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NOTES ON RECENT ADVANCES ON BERGMAN-TYPE PROJECTIONS IN FUNCTION SPACES OF SEVERAL VARIABLES

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Abstract. The intention of this survey is to collect in a single paper many recent results and advances related to Bergman-type projections acting on various spaces of analytic functions of several complex variables in the unit ball, tubular domains over symmetric cones, and bounded strongly pseudoconvex domains.

Various new and interesting extensions of classical results on Bergman projections are presented in our survey. All these results were previously obtained by the first author in various papers. Bergman-type projections have many important applications in the complex function theory of several complex variables in tubular domains over symmetric cones and in bounded strongly pseudoconvex domains. Our results can be seen as direct extensions of previously known results of E. Stein, D. Bekolle, D. Debertol, B. F. Sehba, W. S. Cohn, C. Nana, L. Chen, Sh. Zhang and others, who related function spaces of the same dimension.

In this paper, various new and interesting problems in this research area will also be formulated and posed by the authors. Theorems on Bergman-type projections between analytic function spaces of different dimensions over domains in $\mathbb{C}^n$ may be viewed as a new research area and may have important applications in complex function theory.

1. Introduction, basic facts and assertions on tubular and bounded pseudoconvex domains

The Bergman projection is a classical topic in complex function theory of one and several complex variables. Various aspects and applications of this research area are well known. Recently, new and interesting applications of integral operators of this type were obtained by the first author. In addition, over the last

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two decades the first author has introduced several new scales of function spaces in product domains, spaces of several complex variables with the mixed norm in the unit ball, tubular domains, and bounded strongly pseudoconvex domains, as well as the new analytic Herz spaces in such domains. He has also studied the action of Bergman-type projections and their direct generalizations on these analytic function spaces of several complex variables.

Various significant results on Bergman-type projections or their direct generalizations were also obtained recently in very general Siegel domains of second type by various authors in various Bergman-type analytic spaces of several variables. For these results, we refer the reader to [23] and [25] and various references there. Some results are known for the action of the Bergman projection in multifunctional spaces or multifunctional expressions. We refer the reader, for example, to [12] and [13], as well as to [37] and various references there. All these results were applied later to obtain so-called new sharp decomposition theorems for multifunctional analytic function spaces in several complex variables. We refer the reader to [34], [21], [22], [28], [29], [30] and many other references on this important topic of complex function theory related to the Bergman projection. We also discuss several interesting open problems directly related to this topic.

We define below basic objects and provide basic facts of function theory in tubular domains over the symmetric cones and in bounded strongly pseudoconvex domains, which are essential for the formulation of all our main theorems on projections in tubular and bounded strongly pseudoconvex domains.

A subset Ω of $\mathbb{R}^n$ (or of V with $dim V = n$) is called a cone if $\Lambda x \in \Omega$, holds for all $x \in \Omega, \Lambda > 0$. If, in addition, $\Lambda x + \mu y \in \Omega$ for all $x, y \in \Omega, \Lambda, \mu > 0$, then it is said to be convex. Let $\Omega^* = \{y \in \mathbb{R}^n : (y/x) > 0, x \in \Omega \backslash \{0\}\}$. If $\Omega^* = \Omega$ then Ω is called self-dual (Ω^* is a dual cone).

Let $G(\Omega) = \{g \in Gl(\mathbb{R}^n) : g\Omega = \Omega\}$, where $Gl(\mathbb{R}^n)$ denotes the group of all linear invertible transformations of $\mathbb{R}^n$. If for all $x, y \in \Omega$, $y = gx$, for some $g \in G(\Omega)$, then our open convex cone Ω is homogeneous. If, in addition, $\Omega = \Omega^*$, then it is a symmetric cone.

Let $dv(w)$ and $dv_\alpha(w) = [\Delta^{\alpha - \frac{n}{r}}(v)]dudv, \alpha > \frac{n}{r} - 1$, $w = u + iv$, be standard Lebesgue measures in tubular domains over the symmetric cone T_Ω and weighted Lebesgue measure in the tube, respectively.

Let $T_\Omega = V + i\Omega$ be the tube domain over an irreducible symmetric cone Ω in the complexification $V^{\mathbb{C}}$ of an n-dimensional Euclidean space V. $\mathcal{H}(T_\Omega)$ denotes the space of all holomorphic functions on T_Ω. We denote the rank of the cone Ω by r and the determinant function on V by Δ.

For $\tau \in \mathbb{R}_+$ and the associated determinant function $\Delta(x)$, we set

$$A^\infty_\tau(T_\Omega) = \{F \in \mathcal{H}(T_\Omega) : \|F\|_{A^\infty_\tau} = \sup_{x+iy \in T_\Omega} |F(x+iy)|\Delta^\tau(y) < \infty\}.$$

It can be checked that this is a Banach space. For $1 \leq p, q < \infty$ and $\nu \in \mathbb{R}, \nu > -1$, let $A^{p,q}_\nu(T_\Omega)$ denote the mixed-norm weighted Bergman space consisting of all analytic functions f on T_Ω such that

$$\|F\|_{A^{p,q}_\nu}=\left(\int\limits_{\Omega}\left(\int\limits_{V}|F(x+iy)|^p dx\right)^{\frac{q}{p}}\Delta^\nu(y)dy\right)^{\frac{1}{q}}<\infty.$$

This is a Banach space. Replacing A by L above yields the corresponding larger space of all measurable functions on tube over symmetric cone with the same quasinorm. It is known that the $A^{p,q}_\nu(T_\Omega)$ space is nontrivial if and only if $\nu>-1$. When $p=q$, we write $A^{p,q}_\nu(T_\Omega)=A^p_\nu(T_\Omega)$; this is the classical weighted Bergman space with the usual modification when $p=\infty$.

The (weighted) Bergman projection P_ν is the orthogonal projection from the Hilbert space $L^2_\nu(T_\Omega)$ onto its closed subspace $A^2_\nu(T_\Omega)$, and it is given by the following integral formula

$$P_\nu f(z)=C_\nu\int\limits_{T_\Omega}B_\nu(z,w)f(w)dV_\nu,$$

where

$$B_\nu(z,w)=C_\nu\Delta^{\nu+\frac{n}{r}}((z-\overline{w})/i)$$

is the Bergman reproducing kernel for $A^2_\nu(T_\Omega)$. Here, we used the notation

$$dV_\nu(w)=\Delta^{\nu-\frac{n}{r}}(v)dudv.$$

We also use the following notations: $w=u+iv\in T_\Omega$ and $z=x+iy\in T_\Omega$. Hence, for any analytic function f from $A^2_\nu(T_\Omega)$, the following integral formula holds

$$f(z)=C_\nu\int\limits_{T_\Omega}B_v(z,w)f(w)d_\nu w.$$

We first give basic definitions of new and known analytic function spaces and standard objects in tubular domains over symmetric cones. Then we provide the definitions in the context of bounded strongly pseudoconvex domains with smooth boundary.

Let $T_\Omega\subseteq\mathbb{C}^n$ be a bounded tubular domain over a symmetric cone in $\mathbb{C}^n$. We shall use the following notations:

- $\Delta: T_\Omega\to\mathbb{R}^+$ will denote the determinant function from the boundary, that is $\Delta(z)=\Delta(Imz)$.

Let $d\nu_t(Z)=(\Delta(z))^t d\nu(z), t>-1$. Then,

- ν will be the Lebesgue measure on T_Ω;
- $H(T_\Omega)$ will denote the space of holomorphic functions on T_Ω endowed with the topology of uniform convergence on compact subsets;
- $B: T_\Omega\times T_\Omega\to\mathbb{C}$ will be the Bergman kernel of T_Ω. Note that if B is a kernel of type $t, t\in\mathbb{N}$, then B^s is a kernel of type $st, s\in\mathbb{N}$;
- Given $r\in(0,\infty)$ and $z_0\in T_\Omega$, we shall denote the Bergman ball by $B_{T_\Omega}(z_0,r)$.

