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2-HOP ROMAN DOMINATION OF SOME PRODUCT GRAPHS

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Abstract. In this paper, we introduce the study of 2-hop Roman dominating functions of some product graphs. A 2-hop Roman dominating ($2HRD$) function on a graph $G = (V, E)$ is a function $f : V(G) \to \{0, 1, 2\}$ such that for every vertex v with $f(v) = 0$, there are two vertices x, y with $f(x) = f(y) = 2$ and $d(v, x) = d(v, y) = 2$. The weight of 2HRD function is the sum of its function values over all the vertices. The 2-hop Roman domination number of G denoted by $\gamma_{2hR}(G)$ is the minimum weight of a 2HRD function in G. We present the upper bond of 2-hop Roman domination number of Cartesian product graph. Also, we determine the 2-hop Roman domination number of tensor product graphs.

1. INTRODUCTION

Let $G = (V, E)$ be an undirected simple graph with vertex set V and edge set E. The graph G has order $n = |V|$. The *distance* between two vertices u and v in G, denoted by $d(u, v)$ is the minimum length of a (u, v)-path in G. For every vertex $v \in V$, the *open neighborhood* of v is the set $\{u | uv \in E\}$ denoted by $N(v)$ and the *closed neighborhood* of v is the set $N(v) \cup \{v\}$ denoted as $N[v]$. The cardinality of $N(v)$ is the *degree* of vertex v denoted by $d_G(v) = |N(v)|$. We denote the path, cycle and complete graph with n vertices by P_n, C_n and K_n respectively.

Let G_1 and G_2 be two graphs. The *Cartesian product* of two graphs G_1 and G_2, denoted by $G_1 \Box G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and two vertices $(u_1, v_1), (u_2, v_2) \in G_1 \Box G_2$ are adjacent if either

- $u_1u_2 \in E(G_1)$ and $v_1 = v_2$, or
- $v_1v_2 \in E(G_2)$ and $u_1 = u_2$.

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The *tensor product* $G \times H$ of two graphs G and H is the vertex set $V(G) \times V(H)$ such that two vertices (u_1, v_1) and (u_2, v_2) are adjacent if $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$.

A *dominating set* of G is a set $V \setminus D$ such that every vertex not in D has a neighbor in D. The minimum cardinality of all dominating sets of G is the domination number, denoted as $\gamma(G)$. More on dominating sets can be found in [18].

A subset of $V(G)$ is called a 2-*dominating set* of G if for every vertex $v \in V(G) \setminus S$ is adjacent to at least two vertices in S. The smallest carnality of a 2-dominating set of G is called the 2-domination number of G. Fink and Jacobson [13] introduced the concept of 2-domination number. More results on 2- domination number and its variation can be found in [11, 15, 16].

A set $S \subset V(G)$ is a hop dominating set if for each vertex $u \in V(G) \setminus S$, there exist vertex $v \in S$ such that $d(u, v) = 2$. The minimum cardinality of a hop dominating set is called the hop domination number of G. The concept of hop domination was introduced in [17] and more results on hop domination can be found in [7, 19, 14].

A Roman dominating function of a graph G, denoted as RD-function, is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u with $f(u) = 0$ is adjacent to at least one vertex v with $f(v) = 2$. The weight of a vertex v is its value, $f(v)$, assigned to it under f. The weight, $w(f)$, of f is the sum, $\sum_{u \in V(G)} = f(u)$, of the weights of the vertices. The Roman domination number, denoted $\gamma_R(G)$, is the minimum weight of an RD-function in G; i.e. $\gamma_R(G) = min\{w(f) | f\ is\ an\ RD - function\ in\ G\}$. In [10], Roman domination was first studied, see the papers [2, 3, 6, 8, 9, 12, 21, 22, 23, 25] for more results on Roman domination number and its variation.

A hop Roman dominating function (HRDF) of a graph G is a function $f : V(G) \to \{0, 1, 2\}$ such that for every vertex $u \in V$ with $f(u) = 0$, there is a vertex v with $f(v) = 2$ and $d(u, v) = 2$. The weight $w(f)$ of HRDF f is the sum, $\sum_{v \in V(G)} f(v)$. The minimum weight of a HRDF on G is called the hop Roman domination number of G denoted by $\gamma_{hR}(G)$. See [2, 5, 20, 24] for more details.

Our aim is to apply the concept of 2 dominating function to hop Roman dominating function and establish the variation 2-hop Roman dominating function as follows:

A 2-hop Roman dominating function (2HRD) of a graph G is a function $f : V(G) \to \{0, 1, 2\}$ such that for every vertex v with $f(v) = 0$, there are two vertices x, y with $f(x) = f(y) = 2$ and $d(v, x) = d(v, y) = 2$. The weight $w(f)$ of $2HRD$ function f is the sum, $\sum_{v \in V(G)} f(v)$. The minimum weight of a $2HRD$ function on G is called the 2-hop Roman domination number of G denoted by $\gamma_{2hR}(G)$.

Note that every 2-hop Roman dominating function is a hop Roman dominating function. So, $\gamma_{hR}(G) \leq \gamma_{2hR}(G)$ for every graph.

