than it is usual.

2. Preliminaries

It should be emphasized here that the formulas in this text are written in a standard way, as is usual in mathematical logic, with the standard use of labels for logical functions. Thus, the labels $\wedge$, $\vee$, $\Longrightarrow$, and so on, are labels for the logical functions of conjunction, disjunction, implication, and so on. Brackets in formulas are used in the standard way, too. All formulas appearing in this paper are closed by some quantifier. If one of the formulas is open, then the variables that appear in it should be seen as free variables. In addition to the previous one, the sign =:, in the use of $A =: B$, should be understood in the sense that the letter A is the abbreviation for the formula B.

In this section, in addition to repeating the necessary definitions needed for easier reading of the material presented in Section 4, some assertions about JU-algebras and pseudo-valuations on them are presented here, without which reading the material in Section 4 would be less comfortable. Then some properties of this algebra are listed that are significant for this report. The concepts of ideals and filters in these algebras, mostly taken from [11, 13], are listed. Finally, in this section, the definition of the concept of pseudo-valuation maps on JU-algebras is taken from [1].

Here we first give the description of the concept of JU-algebras.

Definition 2.1 ([1], Definition 1). *An algebra* $\mathfrak{A} =: (A, \cdot, 1)$ *of type* $(2, 0)$ *with a binary operation "$\cdot$" and a fixed element* 1 *is said to be* JU-algebras *if it satisfies the following conditions:*

(JU-1) $(\forall x, y, z \in A)((y \cdot z) \cdot ((z \cdot x) \cdot (y \cdot x)) = 1)$,

(JU-2) $(\forall x \in A)(1 \cdot x = x)$ *and*

(JU-3) $(\forall x, y \in A)((x \cdot y = 1 \ \wedge \ y \cdot x = 1) \Longrightarrow x = y)$.

We denote this axiom system by ***JU***. *If additionally holds*

(KU-2) $(\forall x \in A)(x \cdot 1 = 1)$

then this algebra is called a KU-algebra ([5], *Definition 1*).

If $\mathfrak{A}$ is a JU-algebra, let us define $\varphi : A \longrightarrow A$ ([11]) as follows:

$$(\forall x \in A)(\varphi(x) = x \cdot 1).$$

According to (JU-2), the equality $\varphi(1) = 1$ is valid for mapping φ.

Example 2.1: An example of JU-algebra can be found in [1] (Example 1). An example of a JU-algebra that it is not a KU-algebra is given in the paper [2] (Example 2). □

Example 2.2: Let $\mathbb{N}$ be the semi-ring of all natural numbers. Define the operation $*$ on $\mathbb{N}$ as follows

$$(\forall a, b \in \mathbb{N})(a * b =: \frac{b}{gcd(a,b)})$$

where $gcd(a,b)$ is the greatest common divisor of a and b. Then $(\mathbb{N}, *, 1)$ is a JU-algebra. □

Example 2.3: Let $A = \{1,2,3,4,5,6\}$ and define binary operation $\cdot$ as follows:

$\cdot$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	3	3	5	6
3	1	1	1	2	5	6
4	1	1	1	1	5	6
5	5	5	5	5	1	5
6	1	1	2	1	5	1

Then $(A, \cdot, 1)$ is a JU-algebra but it is not a KU-algebra. □

We call the constant 1 of A the fixed element of $\mathfrak{A}$. For the sake of convenience, we write $\mathfrak{A}$ instead of $(A, \cdot, 1)$ to represent a JU-algebra. We define a relation $\leqslant$ in $\mathfrak{A}$ by

$$(\forall x, y \in A)(x \leqslant y \iff y \cdot x = 1).$$

By direct calculation, it can be verified that the relation defined in this way is a relation of partial order on A left compatible and right anti-compatible with the operation in A ([1], Lemma 2 and Lemma 3):

(JU$_7$) $(\forall x, y, z \in A)(x \leqslant y \implies y \cdot z \leqslant x \cdot z)$, and

(JU$_8$) $(\forall x, y, z \in A)(x \leqslant y \implies z \cdot x \leqslant z \cdot y)$.

An important conclusion can be immediately deduced ([1], Lemma 3) from the axioms of JU-algebra:

(JU$_9$) $(\forall x, y, z \in A)((z \cdot x) \cdot (y \cdot x) \leqslant y \cdot z)$.

Validity of the following formula

(JU$_{10}$) $(\forall x, y \in A)((y \cdot x) \cdot x \leqslant y)$

can be demonstrated as follows: If we put $y = 1$ and $z = y$ in (JU-1), we get

$$(1 \cdot y) \cdot ((y \cdot x) \cdot (1 \cdot x)) = 1, \text{ i.e. } y \cdot ((y \cdot x) \cdot x) = 1$$

due to (JU-2). This means $(y \cdot x) \cdot x \leqslant y$.

For what follows, we need the following result ([1], Lemma 4):

(JU_{12}) $(\forall x, y, z \in A)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$.

On the other hand, since the validity of the formula

(JU_{11}) $(\forall x \in A)(x \cdot x = 1)$

can be deduced from the axiom (JU-1), by choosing $y = 1$ and $z = 1$, the formula (JU_{10}) can be proved by relying on (JU_{12}) by the following method: From $(y \cdot x) \cdot (y \cdot x) = 1$ it follows $y \cdot ((y \cdot x) \cdot x) = 1$ due to (JU_{12}).