We define new Banach mixed norm and analytic Bergman-type spaces on the product of tubular domains over symmetric cones, $T_\Omega \times ...T_\Omega$, as follows. Let $m \geq 1, p_j \in (1, \infty); \nu_j > \frac{n}{r} - 1, j = 1, ...m$. Then

$$A^{\vec{p}}_{\vec{\nu}} = \{f \in H(T_\Omega)^m = H(T_\Omega \times ... \times T_\Omega) :$$

$$\Big(\int_{T_\Omega} ...\Big(\int_{T_\Omega} |f(z_1, ..., z_m)|^{p_1} \Delta^{\nu_1 - \frac{n}{r}}(y_1) dx_1 dy_1\Big)^{\frac{p_2}{p_1}} ... \Delta^{\nu_m - \frac{n}{r}}(y_m) dx_m dy_m\Big)^{\frac{1}{p_m}} < \infty)\}.$$

Let $\mathbb{C}$ denote the set of complex numbers, and let $\mathbb{C}^n = \mathbb{C} \times ... \times \mathbb{C}$ denote the Euclidean space of complex dimension n. The open unit ball in $\mathbb{C}^n$ is the set $B_n = \{z \in \mathbb{C}^n : |z| < 1\}$. We denote by $H(B_n)$ the space of holomorphic functions on the open unit ball in $\mathbb{C}^n$. Moreover, let ν denote the Lebesgue measure on B_n, normalized so that $\nu(B_n) = 1$. For any $\alpha \in \mathbb{R}$, let $d\nu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha \nu(z)$, for $z \in B_n$. Here, if $\alpha \leq -1$, then $c_\alpha = 1$, and if $\alpha > -1$ then $c_\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}$ is the normalizing constant so that ν_α has unit total mass. The Bergman metric on B_n is $\beta(z, w) = \frac{1}{2}\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}$, where φ_z is the Möbius transformation of B_n that interchanges 0 and z. Let $D(a, r) = \{z \in B_n : \beta(z, a) < r\}$ denote the Bergman metric ball in B_n centered at $a \in B_n$ with radius $r > 0$.

We shall use the following notations:

- $\Delta : D \to \mathbb{R}^+$ will denote the Euclidean distance from the boundary, that is $\Delta(z) = d(z, \partial D)$;
- given two non-negative functions $f, g : D \to \mathbb{R}^+$, we write $f \preceq g$ if there exists $C > 0$ such that $f(z) \leq Cg(z)$ for all $z \in D$. The constant C is independent of $z \in D$, but it might depend on other parameters (r, θ, etc);
- given two strictly positive functions $f, g : D \to \mathbb{R}^+$, we write $f \approx g$ if $f \preceq g$ and $g \preceq f$, i.e., if there exists $C > 0$ such that $C^{-1}g(z) \leq f(z) \leq Cg(z)$ for all $z \in D$;
- ν will denote Lebesgue measure;
- $H(D)$ will denote the space of holomorphic functions on D, endowed with the topology of uniform convergence on compact subsets;
- for $1 \leq p \leq \infty$, the Bergman space $A^p(D)$ is the Banach space $L^p(D) \cap H(D)$, endowed with the L^p -norm;
- more generally, for $\beta \in \mathbb{R}$, we introduce the weighted Bergman space

  $$A^p(D, \beta) = L^p(\Delta^\beta \nu) \cap H(D)$$

  with norm

  $$\|f\|_{p,\beta} = [\int_D |f(\zeta)|^p \Delta^\beta(\zeta) d\nu(\zeta)]^{\frac{1}{p}}$$

  for $1 \leq p < \infty$, and

  $$\|f\|_{\infty,\beta} = \|f\Delta^\beta\|_\infty$$

  for $p = \infty$;
- $K : D \times D \to \mathbb{C}$ will be the Bergman kernel of D; K_t is a kernel of type t;

- for each $z_0 \in D$, the normalized Bergman kernel $k_{z_0} : D \to \mathbb{C}$ is defined by
$$k_{z_0}(z) = \frac{K(z,z_0)}{\sqrt{K(z_0,z0)}} = \frac{K(z,z_0)}{\|K(\cdot,z_0)\|_2};$$
- given $r \in (0,1)$ and $z_0 \in D$, we denote by $B_D(z_0,r)$ the Kobayashi ball with center z_0 and radius $\frac{1}{2}\log\frac{1+r}{1-r}$.

For any two m-tuples of real numbers $a = (a_1, ..., a_m)$ and $b = (b_1, ..., b_m)$, we define the integral operator on the unit ball and on products of balls,
$$(S_{a,b}f)(z_1,...,z_m) = \prod_{j=1}^{m}(1-|z_j|^2)^{a_j}\int_{B_n}\frac{f(w)(1-|w|^2)^{-n-1+\sum_{j=1}^m b_j}}{\prod_{j=1}^m(1-\langle z_j,w\rangle)^{a_j+b_j}}d\nu(w),$$
where $z_1,...z_m \in B_n$, and $f \in L^1(B_n, d\nu_{-n-1-\sum_{j=1}^m b_j})$. Note that for such f, the function $S_{a,b}f$ is defined on $(B_n)^m$, the product of m copies of B_n. These new extensions of Bergman projections, namely the Bergman-type integral operators $S_{a,b}$, were used and studied by the first author in trace-type theorems for BMOA type analytic function spaces on the ball tube and bounded strongly pseudoconvex domains. Based on the above definition, we can easily define integral operators of this type $S_{a,b}$ in the context of all mentioned domains in $\mathbb{C}^n$.

For our theorem in pseudoconvex domains (case $p < 1$), we always assume a stronger condition on the weighted Bergman kernels, namely the following property (C): $K_t(z,w) \asymp K_t(a_k,w)$ for any $z \in B_D(a_k,r), r \in (0;1), w \in B_D(a_m,r), r \in (0;1)$ for $t > 0$, and any natural number m (we assume here that the Bergman kernel is positive; otherwise add modulus.)

Let T_Ω be the tube domain over the symmetric cone, and $H(T_\Omega)$ be the space of all analytic functions on the tube. We define Bergman spaces for $1 \le p,q < \infty, \Gamma > \frac{n}{r}-1$. Let
$$A^{p,q}_{\Gamma}(T_\Omega) = \{f \in H(T_\Omega) : \Big(\int_{\Omega}\Big(\int_{\mathbb{R}^n}|f(x+iy)|^p dx\Big)^{\frac{q}{p}}(\Delta^{\Gamma-\frac{n}{r}}(y)dy)\Big)^{\frac{1}{q}} < \infty\}.$$

Replacing A by L, and H by L^1, we get, as usual, the known larger spaces of measurable functions on the tube T_Ω. Note that $A^{p,q}_{\Gamma} = \{0\}$ if $\Gamma < \frac{n}{r}-1$.

Let $f \in H(T_\Omega \times ... \times T_\Omega), f = (z_1,...,z_m), m \ge 1$; i.e. the function f is holomorphic separately in each variable . Let $\vec{f} = (f_1,...,f_m); f = \prod_{j=1}^{m} f_j$,
$$\|\vec{f}\|^p_{B^p_{\vec{\nu}}} = \int_{T_\Omega}|f_1(z)|^p...|f_m(z)|^p\Delta^{\nu_1-\frac{n}{r}}(y)...\Delta^{\nu_m-\frac{n}{r}}(y)dxdy < \infty;$$
$y = Imz; f_j \in H(T_\Omega), 1 \le p < \infty, \nu_j > \frac{n}{r}-1, j = 1,...,m$. We assume $\|\vec{f}\|^p_{B^p_{\vec{\nu}}} < \infty$.

These are multifunctional analytic function spaces on the tube. Similar spaces can be defined on bounded strongly pseudoconvex domains. Many results of this paper can also be extended to such spaces by similar methods.