In Section 2, we present the upper bound for the 2-hop Roman domination number of Cartesian product of paths and paths. In the last section, we determine the 2-hop Roman domination number of tensor product of paths and paths, paths and cycles, complete graphs and complete graphs.

2. 2-HOP ROMAN DOMINATION NUMBER OF CARTESIAN PRODUCT GRAPHS

In this section, we shall consider the 2-hop Roman dominating functions of Cartesian product of paths. Let S_n and BS_n denotes the star graph and bistar graph respectively. We begin with the following results on the Paths, Star and Bistar.

Observation. The $\gamma_{2hR}(P_n)$ is $n+1$.

Theorem 1. *For $n \geq 4$, The $\gamma_{2hR}(S_n) = 5$.*

Proof. Let v_0 be the center vertex of S_n, set $f(v_0) = 1$, $f(x) = f(y) = 2$, where x and y are arbitrary two vertices of degree one in S_n and assign 0 to the remaining vertices in S_n. Thus $\gamma_{2hR}(S_n) = 5$.

Assume that there exist 2HRD function f on S_n such that $w(f) \leq 4$. Let P be the set of vertices with label 0 and H be the set of vertices with label 2. Since $w(f) \leq 4$ and f is 2HRD, we have that $|P| \geq 1$ and $|H| \geq 2$. If $|P| \geq 1$ and $|H| \geq 2$ with $f(v_0) = 1$, then $w(f) > 4$, a contradiction. Also, assume that $w(f) > 5$. We split the problem into the following cases.

Case 1: Set $f(v_0) = 1$, $|P| \geq 1$ and $|H| > 2$. Each vertex in P will be at distance 2 to more than 2 vertices in H, which will give 2HRD function but not with minimum weight.

Case 2: Set $f(v_0) = 1$, $|P| \geq 1$ and $|H| \geq 2$ and set $f(y) = 1$, where y is a vertex with degree one in S_n. The labelling will give 2HRD with $w(f) > 5$ which is not the minimum. Hence $\gamma_{2hR}(S_n) = 5$. □

Theorem 2. *For $n \geq 8$, $\gamma_{2hR}(BS_n) = 8$*

Proof. Let u_0, v_0 be the center vertices of BS_n. Set $f(u_0) = f(v_0) = 2$ and $f(x) = f(y) = 2$, where x and y are arbitrary two vertices of degree one in BS_n such that xu_0 and yv_0 are edges in BS_n. Assign zero (0) to the remaining vertices in BS_n. Thus $\gamma_{2hR}(BS_n) = 8$. Assume that $w(f) \leq 7$ for a $2HRD$ function f on BS_n. Let P and H be the set of vertices with label zero(0) and two(2) respectively. We have that $|P| \geq 1$ and $|H| \geq 2$ since f is a $2HRD$ function. If $|P| \geq 1$ and $|H| \geq 2$ with $f(u_0) = f(v_0) = 2$, then $w(f) > 7$, a contradiction. Hence $w(f) \nleq 7$. Also, assume that $w(f) > 8$. We split the problem into the following cases.

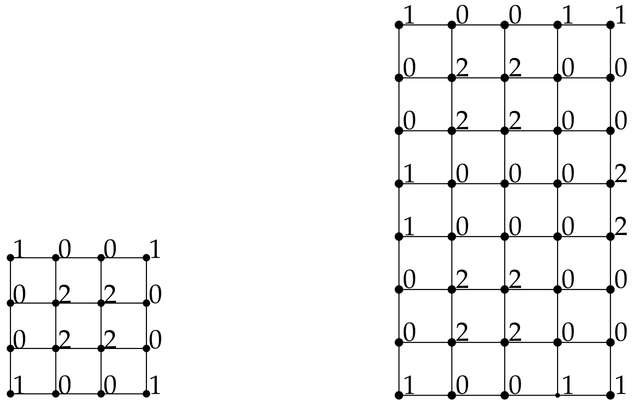
Case 1: Set $f(v_0) = f(u_0) = 1$, $|P| \geq 1$ and $|H| > 2$. Each vertex in P will be at distance 2 to more than 2 vertices in H, which will give 2HRD function but not with minimum weight.

Case 2: Set $f(v_0) = 1$, $|P| \geq 1$ and $|H| \geq 2$ and set $f(w) = 1$, where w is a vertex with degree one in BS_n. The labeling will give 2HRD with $w(f) > 8$ which is not the minimum. Hence $\gamma_{2hR}(BS_n) = 8$. □

The next result gives the upper bound for the 2-hop Roman domination number of Cartesian product of path and path.

Theorem 3. *Let $n, m > 4$ be positive integers. If $G = P_n \Box P_m$, then $\gamma_{2hR}(G) \leq \frac{3}{4}|G|$.*

Proof. Let a_{ij} denotes the vertex in the ith row and jth column in the graph $P_n \Box P_m$, such that $1 \leq i \leq n$ and $1 \leq j \leq m$. The problem will be split into the following cases:



(a) $n, m \equiv 0 \mod 4$ (b) $n \equiv 0 \mod 4,\ m \equiv 1 \mod 4$

FIGURE 1. The function f on $P_n \square P_m$, where $n, m \equiv (0 \mod 4)$ in (a) and $n \equiv (0 \mod 4), m \equiv (1 \mod 4)$ in (b)

Case 1.: $n \equiv 0(mod 4)$ and m a positive integer.