The concept of ideals in a JU-algebra $\mathfrak{A}$ is given by the following definition:

Definition 2.2. *A non-empty subset J of a JU-algebra $\mathfrak{A}$ is called a JU-ideal of $\mathfrak{A}$ if*

(J-1) $1 \in J$ *and*

(J-2) $(\forall x, y \in A)(((x \in J \ \wedge\ x \cdot y \in J) \implies y \in J)$.

In [11] it was shown that for every ideal J in a JU-algebra A holds

(J-3) $(\forall x, y \in A)(A)((x \leqslant y \ \wedge\ y \in J) \implies x \in J)$.

Also, it is shown ([11], Proposition 2) that the condition (J2) is equivalent to the condition

(J-4) $(\forall x, y, z \in A)((x \cdot (y \cdot z) \in J \ \wedge\ y \in J) \implies x \cdot z \in J)$.

Example 2.4: Let A be as in Example 2.2. The order relation $\leqslant$ in this JU-algebra $\mathfrak{A}$ is given by $\leqslant = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(1,2),(1,3),(1,4),(1,6),(2,3),(2,4),(2,6),(3,4),(3,6),(4,6)\}$. The subsets $J_1 = \{1\}$, $J_2 = \{1,2\}$, $J_4 = \{1,2,3,4\}$ and $J_6 = \{1,2,3,4,6\}$ are JU-ideals in $\mathfrak{A}$. The subset $K = \{1,2,3\}$ is not a JU-ideal in $\mathfrak{A}$ because, for example, $3 \in K$ and $3 \cdot 4 = 2 \in K$ but $4 \notin K$. The subset $L = \{1,5\}$ is also not a JU-ideal in $\mathfrak{A}$ because, for example, $5 \in L$ and $5 \cdot 2 = 5 \in L$ but $2 \notin L$. □

The concept of filters in the JU-algebra $\mathfrak{A}$ was introduced in [11] as follows:

Definition 2.3 ([11], Definition 6)**.** *A subset F of a JU-algerba $\mathfrak{A}$ is a JU-filter of $\mathfrak{A}$ if the following hold:*

(F-1) $1 \in F$ *and*

(F-2) $(\forall x, y \in A)((x \cdot y \in F \ \wedge\ y \in F) \implies x \in F)$.

In [13] it was shown that for every filter F in a JU-algebra $\mathfrak{A}$ holds

(F-3) $(\forall x, y \in A)(A)((x \leqslant y \ \wedge\ x \in F) \implies y \in F)$.

Also, it is shown ([13], Proposition 3.5) that the condition (F2) is equivalent to the condition

(F-4) $(\forall x, y, z \in A)(((x \cdot y) \cdot z \in F \ \wedge\ z \in F) \implies x \cdot y \in F)$.

Example 2.5: Let A be as in Example 2.2. The order relation $\leqslant$ in this JU-algebra $\mathfrak{A}$ is given by $\leqslant = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(1,2),(1,3),(1,4),(1,6),(2,3),(2,4),(2,6),(3,4),(3,6),(4,6)\}$. The set $F =: \{1,2,3,4,6\}$ is a JU-filter in $\mathfrak{A}$. However, the set $G =: \{1,5\}$ is not a filter in $\mathfrak{A}$ because, for example, $5 \in G$ and $5 \cdot 2 = 5 \in G$ but $2 \notin G$. □

3. PSEUDO-VALUATIONS ON JU-ALGEBRAS

3.1. **Standard pseudo-valuation.** The concept of pseudo-valuations on JU-algebras was introduced in 2019 in [1] by U. Ali, M. A. Ansari and M. U. Rehman. Since we intend to introduce some other types of pseudo-valuations on JU-algebras, we will recognize this concept as 'standard pseudo-valuation' in all that follows.

Definition 3.1 ([1], Definition 6). *A real-valued function* $v : A \longrightarrow \mathbb{R}$ *on a JU-algebra* $\mathfrak{A}$ *is called a standard pseudo-valuation on* $\mathfrak{A}$ *if it satisfies the following two conditions:*

(V1) $v(1) = 0$,

(V2) $(\forall x, y, z \in A)(v(x \cdot z) \leqslant v(x \cdot (y \cdot z)) + v(y))$.

A standard pseudo-valuation v *on a JU-algebra* $\mathfrak{A}$ *satisfying the following condition:*

(V3) $(\forall x \in A)(v(x) = 0 \implies x = 1)$

is called a valuation on $\mathfrak{A}$.

Example 3.1: Let $A = \{1, 2, 3, 4\}$ be as in Example 2.1. It can be verified that a real valued function v defined on $\mathfrak{A}$ by $v(1) = 0$, $v(2) = v(3) = 1$ and $v(4) = 3$, is a standard pseudo-valuation on $\mathfrak{A}$. □

Example 3.2: Let A be as in Example 2.3 and 2.4. It can be verified that a real valued function v defined on $\mathfrak{A}$ by $v(1) = v(2) = v(3) = v(4) = 0$ and $v(5) = v(6) = 1$, is a standard pseudo-valuation on $\mathfrak{A}$. □

It can easily be deduced that condition (V2) can be replaced by the condition:

(V4) $(\forall y, z \in A)(v(z) \leqslant v(y \cdot z) + v(y))$.