Next, let $1 \leq p < \infty, f = f(z_1, ..., z_m)$. We consider analytic subspaces of $H(T^m_\Omega), T^m_\Omega = T_\Omega \times ... \times T_\Omega, \nu_j > \frac{n}{r} - 1, j = 1, ..., m$. These are spaces $(A^p_\nu)_1, (A^p_\nu)_2$, $(A^p_\nu)_3$ with norms

$$\|f\|^p_{(A^p_{\vec{\nu}})_1} = \int_{T_\Omega} ... \int_{T_\Omega} |f(x_1 + iy_1, ..., x_m + iy_m)|^p \prod_{j=1}^{m} \Delta^{\nu_j - \frac{n}{r}}(y_j) dx_j dy_j < \infty;$$

$$\|f\|^p_{(A^p_{\vec{\nu}})_2} = \int_{\mathbb{R}^n} ... \int_{\mathbb{R}^n} \int_{\Omega} |f(x_1 + iy_1, ..., x_m + iy_m)|^p \Delta^{\nu - \frac{n}{r}}(y) \Big(\prod_{j=1}^{m} dx_j \Big) dy < \infty;$$

$$\|f\|^p_{(A^p_{\vec{\nu}})_3} = \int_{\mathbb{R}^n} \int_{\Omega} ... \int_{\Omega} |f(x_1 + iy_1, ..., x_m + iy_m)|^p \prod_{j=1}^{m} \Delta^{\nu_j - \frac{n}{r}}(y_j) dx dy_j < \infty.$$

Let D be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary and let $d(z) = dist(z, \partial D)$. Then there exists a neighborhood U of $\bar{D}$ and $\rho \in \mathbb{C}^\infty(U)$ such that $D = \{z \in U : \rho(z) > 0\}, |\Delta\rho(z)| \geq c > 0$ for $z \in \partial D, 0 < \rho(z) < 1$ for $z \in D$ and $-\rho$ is strictly plurisubharmonic in a neighborhood U_0 of ∂D.

Then there exists r_0 such that the domains $D_r = \{z \in D : \rho(z) > r\}$ are also smoothly bounded strictly pseudoconvex domains for all $0 < r \leq r_0$. Let $d\sigma_r$ be the normalized surface measure on $\mathbb{C}^n$. For $0 < p < \infty, 0 < q \leq \infty, \Delta > 0, k = 0, 1, r$, we set

$$\|f\|_{p,q,\Delta,k} = \sum_{|\alpha| \leq k} \Big(\int_0^{r_0} \Big(r^\Delta \int_{\partial D_r} |D^\alpha f|^p d\sigma_r \Big)^{\frac{q}{p}} \frac{dr}{r} \Big)^{\frac{1}{q}}, 0 < q < \infty;$$

$$\|f\|_{p,\infty,\Delta,k} = \sup_{0 < r < r_0} \sum_{|\alpha| \leq k} \Big(r^\Delta \int_{\partial D_r} |D^\alpha f|^p d\sigma_r \Big)^{\frac{1}{p}}.$$

The corresponding spaces $A^{p,q}_{\Delta,k} = \{f \in H(D) : \|f\|_{p,q,\Delta,k} < \infty\}$ are complete quasinormed spaces for $p, q \geq 1$. For $p, q \geq 1$, they are Banach spaces. We mainly consider the case $k = 0$, in which we simply write $A^{p,q}_\Delta$ and $\|f\|_{p,q,\Delta}$.

We also consider these spaces for $p = \infty$ and $k = 0$. The corresponding space is denoted by $A^{\infty,p}_\Delta = A^{\infty,p}_\Delta(D)$ and consists of all $f \in H(D)$ such that

$$\|f\|_{\infty,p,\Delta} = \Big(\int_0^{r_0} (\sup_{\partial D_r} |f|)^p r^{\Delta p - 1} dr \Big)^{\frac{1}{p}} < \infty.$$

Also for $\Delta > -1$,the space $A^\infty_\Delta = A^\infty_\Delta(D)$ consists of all $f \in H(D)$ such that

$$\|f\|_{A^\infty_\Delta} = (\sup_{z \in D}) |f(z)| \rho^\Delta(z) < \infty,$$

and the weighted Bergman space $A^p_\Delta = A^{p,p}_{\Delta+1}(D)$ consists of all $f \in H(D)$ such that

$$\|f\|_{A^p_\Delta} = \Big(\int_D |f(z)|^p \rho^\Delta(z) d\nu(z) \Big)^{\frac{1}{p}} < \infty,$$

where dv is the normalized Lebesgue measure on D.

Let $T_\Omega = v + i\Omega$ be the tube domain over an irreducible symmetric cone Ω in the complexification $V^{\mathbb{C}}$ of an n-dimensional Euclidean space $\tilde{V}$. We denote the rank of the cone Ω by r and the determinant function on $\tilde{V}$ by Δ.

Let us introduce some convenient notations regarding multi-indices. If $t = (t_1, ..., t_m)$, then $t^* = (t_m, ..., t_1)$, and for $a \in \mathbb{R}, t + a = (t_1 + a, ..., t_m + a)$. Also, if $t, k \in \mathbb{R}^m$, then $t < k$ means $t_j < k_j$, for all $j = 1, .., m$. We are going to use the following multi-index

$$g_0 = ((j-1)\frac{d}{2})_{1\le j\le r}, \text{ where } (r-1)\frac{d}{2} = \frac{n}{r} - 1.$$

$\mathcal{H}(T_\Omega)$ denotes the space of all holomorphic functions on T_Ω. We denote m-fold Cartesian products of tubes by $T_\Omega^m = T_\Omega \times ... \times T_\Omega$. The space of all analytic functions on this new product domain, which are analytic in each variable separately, will be denoted by $\mathcal{H}(T_\Omega^m)$. By m we denote a natural number bigger than 1. For $\tau \in \mathbb{R}_+$ and the associated determinant function $\Delta(x)$, we set

$$\tilde{A}_\tau^\infty(T_\Omega) = \Big\{F \in \mathcal{H}(T_\Omega) : \|F\|_{A_\tau^\infty} = \sup_{x+iy\in T_\Omega} |F(x+iy)|\Delta^\tau(y) < \infty\Big\}.$$

It can be checked that this is a Banach space.
For $1 \le p, q < +\infty$ and $\nu \in \mathbb{R}, \nu > \frac{n}{r} - 1$, we denote by $\tilde{A}_\nu^{p,q}(T_\Omega)$ the mixed-norm weighted Bergman space consisting of all analytic functions f on T_Ω such that

$$\|F\|_{\tilde{A}_\nu^{p,q}} = \Big(\int_\Omega \Big(\int_{\tilde{V}} |F(x+iy)|^p dx\Big)^{\frac{q}{p}} \Delta^\nu(y)\frac{dy}{\Delta(y)^{\frac{n}{r}}}\Big)^{\frac{1}{q}} < \infty.$$

This is a Banach space.

Replacing $\tilde{A}$ by $\tilde{L}$ above, we obtain, as usual, the corresponding larger space of all measurable functions on the tube over the symmetric cone with the same quasinorm. It is known that the space $\tilde{A}_\nu^{p,q}(T_\Omega)$ is nontrivial if and only if $\nu > \frac{n}{r} - 1$. We assume this throughout. When $p = q$, we write $\tilde{A}_\nu^{p,q}(T_\Omega) = \tilde{A}_\nu^p(T_\Omega)$. This is the classical weighted Bergman space with the usual modification when $p = \infty$. We add some notions on Bergman-type analytic function spaces on products of tubular domains.

Let $T_\Omega^m = T_\Omega \times ... \times T_\Omega$. To define the related two Bergman-type spaces $\tilde{A}_\nu^p(T_\Omega)$ and $\tilde{A}_\tau^\infty(T_\Omega)$ (ν and τ can also be vectors) in m-products of tube domains T_Ω^m, we follow standard procedure which is well known in the case of unit disk and unit ball. Namely, we consider analytic functions $F = F(z_1, ..., z_m)$ which are analytic in each variable, and each variable belongs to the tube T_Ω. We define $H(T_\Omega^m)$ as the space of all such functions. For example, for all $z_j = x_j + iy_j, \tau_j \in \mathbb{R}, j = 1, ..., m, F(z) = F(z_1, ..., z_m), \tau = (\tau_1, ..., \tau_m)$, we set

$$\tilde{A}_\tau^\infty(T_\Omega^m) = \Big\{F \in \mathcal{H}(T_\Omega) : \|F\|_{\tilde{A}_\tau^\infty} = \sup_{x+iy\in T_\Omega^m} |F(x+iy)|\Delta^\tau(y) < \infty\Big\},$$

where

$$|F(x+iy)| = |F(x_1 + iy_1, ..., x_m + iy_m)|,$$

and $\Delta^{\tau}(y)$ is a product of m one-dimensional $\Delta^{\tau_j}(y_j)$ functions, $j = 1, ..., m$. Similarly, the Bergman space $\tilde{A}^p_\tau$ can be defined on products of tubes for all $\tau = (\tau_1, ..., \tau_m), \tau_j > \frac{n}{r} - 1, j = 1, ..., m$. It can be shown that all spaces are Banach spaces. Replacing $\tilde{A}$ by $\tilde{L}$ above, we obtain, as usual, the corresponding larger space of all measurable functions in products of tubes over the symmetric cone with the same quasinorm.