Subcase 1.1.: $n, m \equiv 0(mod\ 4)$. Define a function $f : V(G) \to \{0, 1, 2\}$ as follows (see Figure 1(a)):

$$f(a_{ij}) = \begin{cases} 1, & \text{if } i, j \equiv 0 \text{ or } 1 \text{ (mod 4)} \\ 2, & \text{if } i, j \equiv 2 \text{ or } 3 \text{ (mod 4)} \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that the function f produces 2-hop Roman dominating function with weight $w(f) \leq \frac{3}{4}nm = \frac{3}{4}|G|$.

Subcase 1.2.: $n \equiv 0(mod\ 4)$ and $m \equiv 1(mod\ 4)$. Define a function $f' : V(G) \to \{0, 1, 2\}$ as follows (see Figure 1(b)):

$$f'(a_{ij}) = \begin{cases} 1, & \text{if } i = 1, j = m-1, m \\ 2, & \text{if } i \equiv 0 \text{ or } 1 \text{ (mod 4)}, i \neq 1, n \text{ and } j = m \\ 0, & \text{if } i \equiv 0 \text{ or } 1 \text{ (mod 4)}, i \neq 1, n \text{ and } j = m-1 \\ f(a_{ij}), & \text{otherwise.} \end{cases}$$

Then f' is a $2HRD$ function on G. If $j \equiv 0$ or $1(mod\ 4)$, $\sum_{j=1}^{m-5} f'(a_{ij}) = \frac{n}{4}(m-5)$, if $j \equiv 2$ or $3(mod\ 4)$, $\sum_{j=2} f'(a_{ij}) = n\frac{m-1}{2}$, if $j = m-4$, $\sum f'(a_{ij}) = \frac{n}{2}$, if $j = m-1$, $\sum f'(a_{ij}) = 2$, if $j = m$, $\sum f'(a_{ij}) = n-2$, therefore,

$$\begin{aligned} w(f') &= \frac{n(m-5)}{4} + \frac{n(m-1)}{2} + \frac{n}{2} + 2 + n - 2 \\ &= \frac{nm + 2nm}{4} - \frac{5n}{4} + n \end{aligned}$$

$$\begin{aligned} &= \frac{3nm}{4} - \frac{n}{4} \\ &\leq \frac{3nm}{4} = \frac{3}{4}|G|. \end{aligned}$$

Subcase 1.3.: $n \equiv 0(mod\ \ 4)$ and $m \equiv 2(mod\ \ 4)$. Define a function $f' : V(G) \rightarrow \{0, 1, 2\}$ as follows:

$$f'(a_{ij}) = \begin{cases} 1, & \text{if } (i = 1, n \text{ and } m - 2 \leq j \leq m) \text{ and} \\ & (i = 2, n - 1 \text{ and } j = m) \\ 2, & \text{if } i \equiv 0 \text{ or } 1 \text{ (mod 4)}, 1 \leq i \leq n \text{ and } j = m - 1, m \\ 0, & \text{if } i \equiv 0 \text{ or } 1 \text{ (mod 4)}, 1 \leq i \leq n \text{ and } j = m - 2 \\ f(a_{ij}), & \text{otherwise.} \end{cases}$$

The function f' gives a $2HRD$ on G. If $j \equiv 0 \text{ or } 1(mod4)$, $\sum_j f'(a_{ij}) = \frac{n}{4}(m-6)$, if $j \equiv 2 \text{ or } 3(mod4)$, $\sum_j f'(a_{ij}) = \frac{n}{2}(m-6)$, if $j = m-5$, $\sum f'(a_{ij}) = \frac{n}{2}$, if $m - 4 \leq j \leq m-3$, $\sum_j f'(a_{ij}) = 2n$, if $j = m-2$, $\sum f'(a_{ij}) = 2$, if $j = m-1, m$, $\sum_j f'(a_{ij}) = 2n - 2$, hence,

$$\begin{aligned} w(f') =& \frac{n}{4}(m-6) + \frac{n}{2}(m-6) + \frac{n}{2} + 4n \\ &= \frac{3n(m-6)}{4} - \frac{9n}{2} \\ &\leq \frac{3nm}{4} = \frac{3}{4}|G|. \end{aligned}$$

Subcase 1.4.: $n \equiv 0(mod\ \ 4)$ and $m \equiv 3(mod\ \ 4)$. Define a function $f' : V(G) \rightarrow \{0, 1, 2\}$ as follows:

$$f'(a_{ij}) = \begin{cases} 1, \text{ if} & (i, j \equiv 0 \text{ or } 1 \text{ (mod 4) with } j \leq m - 6) \text{ and} \\ & (i \equiv 0 \text{ or } 1 \text{ mod } 4, j = m) \\ 2, \text{ if} & (i, j \equiv 2 \text{ or } 3 \text{ (mod 4), and } j \leq m - 3) \text{ and} \\ & (i \equiv 2 \text{ or } 3 \text{ (mod 4), and } m - 2 \leq j \leq m - 1) \\ 0, & \text{otherwise.} \end{cases}$$