Indeed, if we put $x = 1$ in (V2), we get (V4) by (JU-2). Conversely, if we take $z =: x \cdot z$ in (V4), we get $v(x \cdot z) \leqslant v(y \cdot (x \cdot z)) + v(y)$. From here it follows that $v(x \cdot z) \leqslant v(x \cdot (y \cdot z)) + v(y)$, due to (JU$_{12}$).

From (V2), that is, from (V4), it can be obtained immediately

$$(\forall x, y \in A)(v(\varphi(x)) \leqslant v(x \cdot \varphi(y)) + v(y)).$$

An important conclusion can be immediately deduced from axioms (V1) and (V2):

(V5) $(\forall x, y \in A)(x \leqslant y \implies v(x) \leqslant v(y))$.

The validity of formula (V5) can be demonstrated in the following way: If $x \leqslant y$, then $y \cdot x = 1$. Now, from (V4), with respect to (V1), for $z = x$, it follows that $v(x) \leqslant v(y \cdot x) + v(y) = v(1) + v(y) = 0 + v(y) = v(y)$.

Also, the validity of the formula

(V6) $(\forall x, y, z \in A)(v((x \cdot (y \cdot z)) \cdot z) \leqslant v(x) + v(y))$

can be demonstrated as follows: If we put $x \cdot (y \cdot z)$ instead of the variable x in (V2), we get $v((x \cdot (y \cdot z)) \cdot z) \leqslant v((x \cdot (y \cdot z) \cdot (y \cdot z)) + v(y)$. On the other hand, we start from the inequality $(x \cdot y) \cdot y \leqslant y$ which we got from (JU$_{10}$) if we replace the variables x and y in it. If we put $y =: y \cdot z$ into this last inequality, we get $(x \cdot (y \cdot z)) \cdot (y \cdot z) \leqslant x$. From here, due to (V5), we get $v((x \cdot (y \cdot z)) \cdot (y \cdot z)) \leqslant v(x)$. This proves the validity of formula (V6).

In addition to the above, we can show a demonstration of the validity of the formula

(V7) $(\forall x, y, z \in A)(y \cdot z \leqslant x \implies v(z) \leqslant v(x) + v(y))$.

From (V4), taking $z = x$ and $y = x \cdot (y \cdot z)$, we get

$$\begin{aligned} v(z) &\leqslant v((x \cdot (y \cdot z)) \cdot z) + v(x \cdot (y \cdot z)) = v((x \cdot (y \cdot z)) \cdot z) + v(1) \\ &= v((x \cdot (y \cdot z)) \cdot z) \leqslant v(x) + v(y) \quad \text{due to (V6).} \end{aligned}$$

Finally, let us show that the following holds

(V8) $(\forall x, y, z \in A)(v(x \cdot z) \leqslant v(x \cdot y) + v(y \cdot z))$.

(JU-1) is equivalent to $(z \cdot x) \cdot (y \cdot x) \leqslant y \cdot z$. If we apply (V7) to this, we get $v(y \cdot x) \leqslant v(y \cdot z) + v(z \cdot x)$. If we include $y = x$, $x = z$ and $z = y$ in this last inequality, we get (V8).

For a real function v designed like this, we have:

Proposition 3.1 ([1], Theorem 16)**.** *If v is a standard pseudo-valuation on a JU-algebra $\mathfrak{A}$, then the set $J_v = \{x \in A : v(x) = 0\}$ is an ideal of $\mathfrak{A}$.*

3.2. **Non-standard pseudo-valuations.** As non-standard pseudo-valuation on the JU-algebra $\mathfrak{A}$ we will observe real-valued functions $f : \mathfrak{A} \longrightarrow \mathbb{R}$ determined so that the kernel $J_f = \{x \in A : f(x) = 0\}$ of this function is an ideal of a special type in the algebra $\mathfrak{A}$. Thus, we will introduce the non-standard pseudo-valuation f on $\mathfrak{A}$ so that J_f is an t-ideal, that is, J_f is a p-ideal in $\mathfrak{A}$.

Definition 3.2. *A real-valued function $w : A \longrightarrow \mathbb{R}$ on a JU-algebra $\mathfrak{A}$ is called a non-standard t-pseudo-valuation on $\mathfrak{A}$ if it satisfies the following two conditions:*

(W1) $w(1) = 0$,

(W2) $(\forall x, y, z \in A)(w(x \cdot z) \leqslant w((x \cdot y) \cdot z) + w(y))$.

First, we show that any non-standard t-pseudo-valuation is a standard pseudo-valuation:

Proposition 3.2. *Any non-standard t-pseudo-valuation on a JU-algebra $\mathfrak{A}$ is a standard pseudo-valuation on $\mathfrak{A}$.*

Proof. If we put $x = 1$, $y = x$ and $z = y$ in (W2), we get the condition (V4). □

In what follows, we need the following notion: A subset J of a JU-algebra $\mathfrak{A}$ is a t-ideal of $\mathfrak{A}$ ([11], Definition 7) if the following holds:

(J-1) $1 \in J$ and

(J-t) $(\forall x, y, z \in A)(((x \cdot y) \cdot z \in J \ \wedge \ y \in J) \implies x \cdot z \in J)$.