The (weighted) Bergman projection P_ν is the orthogonal projection from the Hilbert space $\tilde{L}^2_\nu(T_\Omega)$ onto its closed subspace $\tilde{A}^2_\nu(T_\Omega)$ and it is given by the following integral formula (see [15]),

$$P_\nu f(z) = C_\nu \int_{T_\Omega} B_\nu(z, w) f(w) d\tilde{V}_\nu(w),$$

where

$$B_\nu(z, w) = C_\nu \Delta^{-\nu + \frac{n}{r}}(\frac{z - \bar{w}}{i})$$

is the Bergman reproducing kernel for $\tilde{A}^2_\nu(T_\Omega)$. Here, we used notation $d\tilde{V}_\nu(w) = \Delta^{\nu - \frac{n}{r}}(v) du dv$. Throughout the paper, we will use the following notation: $w = u + iv \in T_\Omega$ and $z = x + iy \in T_\Omega$. Hence, for any analytic function from $\tilde{A}^2_\nu(T_\Omega)$, the following reproducing integral formula holds

$$f(z) = C_\nu \int_{T_\Omega} B_\nu(z, w) f(w) d\tilde{V}_\nu(w).$$

We provide now several results on Bergman-type projections on the tube. The weighted Bergman projection P_ν is the orthogonal projection from the Hilbert space $\tilde{L}^2_\nu(T_\Omega)$ onto its closed subspace $\tilde{A}^2_\nu(T_\Omega)$ and it is given by the integral formula

$$(P_\nu f)(z) = \int_{T_\Omega} B_\nu(z, w) f(w) \Delta^{\nu - \frac{n}{r}}(Im w) dv(w),$$

$z \in T_\Omega$ and $\nu > \frac{n}{r}$.

If P_ν extends to a bounded operator on $\tilde{L}^{p,q}_\nu$, then the topological dual space $(\tilde{A}^{p,q}_\nu)^*$ of the Bergman space $\tilde{A}^{p,q}_\nu$ identifies with $\tilde{A}^{p',q'}_\nu$ under the integral pairing

$$\langle f, g \rangle_\nu = \int_{T_\Omega} f(z) \overline{g(z)} \Delta^{\nu - \frac{n}{r}}(Im z) dv(z),$$

$f \in \tilde{A}^{p,q}_\nu, g \in \tilde{A}^{p',q'}_\nu$ (see [18]). Let $\beta > -1, \Gamma > 0$ and $\alpha > 0$. Let also

$$(T_{\alpha,\beta,\Gamma} f)(z) = \Delta^\alpha(Im z) \int_{T_\Omega} B_\Gamma(z, w) f(w) \Delta^\beta(Im w) dv(w),$$

$$(T^+_{\alpha,\beta,\Gamma} f)(z) = \Delta^\alpha(Im z) \int_{T_\Omega} |B_\Gamma(z, w)| f(w) \Delta^\beta(Im w) dv(w),$$

$z \in T_\Omega, f \in \tilde{L}^1(T_\Omega)$.

We also define a new Herz-type integral operator for positive $\alpha_j, j = 1, ..., m$ and for all $\beta > -1, \Gamma > -1$,

$$[T_{\vec{\alpha},\beta,\Gamma}(g)](z_1, ..., z_m) = \int_{T_\Omega} \int_{B(\tilde{w},R)} \frac{g(w)[\Delta^{\beta}(Imw)]dv(w)}{[\prod_{j=1}^{m} \Delta^{\alpha_j}(\frac{z_j-\overline{w}}{i})]} \Delta^{\Gamma}(Im\tilde{w})dv(\tilde{w}),$$

where $z_j \in T_\Omega, j = 1, ..., m$.

These new objects, quasinorms and function spaces introduced in this section for tubular domains over symmetric cones can also be defined analogously in the unit ball and in bounded strongly pseudoconvex domains with smooth boundary. This can be carried out by the interested reader without difficulty. We remark that the spaces $A^{p,q}_{\alpha}$ of analytic functions defined above can be extended to product domains in two natural ways by incorporating additional integrals into the norm. These extensions lead to new function spaces, and it would be interesting to study and obtain various projection theorems in these settings. We leave these problems for future investigation by the reader.

We warn the reader that some objects appearing in our theorems are not defined in this paper; we refer the reader to the research articles cited in those theorems for their definitions.

Practically all of our results were previously known for very simple domains and for particular values of parameters. All theorems of this paper also admit, with similar proofs, complete analogues in Bloch-type spaces on the tube and pseudoconvex domains defined above.

2. Bergman-type projections in tubular domains and bounded pseudoconvex domains

The main objective of this section is to collect many recent results mainly related to advances of the first author and his coauthors regarding the Bergman projection in various analytic function spaces on rather complicated tubular domains over symmetric cones, as well as bounded strongly pseudoconvex domains with smooth boundary in $\mathbb{C}^n$. Furthermore, we present theorems that simultaneously generalize certain known classical results in two directions, enhancing the theoretical significance of these assertions We pay attention to certain specific aspects of issues related to the Bergman projection, which often appear in relation to the so-called trace problem in analytic function spaces of several variables in $\mathbb{C}^n$. Namely, we are interested in Bergman-type projections acting between spaces of measurable and analytic functions of different dimensions, beginning with a simple illustrative example.

Let X be a space of measurable functions on a fixed domain D in $\mathbb{C}^n$. The question is whether there exists a Bergman-type projection acting between this space and another space of analytic functions but defined on domain $D \times ... \times D$, which at the same time generalizes well-known classical result. We consider the action of such Bergman-type integral operators from various spaces of measurable functions defined on a fixed domain D in $\mathbb{C}$ or $\mathbb{C}^n$ to corresponding analytic function spaces on the same domain. Results concerning Bergman-type projections between function spaces defined on the same domain D are well known to experts and can be found in the literature for simple domains as the unit disk, polydisk, and unit ball, as well as for more complicated domains in $\mathbb{C}^n$. The interesting new issues here are to consider first difficult domains in $\mathbb{C}^n$. Then, to extend these already known results as far as possible to pairs (X, Y) of various interesting function spaces defined on D and $D \times ... \times D$, respectively, namely $X(D)$ and $Y(D \times ... \times D)$.

In the first two theorems below, we give complete descriptions of traces of certain new function spaces in tubular domains over symmetric cones and within these theorems we provide new interesting results on the action of Bergman projection between function spaces of different dimensions on the tube domains. For the definition of the Herz spaces appearing in these theorems, we refer the reader to the relevant papers cited therein.

Bergman projections acting between function spaces with different dimensions are usually closely related to the so-called trace problem in analytic function spaces of several complex variables. We say trace $X = Y$, when X and Y are quasinormed spaces of $H(D)$ and $H(D \times ... \times D)$, for a certain domain D in $\mathbb{C}^n$, if for any $f \in X$, $f(z, ..., z)$ from Y, and the reverse is also true, any function $g \in Y$ can be extended to a function $f \in X$ such that $f(z, ..., z) = g(z)$, $z \in D$.