Thus the function f' gives a $2HRD$ on G. If $j \equiv 0 \text{ or } 1(mod\ \ 4)$, $\sum_j f'(a_{ij}) = \frac{n}{4}(m-7)$, if $j \equiv 2 \text{ or } 3(mod\ \ 4)$, $\sum_j f'(a_{ij}) = \frac{n}{2}(m-3)$, if $j = m-6$, $\sum f'(a_{ij}) = n$, if $j = m-2, m-1$, $\sum_j f'(a_{ij}) = 2n$, then,

$$\begin{aligned} w(f') =& \frac{n}{4}(m-7) + \frac{n}{3}(m-3) + 3n \\ &= \frac{3nm}{4} - \frac{n}{4} \\ &\leq \frac{3nm}{4} = \frac{3}{4}|G|. \end{aligned}$$

Case 2.: $n \equiv 1(mod4)$ and m a positive integer.

Subcase 2.1.: $n \equiv 1(mod\ \ 4)$ and $m \equiv 0(mod\ \ 4)$. Apply subcase (1.2) above by interchanging n and m.

Subcase 2.2.: $n \equiv 1(mod\ \ 4)$ and $m \equiv 1(mod\ \ 4)$. Define a function $f : V(G) \to \{0, 1, 2\}$ as follows:

$$f^*(a_{ij}) = \begin{cases} 0, \text{if} & (i \equiv 0 \text{ or } 1 \text{ (mod 4) with } i \neq 1, n, j = m-1) \text{ and} \\ & (i = n-1, j \equiv 0 \text{ or } 1 \text{ (mod 4)}, j \neq 1, m), \\ 2, \text{ if} & (i = n, j \neq 1, m-1 \text{ and } j \equiv 0 \text{ or } 1 \text{ (mod 4)}) \\ & (i \equiv 0 \text{ or } 1 \text{ (mod 4)}), i \neq 1, j = m \\ f(a_{ij}), & \text{otherwise.} \end{cases}$$

The functions f^* given above, is a $2HRD$ function on G. The weight is analyze below. If $j \equiv 0$ or $1(mod4)$, $\sum_j^{m-5} f^*(a_{ij}) = (\frac{n+1}{2})(\frac{m-5}{2})$, if $j \equiv 2$ or $3(mod4)$, $\sum_j f^*(a_{ij}) = (n-1)\frac{m-1}{2}$, if $j = m-4$, $\sum f^*(a_{ij}) = \frac{n+1}{2}$, if $j = m -$ ‘1, $\sum_j (f^*(a_{ij}) = 2$, if $j = m$, $\sum_j f^*(a_{ij}) = n$, then, the weight of function f^* is given as:

$$\begin{aligned} w(f^*) =& \frac{n+1}{4}(m-5) + \frac{n-1}{2}(m-1) + \frac{n+1}{2} + 2 + n \\ =& \frac{3nm}{4} - \frac{n}{4} - \frac{m}{4} + \frac{7}{4} \\ =& \frac{3nm}{4} - \frac{n+m-7}{4} \\ \leq& \frac{3nm}{4} = \frac{3}{4}|G|, \ \ \textit{provided that} \ \ n+m > 7. \end{aligned}$$

Subcase 2.3.: $n \equiv 1(mod\ \ 4)$ and $m \equiv 2(mod\ \ 4)$. Define a function $f' : V(G) \to \{0, 1, 2\}$ as follows:

$$f'(a_{ij}) = \begin{cases} 0, & \text{if } i = n-1 \text{ and } j > 1 \text{ with } j \equiv 0 \text{ or } 1 \text{ (mod 4)} \\ 1, & \text{if } i = 1 \text{ and } j = m \\ 2, & \text{if } i = n \text{ and } j \equiv 0 \text{ or } 1 \text{ (mod 4)}, j > 1 \\ f(a_{ij}), & \text{otherwise.} \end{cases}$$

Thus f' is a $2HRD$ function on G. If $j \equiv 0$ or $1(mod4)$, $\sum_{j=1}^{m-5} f'(a_{ij}) = (\frac{n-1}{2} + 1)(\frac{m-2}{2}) - 1$, if $j \equiv 2$ or $3(mod4)$, $\sum_j f'(a_{ij}) = (n-1)\frac{m-2}{2}$, if $j = m-4$, $\sum f'(a_{ij}) = \frac{n}{2}$, if $j = m-2$, $\sum_j f'(a_{ij}) = \frac{n-1}{2} + 2$, if $j = m$, $\sum_j f'(a_{ij}) = n-1$, hence,

$$\begin{aligned} w(f') &= \left(\frac{n+1}{2}\right)\left(\frac{m-2}{2}\right) + (n-1)\frac{m-2}{2} + \frac{n+1}{2} + n - 2 \\ &= \frac{3nm}{4} - \frac{m}{4} \leq \frac{3nm}{4} = \frac{3}{4}|G|. \end{aligned}$$