The following theorem gives an important relation between this non-standard t-pseudo-valuation on a JU-algebra $\mathfrak{A}$ and the notion of t-ideals in $\mathfrak{A}$.

Theorem 1. *If w is a non-standard t-pseudo-valuation on a JU-algebra $\mathfrak{A}$, then the set $J_w = \{x \in A : w(x) = 0\}$ is a t-ideal of $\mathfrak{A}$.*

Proof. Let $x, y, z \in A$ be such that $(x \cdot y) \cdot z \in J_w$ and $y \in J_w$. This means $w((x \cdot y) \cdot z) = 0$ and $w(y) = 0$. Since w is a non-standard pseudo-valuation on $\mathfrak{A}$, we have $0 \leqslant w(x \cdot z) \leqslant w((x \cdot y) \cdot z) + w(y) = 0 + 0 = 0$. Therefore, $w(x \cdot z) = 0$, which gives $x \cdot z \in J_w$. □

In the following theorem, we give some of the properties of this non-standard pseudo-valuation:

Theorem 2. *Let $w : A \longrightarrow \mathbb{R}$ be a non-standard t-pseudo-valuation on a JU-algebra A. Then:*

(W3) $(\forall x, z \in A)(w(x \cdot z) \leqslant w(\varphi(x) \cdot z))$,

(W4) $(\forall x, z \in A)(w(\varphi^2(x) \cdot z) \leqslant w(\varphi(x) \cdot z))$ *and*

(W5) $(\forall x \in A)(w(x \cdot \varphi(x)) \leqslant 0)$.

Proof. (i) If we put $y = 1$ in (W2), we get (W3) with respect to (W1).

(ii) If we put $x = \varphi^2(x)$ in (W3), we get $w(\varphi^2(x) \cdot z) \leqslant w(\varphi(\varphi^2(x)) \cdot z)$. Since holds $\varphi^3(x) = \varphi(x)$, we have $w(\varphi^2(x) \cdot z) \leqslant w(\varphi(x) \cdot z)$.

(iii) If we put $y = 1$ and $z = \varphi(x)$ in (W2), we get

$$w(x \cdot \varphi(x)) \leqslant w((x \cdot 1) \cdot \varphi(x)) + w(1) = w(\varphi(x) \cdot \varphi(x)) + 0 = w(1) = 0,$$

by which (W5) is proved. □

With the following definition, we introduce a new class of non-standard pseudo-valuations:

Definition 3.3. *A real-valued function $\phi : A \longrightarrow \mathbb{R}$ on a JU-algebra $\mathfrak{A}$ is called a non-standard p-pseudo-valuation on $\mathfrak{A}$ if it satisfies the following two conditions:*

(Φ1) $\phi(1) = 0$,

(Φ2) $(\forall x, y, z \in A)(\phi(x) \leqslant \phi((z \cdot y) \cdot (z \cdot x)) + \phi(y))$.

Proposition 3.3. *Any non-standard p-pseudo-valuation on a JU-algebra $\mathfrak{A}$ is a standard pseudo-valuation on $\mathfrak{A}$.*

Proof. If we put $z = 1$ in (Φ2), we get (V4). □

In what follows, we need the following notion: A subset J of a JU-algebra $\mathfrak{A}$ is a p-ideal of $\mathfrak{A}$ ([2], Definition 2.7) if the following holds:

(J-1) $1 \in J$ and

(J-p) $(\forall x, y, z \in A)(((z \cdot y) \cdot (z \cdot x) \in J \ \wedge \ y \in J) \implies x \in J)$.

The following theorem gives an important relation between this non-standard p-pseudo-valuation on a JU-algebra $\mathfrak{A}$ and the notion of p-ideals in $\mathfrak{A}$.

Theorem 3. *If ϕ is a non-standard p-pseudo-valuation on a JU-algebra $\mathfrak{A}$, then the set $J_\phi = \{x \in A : \phi(x) = 0\}$ is a p-ideal of $\mathfrak{A}$.*

Proof. Let $x, y, z \in A$ be such that $(z \cdot y) \cdot (z \cdot x) \in J_\phi$. This means $\phi((x \cdot y) \cdot (z \cdot x)) = 0$ and $\phi(y) = 0$. On the other hand, according to (Φ2), we have

$$\phi(x) \leqslant \phi((z \cdot y) \cdot (z \cdot x)) + \phi(y) = 0 + 0 = 0.$$

So, $x \in J_\phi$. According to this, J_ϕ is a p-ideal in $\mathfrak{A}$. □

Theorem 4. *Let $\phi : A \longrightarrow \mathbb{R}$ be a non-standard p-pseudo-valuation on a JU-algebra A. Then:*

(Φ3) $(\forall x, z \in A)(\phi(x) \leqslant \phi(\varphi(z) \cdot (z \cdot x)))$.