Theorem 1. [1] *Let* $f \in A_{\vec{\nu}\vec{p}}(T^m_\Omega), 1 \leq p_j < \infty, \nu \in \mathbb{R}^m, \nu_j > \nu_0$ *for fixed* $\nu_0 = \nu_0(n, p, r, m)$ *and for all* $j = 1, ..., m$. *Then* $f(z, ..., z) \in A^{p_m}_s, s = \nu_m - \frac{n}{r} + \sum_{j=1}^{m-1}(\nu_j + \frac{n}{r})\frac{p_m}{p_j}$ *with related estimates for norms. And for all* $\tilde{p}_1, \frac{1}{\tilde{p}_1} + \frac{1}{p_m} = 1, \frac{n}{r} \leq \tilde{p}_1, j = 1, ..., m,$ *for some fixed large enough* k, $\nu_j > k$, *the reverse is also true. For each function* $g \in A^{p_m}_s(T_\Omega)$, *there is a function* $F(z, ..., z) = g(z), F \in A^{\vec{p}}_{\vec{\nu}}(T_\Omega)$. *Let, in addition,*

$$T_\beta(f)(z_1, ..., z_m) = C_\beta \int_{T_\Omega} f(w)(\prod_{j=1}^{m} \Delta^{-t}(\frac{z_j - \bar{w}}{i}))dV(w), mt = \beta + \frac{n}{r}, z_j \in T_\Omega,$$

$j = 1, ...m$. *Then, the following assertion holds for all* β, *such that* $\beta > \beta_0$ *for some fixed large enough positive number* β_0. *The Bergman-type integral operator* T_β *(expanded Bergman projection) maps* $A^{p_m}_s(T_\Omega)$ *to* $A^{\vec{p}}_{\vec{\nu}}(T^m_\Omega), \nu = (\nu_1, ..., \nu_m), \nu_j > \nu_0, j = 1, ..., m.$

Theorem 2. [1] *Let* $\nu_j > \nu_0$ *and* $\tau_j > \tau_0$ *for some fixed positive numbers* $\nu_0 = \nu_0(p, n, r, m)$ *and* $\tau_0 = \tau_0(p, n, r, m), 1 \leq p < \infty$. *Let* $f \in S^p_{\nu,\tau}(T^m_\Omega)$. *Then* $f(z, ..., z)$ *belongs to* $A^p_s(T_\Omega)$, *where* $s = \sum_{j=1}^{m}(\nu_j + \frac{n}{r})p + \sum_{j=1}^{m}(\tau_j + \frac{n}{r}) - \frac{n}{r}$ *and for every function* $f \in A^p_s(T_\Omega)$ *there exists a function* $F \in S^p_{\nu,\tau}$ *such that* $F(z, ..., z) = f(z)$ *for all* $\frac{n}{r} \leq p', \frac{1}{p} + \frac{1}{p'} = 1$. *Let, in addition,*

$$(T_\beta f)(z_1, ..., z_m) = C_\beta \int_{T_\Omega} f(w) \prod_{j=1}^{m} \Delta^{-t}(\frac{(z_j - \bar{w})}{i})dV_\beta(w),$$

$mt = \beta + \frac{n}{r}, z_j \in T_\Omega, j = 1, ..., m$. Then, the following assertion holds for all β such that $\beta > \beta_0$ for some fixed large enough positive number β_0. The Bergman projection T_β maps $A^p_s(T_\Omega)$ to $S^p_{\nu,\tau}(T^m_\Omega)$, $\nu = (\nu_1, ..., \nu_m)$, $\tau = (\tau_1, ..., \tau_m)$, $\nu_j > \nu_0$, $\tau_j > \tau_0$, $j = 1, ..., m$.

Proofs of Theorems 1 and 2 are analogous to the simpler and well- known case of the unit disk. The role of the so-called r-lattices in tubular domains is very important in these proofs. In view of the known properties of r-lattices in bounded strongly pseudoconvex domains, these results—with very similar proofs—are expected to hold in that setting as well.

Let U be the unit disk and T be the unit circle. We define, as usual, analytic spaces $A^{p,q}_\alpha$, for all positive $p, q, \alpha > -1$, as function spaces with finite quasinorms

$$\|f\|_{p,q,\alpha} = \int_0^1 (\int_T |f(r\xi)|^p d\xi)^{q/p} \times (1-r)^\alpha (dr).$$

These analytic function spaces can be defined similarly on the polydisk. Moreover, they can also be introduced in very general tubular and bounded pseudoconvex domains in $\mathbb{C}^n$, as well as in products of such domains by incorporating additional integrals into the quasinorms. We denote such spaces of measurable functions by $L^{p,q}_\alpha$, and by $\tilde{L}^{p,q}_\alpha$ (with a tilde) when they are defined on products of such domains. Replacing L by A gives the corresponding analytic subspaces. In the following theorems, we present various projection results for function spaces of this type.

Theorem 3. [2] *Define the operator*

$$T_\beta f(z) = \int_\Lambda f(w) K_{\beta+n+1}(z, w)(\Delta^\beta(w)) dv(w),$$

where β is large enough. Then, the operator T_β maps the space $L^{p,q}_\Delta$ into the analytic function space $A^{p,q}_\Delta$ for all $1 < p, q < \infty$ and $\Delta > -1$.

Theorem 4. [2] *Consider the integral operator*

$$(\tilde{T}_\beta f)(z_1, ..., z_m) = \int_\Lambda ... \int_\Lambda (\vec{f}(w)) \Big(\prod_{j=1}^m K_{\beta+n+1}(z_j, w_j) \Big) (\Delta^\beta(w_j)) dv(w_j).$$

Then, for all for all $\beta > \beta_0$ with large enough β_0, the operator $\tilde{T}_\beta$ maps the space $\tilde{L}^{p,q}_\Delta$ into the space $\tilde{A}^{p,q}_\Delta$ for all $z_j \in \Lambda, j = 1, ..., m, 1 < p, q < \infty$, and $\Delta > 0$.

Theorem 5. [2] *Consider the integral operator*

$$(T_\tau f)(z) = \int_{T_\Lambda} (f(\tilde{w})) B_{\tau + \frac{2n}{r}}(r, \tilde{w}) [\Delta^\tau (Im \tilde{w})] dv(\tilde{w}).$$

Then, for all $\tau > \tau_0$, with large enough τ_0, the operator (T_τ) maps $L^{p,q}_\nu(T_\Lambda)$ to $A^{p,q}_\nu(T_\Lambda)$, where

$$(A^{p,q}_\nu) = \Big\{ f \in H(T_\Lambda) : \int_\Lambda \Big(\int_{R_n} |f(x+iy)|^p dx \Big)^{\frac{q}{p}} \Delta^\nu (Imz) dxdy < \infty \Big\},$$

$\nu > -1$ and $1 < p, q < \infty$.

In the following theorem 6, we formulate a new result on Bergman-type integral operators which related spaces with different dimension in tubular domains and bounded strongly pseudoconvex domains. This theorem is well known for particular case, $m = 1$, on the tube and pseudoconvex domains. The definition of expanded Bergman projection $\mathbb{R}_x$ can be found in [7]. This theorem also has complete analogues in various domains in $\mathbb{C}^n$ in particular in bounded symmetric domains, Siegel domains of second type, matrix domains, etc., all with the same proof.

Theorem 6. [7] *1) Let $1 < p < \infty, s_j > \frac{n}{r} - 1, j = 1, ..., m$. Then, for some fixed large enough X_0 and all $X_j > X_0, j = 1, ..., m$, there exists a constant $C > 0$ such that*

$$\int_{T_\Omega} |R_{\overrightarrow{X}} g(w)|^p \Delta^{(\sum\limits_{j=1}^{m}) s_j + \frac{2n}{r}(m-1)} (Imw) dudv \le \tilde{C} \|g\|_{L^p_{\overrightarrow{s}}} (T^m_\Omega).$$

2) Let $1 < p < \infty, s_j > \frac{n}{r} - 1, j = 1, ..., m$. Then, for some fixed large enough X_0, and all $X_j > X_j, j = 1, ..., m$, there exists a constant $C > 0$ such that

$$\int_\Lambda |\tilde{R}_{\overrightarrow{X}} g(w)|^p \Delta^\tau (w) dv(w) \le \tilde{C} \|g\|_{L^p_{\overrightarrow{s}}} (\Lambda^m),$$

$\tau = (\sum\limits_{j=1}^{m} s_j) + (m-1)(n+1)$.