Subcase 2.4.: $n \equiv 1(mod\ \ 4)$ and $m \equiv 3(mod\ \ 4)$. Define a function $f' : V(G) \to \{0, 1, 2\}$ as follows:

$$f'(a_{ij}) = \begin{cases} 0, \text{if} & (i = n-1 \text{ and } j > 1 \text{ with } j \equiv 0 \text{ or } 1 \text{ (mod 4)}), \\ & (i \equiv 0 \text{ or } 1 \text{ (mod 4) with } 1 < i < n \text{ and } j = m-2) \text{ and} \\ & (i \equiv 2 \text{ or } 3 \text{ (mod 4) with } 1 < i < n-1 \text{ and } j = m) \\ 1, \text{if} & (i = 1 \text{ and } j = m-1, m) \text{ and} \\ & (i = n-1, n \text{ and } j = m) \\ 2, \text{if} & (i = n \text{ and } j \equiv 0 \text{ or } 1 \text{ (mod 4)}, j > 1) \text{ and} \\ & (i \equiv 0 \text{ or } 1 \text{ (mod 4) with } 1 < i < n \text{ and } j = m) \\ f(a_{ij}), & \text{otherwise.} \end{cases}$$

Therefore, f' is a $2HRD$ function on G. If $j \equiv 0 \text{ or } 1(mod 4)$, $\sum_{j=1}^{m-3} f'(a_{ij}) = (\frac{n+1}{2})(\frac{m-3}{2})$, if $j \equiv 2 \text{ or } 3(mod 4)$, $\sum_j f'(a_{ij}) = (n-1)\frac{m-3}{2}$, if $j = m-2$, $\sum_j f'(a_{ij}) = 3$, if $j = m-1$, $\sum_j f'(a_{ij}) = n$, if $j = m$, $\sum_j f'(a_{ij}) = n-2$. Hence,

$$\begin{aligned} w(f') &= \left(\frac{n+1}{2}\right)\left(\frac{m-3}{2}\right) + (n-1)\frac{m-3}{2} + 2n + 1 \\ &= \frac{3nm}{4} - \frac{n}{4} - \frac{m}{4} + \frac{7}{4}. \\ &= \frac{3nm}{4} - \frac{1}{4}(n + m - 7), \text{ provided } n + m > 7 \\ &\leq \frac{3nm}{4} = \frac{3}{4}|G|. \end{aligned}$$

Case 3.: $n \equiv 2(mod 4)$ and m a positive integer.

Subcase 3.1.: $n \equiv 2(mod\ \ 4)$ and $m \equiv 0(mod\ \ 4)$. Apply subcase (1.3) above by interchanging n and m. This gives the desired result.

Subcase 3.2.: $n \equiv 2(mod\ \ 4)$ and $m \equiv 1(mod\ \ 4)$. Apply subcase (2.3) above by interchanging n and m.

Subcase 3.3.: $n \equiv 2(mod\ \ 4)$ and $m \equiv 2(mod\ \ 4)$. Define a function $f' : V(G) \to \{0, 1, 2\}$ as follows:

$$f'(a_{ij}) = \begin{cases} 0, \text{if} & (i \equiv 0 \text{ or } 1 \text{ (mod 4) and } j = m-2) \text{ and} \\ & (i \not\equiv 1 \text{ or } 2 \text{ (mod 6) and } j = m) \\ 1, \text{if} & i = n \text{ and } j = 1 \\ 2, \text{ if} & (i > 1 \text{ with } i \equiv 0 \text{ or } 1 \text{ (mod 4), and } j \leq m-1) \text{ and} \\ & (i \equiv 1 \text{ or } 2 \text{ (mod 6), and } j = m) \\ f(a_{ij}), & \text{otherwise.} \end{cases}$$

Then, the function f' gives a $2HRD$ on G. If $j \equiv 0 \text{ or } 1(mod 4)$, $\sum_j f'(a_{ij}) = (\frac{n}{2})(\frac{m-4}{2}) + 1$, if $j \equiv 2 \text{ or } 3(mod 4)$, $\sum_j f'(a_{ij}) = n(\frac{m-2}{2})$, if $j = m-2$, $\sum f'(a_{ij}) = 0$, if $j = m-1$ $\sum_j f'(a_{ij}) = n-1$, if $j = m$ $\sum_j f'(a_{ij}) = n-2$, thus,

$$w(f') = \frac{n}{4}(m-4) + \frac{n}{2}(m-2) + 2n - 2$$

$$= \frac{3nm}{4} - 2 \leq \frac{3nm}{4} = \frac{3}{4}|G|.$$

Subcase 3.4.: $n \equiv 2(mod\ \ 4)$ and $m \equiv 3(mod\ \ 4)$. Define a function $f' : V(G) \to \{0, 1, 2\}$ as follows:

$$f'(a_{ij}) = \begin{cases} 1, \text{if} & (i = n \text{ and } j = 1, m), \\ & (i, j \not\equiv 0 \text{ or } 1 \text{ (mod 4) and } j \leq m - 6) \text{ and} \\ & (i, j \not\equiv 0 \text{ or } 1 \text{ (mod 4) and } j = m) \\ 2, \text{ if} & (i, j \equiv 2 \text{ or } 3 \text{ (mod 4), and } j \leq m - 3) \text{ and} \\ & (i \equiv 2 \text{ or } 3 \text{ (mod 4), and } m - 2 \leq j \leq m - 1) \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the function f' gives a $2HRD$ on G. If $j \equiv 0 \text{ or } 1(mod\ \ 4)$, $\sum_j f'(a_{ij}) = (\frac{n}{2})(\frac{m-5}{2}) + 1$, if $j \equiv 2 \text{ or } 3(mod\ \ 4)$, $\sum_j f'(a_{ij}) = (\frac{n}{2})(m-3)$, if $j = m-2, m-1$, $\sum f'(a_{ij}) = 2n$, if $j = m$, $\sum_j f'(a_{ij}) = \frac{n}{2} + 1$. Hence,

$$w(f') = \frac{n}{2}\left(\frac{m-5}{2}\right) + \frac{n}{2}(m-2) + 2n + 2 = \frac{n}{2}\left(\frac{3m-9}{2}\right) + 2n + 2$$
$$= \frac{3nm}{4} - \frac{n}{4} + 2 \leq \frac{3nm}{4} = \frac{3}{4}|G|, \text{ provided that } n > 6.$$

Case 4.: $n \equiv 3(mod 4)$ and m a positive integer.

Subcase 4.1.: $n \equiv 3(mod\ \ 4)$ and $m \equiv 0(mod\ \ 4)$. Apply subcase (1.4) above by interchanging n and m. This gives the desired result.

Subcase 4.2.: $n \equiv 3(mod\ \ 4)$ and $m \equiv 1(mod\ \ 4)$. Apply subcase (2.4) above by interchanging n and m.

Subcase 4.3.: $n \equiv 3(mod\ \ 4)$ and $m \equiv 2(mod\ \ 4)$. Apply subcase (3.4) above by interchanging n and m.

Subcase 4.4.: $n \equiv 3(mod\ \ 4)$ and $m \equiv 3(mod\ \ 4)$. Define a function $f' : V(G) \to \{0, 1, 2\}$ as follows:

$$f'(a_{ij}) = \begin{cases} 1, \text{if} & (i, j \equiv 0 \text{ or } 1 \text{ (mod 4) and } j < m - 4), \\ & (i = 1 \text{ and } m - 3 \leq j \leq m - 1), \\ & (i = n \text{ and } j = m - 1), \text{ and} \\ & (i \neq 3 \text{ with } i \equiv 2 \text{ or } 3 \text{ (mod 4) and } j = m) \\ 2, \text{ if} & (i, j \equiv 2 \text{ or } 3 \text{ (mod 4), and } j < m - 4) \\ & (i = 1 \text{ and } j = m) \text{ and} \\ & (i > 1 \text{ with } i \equiv 0 \text{ or } 1 \text{ (mod 4), and } m - 1 \leq j \leq m - 2) \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the function f' gives a $2HRD$ on G. If $j \equiv 0 \text{ or } 1(mod\ \ 4)$, $\sum_j^{m-6} f'(a_{ij}) = (\frac{n-1}{2})(\frac{m-5}{2})$, if $j \equiv 2 \text{ or } 3(mod\ \ 4)$, $\sum_j^{m-4} f'(a_{ij}) = (n+1)(\frac{m-3}{2})$, if $j = m-3$, $\sum f'(a_{ij}) = 1$, if $j = m-2$, $\sum_j f'(a_{ij}) = n-2$, if $j = m-1$, $\sum f'(a_{ij}) = n-1$, if

$j = m,\ \sum_j f'(a_{ij}) = \frac{n+3}{2}$. Therefore,

$$\begin{aligned} w(f') =& \frac{n-1}{2}\left(\frac{m-5}{2}\right) + (n+1)\frac{m-3}{2} + \frac{5n+3}{2} - 2 \\ =& \frac{3nm}{4} - \frac{nm}{4} + \frac{m-3}{4} = \frac{3nm}{4} - \frac{1}{4}(n-m+3) \\ \leq& \frac{3nm}{4} = \frac{3}{4}|G|, \text{ provided that } n-m \geq 3. \end{aligned}$$

□

3. 2-HOP ROMAN DOMINATION NUMBER OF TENSOR PRODUCT GRAPHS

In this section, we shall consider the 2-hop Roman domination number of tensor product of some graphs. We begin with the following result on the tensor product of paths and paths.

Theorem 4. *Let $n, m > 4$ be positive integers. If $G = P_n \times P_m$, then $\gamma_{2hR}(G) = n(m-1)$.*

Proof. Let the partite sets (row) of the n-partite graph $P_n \times P_m$ be u_{ij}, $1 \leq i \leq n,\ 1 \leq j \leq m$. We assume that the vertices having the same subscript j are the corresponding vertices of the same set. Let u_{ij} denotes the vertex in the ith row and jth column. Note that vertices on the same row or column are not adjacent to each other which follows from the definition of tensor product graphs. Next, we split the problem into the following cases for all n.