(Φ4) $(\forall x, y \in A)(\phi(x) \leqslant \phi(\varphi(x \cdot y)) + \phi(y))$.

(Φ5) $(\forall x \in A)(\phi(x) = \phi(\varphi^2(x)))$.

Proof. (i) If we put $y = 1$ in (Φ2), we get (Φ3).

(ii) If we put $z = x$ in (Φ2), we get (Φ4).

(iii) If we include $y = 1$ in (Φ4), we get $\phi(x) \leqslant \phi(\varphi^2(x))$. On the other hand, according to (9) in [11], we have $\varphi^2(x) \leqslant x$. From here, according to (V5), since ϕ is a standard pseudo-valuation on $\mathfrak{A}$, we have $\phi(\varphi^2(x) \leqslant \phi(x)$. Hence, we have ($\Phi$5). □

Example 3.3: Let $A = \{1, a, b\}$ and define binary operation $\cdot$ as follows:

$\cdot$	1	a	b
1	1	a	b
a	1	1	b
b	b	b	1

Then $\mathfrak{A} = (A, \cdot, 1)$ is a JU-algebra ([2], Example 5). If we put $\phi(1) = \phi(a) = 0$ and $\phi(b) = 1$, then ϕ is a non-standard p-pseudo-valuation on $\mathfrak{A}$ and the ideal $J_\phi = \{0, a\}$ is a p-ideal in $\mathfrak{A}$. □

4. On congruence induced by pseudo-metric

For a real-valued function v on a JU-algebra $\mathfrak{A}$, let us define a mapping

$$d_v : A \times A \ni (x, y) \longmapsto d_v(x, y) =: v(x \cdot y) + v(y \cdot x).$$

The d_v is called pseudo-metric induced by the pseudo-valuation v. We have the following result ([1], Theorem 17):

Proposition 4.1. *Let $\mathfrak{A}$ is a JU-algebra. If a real-valued function v on $\mathfrak{A}$ is a (standard) pseudo-valuation on $\mathfrak{A}$, then d_v is a pseudo-metric on $\mathfrak{A}$, and so (A, d_v) is a pseudo-metric space. This means that d_v satisfies the following conditions:*

(d0) $(\forall x, y \in A)(d_v(x, y) \geqslant 0)$,
(d1) $(\forall x \in A)(d_v(x, x) = 0)$,
(d2) $(\forall x, y \in A)(d_v(x, y) = d_v(y, x))$,
(d3) $(\forall x, y, z \in A)(d_v(x, z) \leqslant d_v(x, y) + d_v(y, z))$.

By congruence θ on a JU-algebra $\mathfrak{A}$ we mean an equivalence relation on A compatible with the multiplication in $\mathfrak{A}$ in the following sense:

$$(\forall x, y, z \in A)((x, y) \in \theta \implies ((z \cdot x, z \cdot y) \in \theta \wedge (x \cdot z, y \cdot z) \in \theta)).$$

In this section, a congruence relation on a JU-algebra $\mathfrak{A}$ induced by the (standard) pseudo-valuation on $\mathfrak{A}$ is created. Then, some of the important features of this designed congruence are analyzed.

Theorem 5. *The relation θ on a JU-algebra $\mathfrak{A}$, defined by*

$$(\forall x, y \in A)((x, y) \in \theta \iff d_v(x, y) = 0),$$

is a congruence relation on $\mathfrak{A}$.

Proof. It is clear that θ is an equivalence relation on $\mathfrak{A}$. The reflexivity of the relation θ holds due to (d1). The symmetry of the relation θ follows from (d2). Let $x, y, z \in A$ be such that $(x, y) \in \theta$ and $(y, z) \in \theta$. This means $d_v(x, y) = 0$ and $d_v(y, z) = 0$. On the other hand, according to (V8), we have

$$\begin{aligned} 0 \leqslant d_v(x, z) = v(x \cdot z) + v(z \cdot x) &\leqslant v(x \cdot y) + v(y \cdot z) + v(z \cdot y) + v(y \cdot x) \\ &= d_v(x, y) + d_v(y, z) = 0 + 0 = 0. \end{aligned}$$

Thus $d_v(x, z) = 0$. Hence $(x, z) \in \theta$.

Let us show that the relation θ compatible with the multiplication in $\mathfrak{A}$. Let $x, y, z \in A$ be arbitrary elements such that $(x, y) \in \theta$. This means $d_v(x, y) = 0$.

On the other hand, from (JU-1) $(y \cdot z) \cdot (x \cdot z) \leqslant x \cdot y$ and $(x \cdot z) \cdot (y \cdot z) \leqslant y \cdot x$ follow. From here, according to (V5), we get $v(x \cdot y) \geqslant v((y \cdot z) \cdot (x \cdot z))$ and $v(y \cdot x) \geqslant v((x \cdot z) \cdot (y \cdot z))$. Adding these inequalities, we get:

$$\begin{aligned} 0 = d_v(x, y) = v(x \cdot y) + v(y \cdot x) &\geqslant v((y \cdot z) \cdot (x \cdot z)) + v((x \cdot z) \cdot (y \cdot z)) \\ &= d_v(x \cdot z, y \cdot z) \geqslant 0. \end{aligned}$$

Therefore, $(x \cdot z, y \cdot z) \in \theta$.