Let further $dv_\alpha = (\Delta^\alpha) dv(z)$ where $\Delta(z) = \rho(z)$. Since $|f(x)|^p$ is subharmonic (even plurisubharmonic) for a holomorphic function f, we have $A^p_s(D) \subset A^\infty_t(D)$, for $0 < p < \infty, sp > n$ and $t = s$. Also $A^p_s(D) \subset A^1_s(D)$ for $0 < p \le 1$ and $A^p_s(D) \subset A^1_t(D)$ for $p > 1$, and sufficiently large t.

Therefore, we have an integral representation

$$f(z) = \int_D f(\xi) K_{\tilde{t}}(z, \xi) \rho^t(\xi) dv(\xi),$$

for $f \in A^1_t(D), z \in D, \tilde{t} = t + n + 1$. Here $K_{\tilde{t}}(z, \xi)$ is a kernel of type t that is a smooth function on $D \times D$ such that

$$|K_{\tilde{t}}(z, \xi)| \le C_1 |\widetilde{\Phi}(z, \xi)|^{-(n+1+t)},$$

where $\widetilde{\Phi}(z, \xi)$ is so-called Henkin-Ramirez function on D.

In the following two theorems, which are formulated in the context of bounded strongly pseudoconvex domains D (where dv_β denotes the usual weighted Lebesgue measure on D), we present direct extensions of known results from the polydisk to these general domains D in $\mathbb{C}^n$. These results have the same proofs and they are valid in tubular domains, Siegel domain of second type, bounded symmetric domains and various matrix domains.

Theorem 7. [8] *Let* $0 < p, q \leq 1, \alpha > \tilde{\alpha}_0, \tilde{\alpha}_0$ *be large enough. Then*

$$(T_\alpha f)(w) = \int\limits_{\Lambda} K_{\alpha_0}(z,w) f(z) dv_\alpha(z), \alpha_0 = \alpha + n + 1,$$

maps $A^{p,q}_\beta$ *into* $A^{p,q}_\beta$ *for all* $\beta, \beta > 0, dv_\alpha = \Delta^\alpha dv.$

Theorem 8. [8] *Let* $p_i > 1, i = 1, ..., m, \alpha_j > -1, j = 1, ..., m..$ *Then, we have that the operator* $V_{\overrightarrow{\beta}}(f)$,

$$V_{\overrightarrow{\beta}}(f) = \int\limits_D \dots \int\limits_D f(z_1, \dots, z_m) \prod_{j=1}^m K_{\beta_j+n+1}(z_j, w_j) dv_{\beta_1}(z_1) \dots dv_{\beta_m}(z_m),$$

$\beta_j > \beta_0, j = 1, ..., m, \beta_0$ *is large enough, maps* $L^{p_1,\dots,p_n}_{\alpha_1,\dots,\alpha_n}$ *into* $A^{p_1,\dots,p_n}_{\alpha_1,\dots,\alpha_n}$.

The (weighted) Bergman projection P_ν is the orthogonal projection from the Hilbert space $L^2_\nu(T_\Omega)$ onto its closed subspace $A^2_\nu(T_\Omega)$ and it is given by the following integral formula

$$P_\nu f(z) = C_\nu \int\limits_{T_\Omega} B_\nu(z,\Omega) f(\Omega) dV_\nu(\Omega),$$

where $B_\nu(z,\Omega) = C_\nu \Delta^{-\nu+\frac{n}{r}}(\frac{z-\bar{\Omega}}{i})$ is the Bergman reproducing kernel for $A_\nu(T_\Omega)$. Here, we use the notation $dV_\nu(\Omega) = \Delta^{\nu-\frac{n}{r}}(\nu)dudv$. We denote by $dV(\Omega)$ or $dv(\Omega)$ the Lebegues measure on tubular domain over symmetric cone. Throughout the paper, we shall use the following notation: $\Omega = u + iv \in T_\Omega$ and $z = x + iy \in T_\Omega$. For any function f from A^∞_τ for large enough ν, we have

$$f(z) = C_\nu \int\limits_{T_\Omega} B_\nu(z,\Omega) f(\Omega) dv(\Omega).$$

We assume that for weighted Bergman kernel $(K_\tau(z,\Omega))$ the following estimate holds,

$$\int\limits_{B(z,\tau)} (\Delta(\tilde{z})^\alpha |K_\tau(\tilde{z},\Omega)| dV(\tilde{z}) \leq C|K_{\tau+n+1}(\Omega,z)|\Delta(z)^\alpha; \Omega, z \in D$$

for $\alpha > 0, \tau \geq 0$.

Let X be Herz-type spaces in pseudo convex or tube domains. Let $f \in H(D_1)$, where D_1 is a unit disk and $p \leq 1$. Then, we denote

$$Y = \tilde{X}_{p,q,\alpha,\alpha_1} = \tilde{X}(D_1)\{f \in H(D_1) : \int\limits_{D_1} (\int\limits_{D(z,r)} |f(\Omega)|^p dV_\alpha(\Omega))^{\frac{q}{p}} dV_{\alpha_1(z)} < \infty\},$$

where $H(D_1)$ is a space of all analytic functions in D_1, $dV(\Omega)$ is a Lebegues measure on $D_1; dV_\alpha(z) = (1-|z|)^\alpha dV(z), \alpha > -1$ and $D(z,r)$ is the Bergman ball in the unit disk.

Note that these analytic Herz spaces can be defined similary in tubular and bounded strongly pseudoconvex domains for the same values of parameters, using the definition of the Bergman ball in these settings. This clearly yields a direct generalization of the classical Bergman spaces in the domains mentioned above. Moreover, these Herz-type spaces can also be defined for all positive values of p and q.

In the following theorem, which is taken from the paper of Shamoyan and Shipka, we present a new result on the Bergman projection in these spaces under an additional condition on the Bergman kernel (as mentioned above). This result generalizes the classical Bergman projection theorem to the aforementioned domains and has interesting applications in the same paper. The result, with a very similar proof, may be also valid in many of the other difficult domains discussed earlier. In theorem 10, we denote by X and $\tilde{X}$ the Herz spaces (omitting indexes) in bounded pseudoconvex and tubular domains.

Theorem 9. [9] *The integral operator P^+_β (the Bergman projection with positive kernel) for $\beta > \beta_0$, where β_0 is large enough, maps from X to X and from $\tilde{X}$ to $\tilde{X}$ for all $q \leq p \leq 1, \alpha > -1, \alpha_1 > 1, \tilde{\alpha} \geq 0$.*

In the next two theorems, first for $p > 1$ and then for other positive values of p, we prove the boundedness of the Bergman-type integral operators $S_{a,b}$ between BMOA-type spaces in the unit ball and the polyball (i.e. spaces of different dimensions). These results, whose proofs are not very difficult and based on the so-called composition formula, are expected to remain valid, with the same arguments, in Siegel domains of the second type, bounded symmetric domains, various matrix domains, and also in the polydisk. An appropriate modification of the operator $S_{a,b}$ must first be defined in each setting..

Theorem 10. [10] *Let $1 < p < \infty$. Suppose $s_1, ..., s_m > -1$ and $r_1, ..., r_m \geq 0$ are such that for each $j = 1, ..., m$, we have $-pa_j < \min\{s_j + 1, s_j + 1 + n - r_j\}$ and $ms_j + 1 < p(mb_j - n) - (m-1)(n+1)$. Let $t = (m-1)(n+1) + \sum_{j=1}^m s_j$. Then, there exists a constant $C > 0$ such that*

$$\int_{B_n} \dots \int_{B_n} |S_{a,b} f(z_1, ..., z_m)|^p \prod_{j=1}^{m} \frac{(1-|z_j|^2)^{s_j}}{|1-\langle u_j, z_j\rangle|^{r_j}} dv(z_1)...dv(z_m) \leq$$

$$\leq C \int_{B_n} |f(w)|^p \frac{|1-|w|^2|^t}{\prod_{j=1}^m (1-\langle u_j, w\rangle)^{r_j}} dv(w),$$

for all $f \in L^p(B_n, dv_t)$ and $u_1, ..., u_m \in B_n$.

For the case $0 < p \leq 1$, we have the following result.