Case 1: For all positive integer n, with $m \equiv 0 \pmod 4$. Define a function $f : V(G) \to \{0,1,2\}$ as follows (see Figure 2):

$$f(u_{ij}) = \begin{cases} 1, & \text{if } 1 \leq i \leq n, m-3 \leq j \leq m \\ 2, & \text{if } 1 \leq i \leq n, j \equiv 2 \text{ or } 3 \text{ (mod 4)} \\ 0, & \text{otherwise .} \end{cases}$$

Thus the function f gives a $2HRD$ on G. If $j \equiv 2 \ or \ 3 \ (mod \ 4)$, $\sum_{j=1}^{m-3} f(u_{ij}) = 4n(\frac{m}{4} - 1)$. If $m-3 \leq j \leq m$, $\sum_j f(u_{ij}) = 3n$. Then,

$$w(f) = 4n\left(\frac{m}{4} - 1\right) + 3n = nm - n = n(m-1).$$

Case 2: For all positive integer n, with $m \equiv 1 \pmod 4$. Define a function $f : V(G) \to \{0,1,2\}$ as follows:

$$f(u_{ij}) = \begin{cases} 2, & \text{if } 1 \leq i \leq n, j \equiv 2 \text{ or } 3 \text{ (mod 4)} \\ 0, & \text{otherwise .} \end{cases}$$

Thus the function f gives a $2HRD$ on G. If $j \equiv 2 \ or \ 3 \ (mod \ 4)$, $\sum_{j=1} f(u_{ij}) = 4n(\frac{m}{4} - 1)$. Then,

$$w(f) = 4n\left(\frac{m-1}{4}\right) = n(m-1).$$

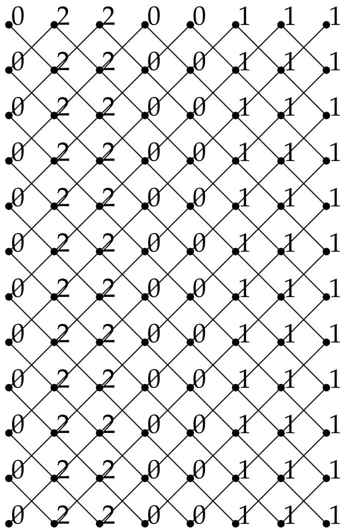


FIGURE 2. The function f on $P_n \times P_m$, where $n, m \equiv 0 \mod 4$ with $n > m$.

Case 3: For all positive integer n, with $m \equiv 2 \pmod 4$. Define a function $f : V(G) \to \{0,1,2\}$ as follows:

$$f(u_{ij}) = \begin{cases} 1, & \text{if } 1 \leq i \leq n, j = m \\ 2, & \text{if } 1 \leq i \leq n, j \equiv 2 \text{ or } 3 \text{ (mod 4)} \\ 0, & \text{otherwise} . \end{cases}$$

Thus the function f gives a $2HRD$ on G. If $j \equiv 2 \; or \; 3 \; (mod \; 4)$, $\sum_{j=1}^{m-1} f(u_{ij}) = 4n(\frac{m}{4} - 1)$. If $j = m$, $\sum_j f(u_{ij}) = n$. Then,

$$w(f) = 4n\left(\frac{m-2}{4}\right) + n = nm - n = n(m-1).$$

Case 4: For all positive integer n, with $m \equiv 3 \pmod 4$. Define a function $f : V(G) \to \{0,1,2\}$ as follows:

$$f(u_{ij}) = \begin{cases} 1, & \text{if } 1 \leq i \leq n, j = m-1, m \\ 2, & \text{if } 1 \leq i \leq n, j \equiv 2 \text{ or } 3 \text{ (mod 4)} \\ 0, & \text{otherwise} . \end{cases}$$

Thus the function f gives a $2HRD$ on G. If $j \equiv 2 \; or \; 3 \; (mod \; 4)$, $\sum_{j=1}^{m-2} f(u_{ij}) = 4n(\frac{m-3}{4})$. If $j = m-1, m$, $\sum_j f(u_{ij}) = 2n$. Then,

$$w(f) = 4n(\frac{m-3}{4}) + 2n = nm - n = n(m-1).$$

To show that $\gamma_{2hR}(G) = n(m-1)$. Assume that G admits a $2HRD$ function g with weight $w(g) < n(m-1)$. Let x be an arbitrary vertex in G. Then the following holds:

(1) $g(x) \cap \{0\} \neq \phi$ for $f(x) = 1$;
(2) $g(x) \cap \{0\} \neq \phi$ for $f(x) = 2$.
(3) $g(x) \cap \{1\} \neq \phi$ for $f(x) = 2$.

If (1) holds, then there exist two vertices u, v such that $d(x, u) = d(x, v) = 2$ and $g(u) = g(v) = 2$. Therefore,

$$w(g) = w(f) - 1 + 4 = n(m-1) - 1 + 4 = n(m-1) + 3 > n(m-1).$$

This is a contradiction to the hypothesis. If (2) holds, then there exist vertex $\{y\} \in G$ such that $g(y) = 1$, with $d(x, y) = 2$ and $f(y) = 0$. Also, there exist two vertices u, v such that $d(x, u) = d(x, v) = 2$ and $g(u) = g(v) = 2$. Thus,

$$\begin{aligned} w(g) &= w(f) - 2 + \{g(y)\} + 4 = n(m-1) - 2 + \{g(y)\} + 4 \\ &= n(m-1) + \{g(y)\} + 2 > n(m-1) \text{ since } \{g(y)\} \geq 1. \end{aligned}$$