By corresponding substitution of variables in (JU-1), we get

$$(z \cdot x) \cdot ((x \cdot y) \cdot (z \cdot y)) = 1 \text{ and } (z \cdot y) \cdot ((y \cdot x) \cdot (z \cdot x)) = 1.$$

If we apply (V2) to the variables $z \cdot x$, $x \cdot y$ and $z \cdot y$, we get

$$v((z \cdot x) \cdot (z \cdot y)) \leqslant v((z \cdot x) \cdot ((x \cdot y) \cdot (z \cdot y))) + v(x \cdot y) = v(x \cdot y).$$

Analogously, if we apply (V2) to the variables $z \cdot y$, $y \cdot x$ and $z \cdot x$, we get

$$v((z \cdot y) \cdot (z \cdot x)) \leqslant v((z \cdot y) \cdot ((y \cdot x) \cdot (z \cdot x))) + v(y \cdot x) = v(y \cdot x).$$

Adding these two inequalities, we get

$$\begin{aligned} 0 &\leqslant d_v(z \cdot x, z \cdot y) = v((z \cdot x) \cdot (z \cdot y)) + v((z \cdot y) \cdot (z \cdot x)) \\ &\leqslant v(x \cdot y) + v(y \cdot x) = d_v(x, y) = 0. \end{aligned}$$

So, $(z \cdot x, z \cdot y) \in \theta$. □

It can be shown that the following statement is valid:

Proposition 4.2. *Let v be a (standard) pseudo-valuation on a $\mathfrak{A}$. Then holds*

$$(\forall x, y \in A)((x, y) \in \theta \iff (x \cdot y \in J_v \wedge y \cdot x \in J_v)).$$

Proof. Let $x, y \in A$ be such that $(x, y) \in \theta$. Then $d_v(x, y) = 0$. This means $v(x \cdot y) + v(y \cdot x) = 0$. This is possible only if $v(x \cdot y) = 0$ and $v(y \cdot x) = 0$ since the sum of two non-negative real numbers is equal to 0 if and only if both of these numbers are equal to zero. Thus $x \cdot y \in J_v$ and $y \cdot x \in J_v$. The reverse implication is clear. This proves the proposition. □

The following theorem shows that the core $\{u \in A : (u, 1) \in \theta\}$ of this congruence on $\mathfrak{A}$ is an ideal in $\mathfrak{A}$.

Theorem 6. *Let* $v : A \longrightarrow \mathbb{R}$ *be a pseudo-valuation on a JU-algebra* $\mathfrak{A}$*,* d_v *be an induced pseudo-metric on* $\mathfrak{A}$ *and* θ *be a congruence on* $\mathfrak{A}$ *induced by* d_v*. Then the class* $[1]_\theta =: \{x \in A : (x, 1) \in \theta\}$*, generating by the congruence* θ *on* $\mathfrak{A}$*, is an ideal in* $\mathfrak{A}$*.*

Proof. It is obvious that $1 \in [1]_\theta$ holds due to the reflexivity of the congruence relation θ. Let $x, y, z \in A$ be arbitrary elements such that $x \cdot (y \cdot z) \in [1]_\theta$ and $y \in [1]_\theta$. This means $(x \cdot (y \cdot z), 1) \in \theta$ and $(y, 1) \in \theta$. On the other hand, from $(y, 1) \in \theta$ follows $(x \cdot (y \cdot z), x \cdot (1 \cdot z)) \in \theta$ due to the compatibility of the relation θ with multiplication in $\mathfrak{A}$. From here we get $(x \cdot (1 \cdot y), 1) \in \theta$ due to the transitivity of the relation θ. Thus, $(x \cdot z, 1) \in \theta$ with respect to (JU-2). Therefore, $x \cdot z \in [1]_\theta$. □

Let θ be a congruence relation on a JU-algebra $\mathfrak{A}$ and let $[x]_\theta =: \{y \in A : (x, y) \in \theta\}$. Let us define the operation $*$ in the family $\mathfrak{A}/\theta =: \{[x]_\theta : x \in A\}$ as follows

$$(\forall x, y \in A)([x]_\theta * [y]_\theta =: [x \cdot y]_\theta).$$

Let us show that the multiplication operation $*$, determined in this way in $\mathfrak{A}/\theta$, is well-defined and that $(A/\theta, *, [1]_\theta)$ is a JU-algebra.

Theorem 7. *The operation* $*$ *is well-defined and* $(A/\theta, *, [1]_\theta)$ *is a JU-algebra.*

Proof. Let $x, y, u, v \in A$ be such that $[x]_\theta = [u]_\theta$ and $[y]_\theta = [v]_\theta$. Then $(x, y) \in \theta$ and $(y, v) \in \theta$. From here we get $(x \cdot y, u \cdot y) \in \theta$ and $(u \cdot y, u \cdot v) \in \theta$ due to the computability of the relation θ with the multiplication in A. Hence $(x \cdot y, u \cdot v) \in \theta$ due to the transitivity of the relation θ. Now, for $t \in [x]_\theta * [y]_\theta = [x \cdot y]_\theta$, we have $(t, x \cdot y) \in \theta$. Then $(t, u \cdot v) \in \theta$ due to the transitivity of the relation θ. Thus $t \in [uv]_\theta = [u]_\theta * [v]_\theta$. Hence $[x]_\theta * [y]_\theta \subseteq [u]_\theta * [v]_\theta$. Reverse inclusion can be proved analogously to the previous one. This proves that the operation $*$ is well-defined.