Theorem 11. [10] *Let $0 < p \leq 1$. Suppose $s_1, ...s_m > -1$ and $r_1, ..., r_m \geq 0$ are such that for each $j = 1, ..., m$, we have $-pa_j < \min\{s_j + 1, s_j + 1 + n - r_j\}$ and $s_j + 1 <$*

$pb_j - n$. Denote $t = (m-1)(n+1) + \sum_{j=1}^{m} s_j$. Then, there exists a constant $C > 0$ such that

$$\int_{B_n} \dots \int_{B_n} |(S_{a,b}f)(z_1,\dots,z_m)|^p \prod_{j=1}^{m} \frac{(1-|z_j|^2)^{s_j}}{|1-\langle u_j, z_j\rangle|^{r_j}} d\nu(z_1)\dots d\nu(z_m) \leq$$

$$\leq C \int_{B_n} |f(w)|^p \frac{(1-|w|^2)^t}{\prod_{j=1}^{m} |1-\langle u_j, w\rangle|^{r_j}} d\nu(w),$$

for all $f \in A^p(B_n, d\nu_t)$ and $u_1,\dots,u_m \in B_n$.

We need the following estimates for our results (see [16]).

$$\int_{\Omega} \Delta^t(z) K_{n+1+t+s}(z,w) K_r(z,w) d\nu(z) \leq c_1 \Delta^{-s} K_r(w,\nu), \tag{2.1}$$

where $w, \nu \in \Omega$, the kernel K_t is a function defined via the following estimate of the Henkin-Ramirez function Φ,

$$|K_t(z,w)| \leq c|\Phi(z,w)|^{-1}, z, w \in \Omega,$$

see [18]. In the case of tubular domain for the Bergman kernel $B_\alpha(z,w)$, we need the following estimate

$$\int_{T_\Omega} \frac{\Delta^t(Im(w)) d\tilde{V}(w)}{\Delta(\frac{z-\bar{w}}{i})^{\frac{2n}{\tau}+t+s} \Delta(\frac{\nu-w}{i})^r} \leq c_2 \frac{\Delta^{-s}(Im(w))}{\Delta^r(\frac{w-\nu}{i})}, w, \nu \in T_\Omega. \tag{2.2}$$

Here, c_2 and c_1 are constants, and $t > -1, s > 0, 0 \leq r \leq n+1+t$ in the pseudoconvex domains, and $t > -1, s > 0, 0 \leq r < \frac{2n}{\tau} + t$ in the tubular domain, where τ is the rank of our tube domain.

We always assume that $K_t, t \in \mathbb{N}$, so we consider the Bergman kernel in pseudoconvex domains only with natural index. We define the BMOA-type space in products of bounded pseudoconvex domains as a subspace of $H(\Omega^m)$ with the following finite quasinorm,

$$\int_{\Omega} \dots \int_{\Omega} |(f)(z_1,\dots,z_m)|^p \prod_{j=1}^{m} \Delta^{s_j}(z_j) \prod_{j=1}^{m} |K_{r_j}(z_j,u_j)| d\nu(z_1)\dots d\nu(z_m),$$

for positive values of involved parameters, and for all $s_j > -1, r_j \in \mathbb{N}$ and j. Setting $m = 1$ in this quasinorm yields new analytic function spaces on pseudoconvex domains Ω. We now define the following Bergman-type projections on such domains, for sufficiently large parameters,

$$S_{a,b}(f)(z_1,\dots,z_m) = \prod_{j=1}^{m} \Delta^{a_j}(z_j) \int_{\Omega} f(w) \prod_{j=1}^{m} |K_{a_j+b_j}(z_j,w)| (\Delta(w))^{-n-1+\sum_{j=1}^{m} b_j} d\nu(w).$$

Under certain additional conditions on the Bergman kernel, in Theorems 12, 13, we provide new results on the boudedness of the Bergman-type integral operators $S_{a,b}$ acting on certain BMOA-type function spaces of analytic functions in tubular domains and bounded strongly pseudoconvex domains. We remark that the first author previously proved an analogous result in the unit ball without any additional conditions. The proofs in all these domains are similar.

Theorem 12. [10] *Let* $1 < p < \infty, s_j > -1, r_j \in \mathbb{N}, a_j > a_0, b_j > b_0, a_0 = a_0(s_1, ..., s_m, p, m, n), b_0 = b_0(s_1, ..., s_m, p, m, n), j = 1, ..., m$. *Let* $t = (m-1)(n+1) + \sum_{j=1}^{m} s_j$ *and (2.1) holds. Then, there exists a constant* $C > 0$ *such that*

$$\int_{\Omega} ... \int_{\Omega} |S_{a,b}(f)(z_1, ..., z_m)|^p \prod_{j=1}^{m} \Delta^{s_j}(z_j) \prod_{j=1}^{m} |K_{r_j}(z_j, u_j)| d\nu(z_1)...d\nu(z_m) \leq$$

$$\leq C \int_{\Omega} (f(w))^p \Delta^t(w) \prod_{j=1}^{m} |K_{r_j}(u_j, w) d\nu(w)|.$$

The conclusion of the previous part of the theorem is also valid for $0 < p \leq 1$. *For other values of the parameters* a_0, b_0, *we assume in addition to (2.1) that property (C) is valid and that* $\frac{r_j}{p} \in \mathbb{N}$.

We define the analytic BMOA-type space in products of tubular domains over symmetric cones as a subspace of $H(T_{\Omega}^{m})$ with the following finite quasinorm,

$$\int_{T_\Omega} ... \int_{T_\Omega} |(f)(z_1, ..., z_m)|^p \prod_{j=1}^{m} \Delta^{s_j}(Im(z_j)) \prod_{j=1}^{m} \Delta^{r_j}\left(\frac{\bar{z}_j - u_j}{i}\right) dV(z_1)...dV(z_m),$$

for positive values of involved parameters, and for all $s_j > -1$ and j. Putting $m = 1$ in this quasinorm, we get such spaces in tubular domains over symmetric cones T_Ω. We define new Bergman-type integral operators in tubular domains as follows:

$$S_{a,b}(f)(z_1, ..., z_m) =$$

$$= \prod_{j=1}^{m} \Delta^{a_j}(Im(z_j)) \int_{T_\Omega} f(w) \prod_{j=1}^{m} \Delta^{-(a_j+b_j)}\left(\frac{z_j - w_j}{i}\right) [\Delta(Im(w))]^{-\frac{2n}{r} + \sum_{j=1}^{m} b_j} dV(w),$$

$z_j \in T_\Omega, j = 1, ..., m$, and for large enough involved parameters.

Theorem 13. [10] *Let* $1 < p < \infty, s_j > -1, r_j \geq 0, a_j > a_0, b_j > b_0, a_0 = a_0(s_1, ..., s_m, p, m, n)$, $b_0 = b_0(s_1, ..., s_m, p, m, n), j = 1, ..., m$. *Let* $t = (m-1)\frac{2n}{r} + \sum_{j=1}^{m} s_j$ *and (2.2) holds. Then, there exists a constant* $C > 0$ *such that*

$$\int_{T_\Omega} ... \int_{T_\Omega} |S_{a,b}(f)(z_1, ..., z_m)|^p \prod_{j=1}^{m} \Delta^{s_j}(Im(z_j)) \prod_{j=1}^{m} \Delta^{-r_j}\left(\frac{\bar{z}_j - u_j}{i}\right) dV(z_1)...dV(z_m) \leq$$

$$\leq C \int_{T_\Omega} |f(w)|^p \Delta^t(Im(w)) \prod_{j=1}^{m} \Delta^{-r_j}\left(\frac{u_j - \bar{w}}{i}\right) dV(w),$$

where $u_j \in T_\Omega, j = 1, ..., m$.

These general results may also hold for all $p \leq 1$, with essentially the same proof, in many other complicated domains. In the following theorem, we extend an already known result of B. Sehba originally established for classical Bergman spaces on the tube in two directions simultaneously, namely to product domains and mixed norm function spaces. Earlier, this result was obtained by O. Yaroslavceva for the simple case of the unit disk. The proofs in both cases are similar.