This is a contradiction. If (3) holds, then there exist vertex $\{y\} \in G$ such that $g(y) = 1$, with $d(x, y) = 2$ and $f(y) = 0$. Therefore,

$$\begin{aligned} w(g) &= w(f) - 1 + \{g(y)\} = n(m-1) + \{g(y)\} - 1 \\ &= n(m-1) + \{g(y)\} - 1 > n(m-1) \text{ since } \{g(y)\} > 1. \end{aligned}$$

This is a contradiction to the hypothesis. Also, assume that G admits a $2HRD$ function g with weight $w(g) > n(m-1)$. Let x be an arbitrary vertex in G. Then the following holds:

(1) $g(x) \cap \{1\} \neq \phi$ for $f(x) = 0$;
(2) $g(x) \cap \{2\} \neq \phi$ for $f(x) = 1$.

If (1) holds, then $g(u) = g(v) = 1$ whenever $f(u) = f(v) = 2$ and $d(x, u) = d(x, v) = 2$. Therefore,

$$w(g) = w(f) + 1 - 2 = n(m-1) + 1 - 2 = n(m-1) - 1 < n(m-1).$$

This is a contradiction to the hypothesis. If (2) holds, there exist vertex $\{y\} \in G$ such that $g(y) = 2$ whenever $f(x) = 1$. This implies that there exist a vertex x in G with $g(x) = 0$ and at distance two to more than two(2) vertices which will not give the minimum weight. This contradict the definition of $2HRD$ number. Hence, $w(g) = n(m-1)$. Therefore, $\gamma_{2hR}(G) = n(m-1)$. □

Furthermore, the next result gave the $2HRD$ of tensor product of paths and cycles.

Theorem 5. *Let n, m be any positive integers such that $m > n$ and $m > 5$, then $\gamma_{2hR}(P_n \times C_m) = n(m-2)$.*

Proof. Let n, m positive integers with $m > 5$. Define a function $f' : V(G) \to \{0, 1, 2\}$ as follows:

$$f'(u_{ij}) = \begin{cases} 1, & \text{if } i = 1, n, \text{ and } j = m \\ f(u_{ij}), & \text{otherwise .} \end{cases}$$

Therefore, the function f' gives a $2HRD$ on G. Applying Theorem 4 in all cases gives the desired result. ☐

The following result gave the $2HRD$ number of tensor product of complete graphs.

Theorem 6. *Let m, n be positive integers , then $\gamma_{2HR}(K_n \times K_m) = m + 2n$.*

Proof. Let $\alpha = \{0, 1, 2\}$ and for any vertex v_{ij} in $K_n \times K_m$, we denote the set of vertices with $f(v_{ij}) = \alpha$ by $V_{ij(\alpha)}$. For arbitrary row i and column $j, j+1$, let $f(v_{ij}) = 2$. Also, let $f(v_{ij}) = 1$ whenever v_{ij} is in the same row or column with $V_{ij(2)}$. Lastly, assign zero to the remaining vertices of the graph $K_n \times K_m$. Clearly, the function f gives a $2HRD$ of $K_n \times K_m$ since all the vertices are adjacent except the vertices that belongs to the the same row and column, this follows from the definition of the tensor product graphs $K_n \times K_m$. Thus,

$$\begin{aligned} |V_{ij(2)}| &= 2 \\ |V_{ij(1)}| &= m - 2 + 2n - 2 = m + 2n - 4 \\ |V_{ij(0)}| &= nm - m - 2n + 2 = m(n-1) - 2(n-1) \\ &= (n-1)(m-2). \end{aligned}$$

Therefore,

$$w(f) = \sum_j V_{ij(2)} + \sum_j V_{ij(1)} = 4 + m + 2n - 4 = m + 2n.$$

Next, we establish that $\gamma_{2hR}(G) = m + 2n$. Assume that G admits a $2HRD$ function g with weight $w(g) < m + 2n$. Let x be an arbitrary vertex in G. Then the following holds:

(1) $g(x) \cap \{0\} \neq \phi$ for $f(x) = 1$.
(2) $g(x) \cap \{0\} \neq \phi$ for $f(x) = 2$.
(3) $g(x) \cap \{1\} \neq \phi$ for $f(x) = 2$.

Applying the arguments in Theorem 4, gives the desired result. Also, assume that G admits a $2HRD$ function g with weight $w(g) > m + 2n$. Let x be an arbitrary vertex in G. Then the following holds:

(1) $g(x) \cap \{1\} \neq \phi$ for $f(x) = 0$.
(2) $g(x) \cap \{2\} \neq \phi$ for $f(x) = 1$.

Applying the arguments in Theorem 4, gives the desired result. Hence, $\gamma_{2hR}(G) = m + 2n$. ☐

Corollary 6.1. *Let $n, m \geq 4$ be positive integers and G and H be two graphs, then $\gamma_{2hR}(G \square H) \leq \gamma_{2hR}(G \times H)$.*

Proof. The result follows from Theorems 3 and 4. □

In conclusion, we observe that the 2-hop Roman domination number depends on the size of the graph. That is, the higher the size of the graph, the lower the 2-hop Roman domination number.

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