Since the conditions (JU-1) and (JU-2) for $\mathfrak{A}/\theta$ can be proven by a direct verification, it remains to prove the validity of condition (JU-3). Let $x, y \in A$ be such $[x]_\theta * [y]_\theta = [1]_\theta = [y]_\theta * [x]_\theta$. This means $(x \cdot y, 1) \in \theta$ and $(y \cdot x, 1) \in \theta$. Then $(x \cdot y) \cdot 1 \in J_v$, $x \cdot y = 1 \cdot (x \cdot y) \in J_v$, $(y \cdot x) \cdot 1 \in J_v$ and $y \cdot x = 1 \cdot (y \cdot x) \in J_v$ according to (JU-2). This $(x, y) \in \theta$. Hence $[x]_\theta = [y]_\theta$. □

Example 4.1: Let $A = \{1, a, b, c, d\}$ and operation '$\cdot$' defined on A as follows:

·	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then $A = \langle A, \cdot, 1\rangle$ is a JU-algebra ([2]) where the relation '$\leqslant$' is defined as follows

$$\leqslant := \{(1,1), (1,a), (1,b), (1,c), (1,d), (a,a), (b,b), (b,d),$$

$$(c,c),(a,c),(d,d),(c,d),(a,d)\}.$$

Now, define a real-valued function $v : A \longrightarrow \mathbb{R}$ by $v(1) = v(a) = v(b) = 0$, $v(c) = 3$ and $v(d) = 3$. The function v is a pseudo-valuation on A. Then $J_v =: \{u \in A : v(u) = 0\} = \{1, a, b\}$ is the ideal of A induced by v. By direct checking, it can be determined that it is

$$\theta = \{(1,1),(a,1),(b,1),(a,a),(b,b),(c,d),(d,c)\}.$$

Hence $[1]_\theta = \{1, a, b\}$ and $[c]_\theta = \{c, d\}$ and $A/\theta = \{\{1, a, b\}, \{c, d\}\}$. □

Example 4.2: Let A be as in the example 2.3. If we determine $v : A \longrightarrow \mathbb{R}$ as follows $v(1) = v(2) = 0$, $v(3) = 3$, $v(4) = 4$, $v(5) = 1$ and $v(6) = 6$ Then v is a pseudo-valuation on $\mathfrak{A}$. Thus $J_v = \{1, 2\}$ is an JU-ideal in $\mathfrak{A}$. The relation

$$\theta = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(1,2),(3,4)\}$$

is a congruence on $\mathfrak{A}$. Thus $[1]_\theta = \{1, 2\}$, $[3]_\theta = \{3, 4\}$, $[5]_\theta = \{5\}$ and $[6]_\theta = \{6\}$. □

Let v and w be two pseudo-valuations on a JU-algebra $\mathfrak{A}$. They induce the congruences θ and ϑ on $\mathfrak{A}$ according to the Theorem 5, and the corresponding quotient JU-algebras $(A/\theta, *, [1]_\theta)$ and $(A/\vartheta, *, [1]_\vartheta)$, respectively.

Theorem 8. *Let pseudo-valuations v and w induce congruences θ and ϑ on a JU-algebra $\mathfrak{A}$ respectively. If $[1]_\theta = [1]_\vartheta$, then congruences θ and ϑ coincide.*

Proof. Let $x, y \in A$ be arbitrary elements such that $(x, y) \in \theta$. Then $(x \cdot y, 1) = (x \cdot y, y \cdot y) \in \theta$ and $(y \cdot x, 1) = (y \cdot x, y \cdot y) \in \theta$ due to the compatibility of the congruence θ with multiplication in $\mathfrak{A}$. Thus $x \cdot y$, $y \cdot x \in [1]_\theta = [1]_\vartheta$, that is $(x \cdot y, 1)$, $(y \cdot x, 1) \in \vartheta$. Therefore, $[x]_\vartheta * [y]_\vartheta = [x \cdot y]_\vartheta = [1]_\vartheta$ and $[y]_\vartheta * [x]_\vartheta = [y \cdot x]_\vartheta = [1]_\vartheta$. Hence, $[x]_\vartheta = [y]_\vartheta$ due to (JU-3) in the JU-algebra $(A/\vartheta, *, [1]_\vartheta)$. So, the last means $(x, y) \in \vartheta$. With this, the inclusion of $\theta \subseteq \vartheta$ is proved.

Reverse inclusion can be proved in an analogous way. □

Let v be a pseudo-valuation on a JU-algebra $\mathfrak{A}$ and θ induced congruence on $\mathfrak{A}$ by v. If K is a subset of A, we denote $K^v =: \{[u]_\theta : u \in K\}$. We conclude this section with the following two theorems:

Theorem 9. *Let K be an ideal (a filter) of a JU-algebra $\mathfrak{A}$ and let v be an pseudo-valuation map of a such that $[1]_\theta \subseteq K$. Then the following assertions are valid:*

(1) $(\forall x \in A)(x \in K \Longleftrightarrow [x]_\theta \in K^v)$.

(2) *The set K^v is an ideal (a filter) in $\mathfrak{A}/\theta$.*

Proof. (1) It is clear that if $x \in K$, then $[x]_\theta \in K^v$. Let $x \in A$ be such that $[x]_\theta \in K^v$. Then, there exists an element $y \in K$ such that $[x]_\theta = [y]_\theta$. Hence, $(x, y) \in \theta$ and so $(y \cdot x, 1) = (y \cdot x, y \cdot y) \in \theta$ since θ is a congruence. It follows that $y \cdot x \in [1]_\theta \subseteq K$. Now, from $y \cdot x \in K$ and $y \in K$, it follows that $x \in K$ according to (J-2).

(2) It is clear that $[1]_\theta \in K^v$, since $1 \in K$. Let $x, y \in A$ be such that $[x]_\theta * [y]_\theta \in K^v$ and $[x]_\theta \in K^v$. Then $x \in K$ by (1) and $[x \cdot y]_\theta = [x]_\theta * [y]_\theta \in K^v$. Thus $x \cdot y \in K$ by (1) again. Since K is an ideal of $\mathfrak{A}$, it follows that $x \in K$ and so that $[x]_\theta \in K^v$. Therefore, K^v is an ideal of $\mathfrak{A}/\theta$.

If K is a filter in a JU-algebra $\mathfrak{A}$, the proofs of claims (1) and (2) can be demonstrated symmetrically with respect to the proofs when K is an ideal in $\mathfrak{A}$. For illustration, we show the proof of (1):

It is clear that if $x \in K$, then $[x]_\theta \in K^v$. Let $x \in A$ be such that $[x]_\theta \in K^v$. Then, there exists an element $y \in K$ such that $[x]_\theta = [y]_\theta$. Hence, $(x, y) \in \theta$ and so $(x \cdot y, 1) = (x \cdot y, y \cdot y) \in \theta$ since θ is a congruence. It follows that $x \cdot y \in [1]_\theta \subseteq K$. Now, from $x \cdot y \in K$ and $y \in K$, it follows that $x \in K$ according to (F-2). □

Of course, the inverse of the previous theorem can also be proved.

Theorem 10. *If K^* is an ideal (a filter) in $\mathfrak{A}/\theta$, then the set $K = \{x \in A : [x]_\theta \in K^*\}$ is an ideal (a filter) in $\mathfrak{A}$ that contains the core $[1]_\theta$.*

Proof. Suppose that K^* is a filter in a quotient JU-algebra $\mathfrak{A}/\theta$, generated by the pseudo-valuation v on a JU-algebra $\mathfrak{A}$. Let $x, y \in A$ be such that $x \cdot y \in K$ and $y \in K$. Then, $[x]_\theta * [y]_\theta = [x \cdot y]_\theta \in K^*$ and $[y]_\theta \in K^*$. Since K^* is a filter in $\mathfrak{A}/\theta$, it follows $[x]_\theta \in K^*$ from here. Thus, $x \in K$. As it is obvious that $1 \in K$, since $[1]_\theta \in K^*$, we conclude that K is a filter in A. For any $t \in [1]_\theta$, we have $[t]_\theta = [1]_\theta \in K^*$. Therefore, $t \in K$. This proves that $[1]_\theta \subseteq K$.

In the case when K^* is an ideal in $\mathfrak{A}/\theta$, the proof that K is an ideal in $\mathfrak{A}$ is carried out symmetrically to the previous one. □

5. Conclusion

The JU-algebra is a weakened KU-algebra because the axiom $(\forall x \in A)(x \cdot 1 = 1)$ is omitted from the axiomatic system for KU-algebras. This type of algebra was the subject of interest of several authors (for example, see [1, 2]) and this author as well ([11, 13, 14]). Pseudo-valuation maps on JU-algebras were studied by U. Ali, M. A. Ansari and M. Ur Rehman in [1]. The aim of this paper was to study the quotient structures of JU-algebras induced by the pseudo-valuation on them. Additionally, some of the basic properties of this quotient structure were determined.

In the article [1], the concept of quasi-valuation maps on a JU-algebra is based on the concept of ideals in a JU-algebra. Analogously, some other concept of quasi-valuation map $w : A \longrightarrow \mathbb{R}$ on a JU-algebra $\mathfrak{A} = (A, \cdot, 1)$ could be determined on a JU-algebra $\mathfrak{A}$ by basing it, for example, on a filter in A such that the subset $F_w = \{x \in A : w(x) = 1\}$ is a filter in $\mathfrak{A}$. One of our next scientific articles could be a report on the obtained results of such research on JU-algebras.

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Daniel A. Romano
International Mathematical Virtual Institute,
6, Kordunaška Street, 78000 Banja Luka,
Bosnia and Herzegovina
ORCID: 0000-0003-1148-3258
Email address: `daniel.a.romano@hotmail.com`

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