Theorem 14. [5] *Let*

$$T_{\overrightarrow{\beta}} f(\overrightarrow{z}) = \int\limits_{T_\Omega^m} \frac{f(w_1, ..., w_m) \prod\limits_{j=1}^{m} \Delta^{\beta_j - \frac{n}{r}}(w_j) dv(w_j)}{\Delta^{\beta_1 + \frac{n}{r}}(\frac{z_1 - \overline{w}_1}{i}) ... \Delta^{\beta_m + \frac{n}{r}}(\frac{z_m - \overline{w}_m}{i})},$$

$dv(w) = dudv, w = u + iv \in T_\Omega$, $\overrightarrow{z} = (z_1, ..., z_m) \in T_\Omega$. *Let* $\beta_j > \beta_0, j = 1, ..., m,$ *for some fixed* β_0 *large enough. Then the operator* $T_{\overrightarrow{\beta}}$ *maps* $L^{\overrightarrow{p}}_{\overrightarrow{\nu}}(T_\Omega^m)$ *into* $A^{\overrightarrow{p}}_{\overrightarrow{\nu}}(T_\Omega^m)$, $p_j > 1, \nu_j > \frac{n}{r} - 1, j = 1, ..., m$.

Theorem 15. [6] *Let*

$$[R_{x,y}(g)(w)] = [(Imw)^{-m\frac{2n}{r} + \sum\limits_{j=1}^{m} y_j}] \int\limits_{T_\Omega} ... \int\limits_{T_\Omega} g(z_1, ..., z_m) \frac{[\prod\limits_{j=1}^{m} (Imz_j)^{x_j}] dv(z_1) ... dv(z_m)}{\prod\limits_{j=1}^{m} |\Delta^{x_j + y_j}(\frac{w - z_j}{i})|},$$

for $g \in L^1(T_\Omega^m, dv(z_1), ..., dv(z_m)), w \in T_\Omega$. *Let* $1 < p < \infty, s_j > (\frac{n}{r} - 1), j = 1, ..., m$. *Let* $x_j > x_0$, $y_j > y_0$, *for each* $j = 1, ..., m$, *where* x_0, y_0 *are large enough positive numbers. Then, there exists a constant* $C > 0$ *such that*

$$\int\limits_{T_\Omega} |(R_{x,y}(w))|^p (Imw)^{(m-1)(\frac{2n}{r}) + \sum\limits_{j=1}^{m} (s_j - \frac{n}{r})} dv(w) \leq$$

$$\leq c \int\limits_{T_\Omega} ... \int\limits_{T_\Omega} |g(z_1, ... z_m)|^p \prod_{j=1}^{m} (Imz_j)^{s_j - \frac{n}{r}} dv(z_j).$$

Let $\vec{\alpha} = (\alpha_1, ..., \alpha_m), \vec{\beta} = (\beta_1, ..., \beta_m)$ or $\vec{\beta} = (\beta, ..., \beta)$. We modify the operator $R_{x,y}$ defined above. Let for $g \in L^1(T_\Omega^m)$,

$$[G_{\vec{\alpha},\beta}(g)](x + iy_1, ..., x + iy_m) = \int\limits_{T_\Omega} \frac{g(w)[\Delta^\beta(Imw)] dv(w)}{\left[\prod\limits_{j=1}^{m} \Delta^{\alpha_j}(\frac{w - (\bar{x} + \bar{y}_j)}{i})\right]};$$

$x \in \mathbb{R}^n, y_j \in \Omega, j = 1, ..., m, \alpha_j > 0, \beta > \frac{n}{r} - 1, j = 1, ..., m$.

$$[G_{\vec{\alpha},\vec{\beta}}(g)](x_1 + iy, ..., x_m + iy) = \int\limits_{T_\Omega} \frac{g(w)[\Delta^\beta(Imw)] dv(w)}{\left[\prod\limits_{j=1}^{m} \Delta^{\alpha_j}(\frac{w - (\bar{x}_j + i\bar{y}_j)}{i})\right]};$$

$\beta > \frac{n}{r} - 1, j = 1, ...m, \alpha_j > 0, x_j \in \mathbb{R}^n, j = 1, ..., m, y \in \Omega.$

In the following theorem, we provide various new estimates for several integral operators of Bergman-type on semiproducts and products of tube domains. These estimates extend known results from the unit disk and may also have complete analogues in bounded strongly pseudoconvex domains; we leave this as an open problem. For the definitions of these operators, we refer the reader to the papers of the first author with S. Kurilenko and to the related references cited therein. One of these operators ($G_{x,y}$ operators) was defined above.

Theorem 16. [6] *For* $1 \leq p < \infty$, *some* $\alpha_j \in (\alpha_0, \alpha_0^{'})$; $\beta_j \in (\beta_0, \beta_0^{'})$; $\beta \in (\tilde{\beta}_0, \tilde{\beta}_0^{'})$, $\nu_j > (\frac{n}{r} - 1)$, $j = 1, ..., m$, *and for some fixed positive* $\alpha_0^j, (\alpha_0^j)', \beta_0^j, (\beta_0^j)', \tilde{\beta}_0^j, (\tilde{\beta}_0^j)', j = 1, ..., m$, *the following estimates hold:*

1) $\|G_{\vec{\alpha},\beta}(g)\|_{(A^p_{\vec{\nu}})_3} \leq c_1 \|g\|_{(A^p_\tau)(T_\Omega)}$; *for some values* $\vec{\nu}$ *and* τ,
2) $\|G_{\vec{\alpha},\beta}(g)\|_{(A^p_\nu)_2} \leq c_2 \|g\|_{(A^p_\tau)(T_\Omega)}$; *for some values* $\vec{\nu}$ *and* τ,
3) $\|V_{\vec{\alpha},\vec{\beta}}(g)\|_{(A^p_\nu)_1} \leq c_3 \|g\|_{(A^p_{\vec{\tau}})_3(T_\Omega)}$; *for some values* ν *and* $\vec{\tau}$,
4) $\|U_{\vec{\alpha},\vec{\beta}}(g)\|_{(A^p_\nu)_1} \leq c_4 \|g\|_{(A^p_{\vec{\tau}})_2(T_\Omega)}$; *for some values* ν *and* $\vec{\tau}$,

where $\alpha_0^j, ..., (\beta_0^j)'$ *depend on* $\nu_j, \tau_j, p, n, \nu, \tau, j = 1, ..., m$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$.

In the following theorem, we formulate a result that relates spaces of different dimensions on tube domains and pseudoconvex domains. We note that in the case $m = 1$, the additional condition in Theorem 17 disappears, and we recover a known result for the aforementioned domains. The theorem is stated in the context of tube domains and bounded pseudoconvex domains; the proof arguments are adapted from earlier work on the unit disk and polydisk by other authors. Nevertheless, the result is of independent interest in this more general setting.

Theorem 17. [7] *1) Let* $1 < p_j < \infty, j = 1, ..., m$, *and let*

$$T_\beta f(\overrightarrow{z}) = \int_{T_\Omega} \frac{f(w) d\nu_{\beta_1}(w)}{\prod_{j=1}^m \Delta^{\beta + \frac{2n}{r}}(\frac{z_j - w}{i})},$$

where $\overrightarrow{z} = (z_1, ..., z_m), z_j \in T_\Omega, j = 1, ..., m$. *Let* $\beta > \beta_0, j = 1, ..., m$, *for some fixed* β_0 *large enough. Then,* T_β *maps* $A^{p_m}_\nu(T_\Omega)$ *or* $L^{p_m}_\nu$ *to* $A_{\overrightarrow{\nu}\overrightarrow{p}}(T^m_\Omega)$,

$$\nu = \sum_{j=1}^{m-1} [\nu_j + \frac{2n}{r}](\frac{p_m}{p_j}) + \nu_m.$$

Here $A^p_\nu = A^p_\nu(T^1_\Omega)$, *if for* $F \in A^{\overrightarrow{q}}(\overrightarrow{\alpha})(T^m_\Omega) F(w, ..., w) \in A^{q_m}(\alpha)(T_\Omega)$, *where*

$$\alpha = \beta - \frac{\tau q_m}{p_m}, \tau = \alpha_m + \sum_{j=1}^{m-1} (\alpha_j + \frac{2n}{r}) \frac{p_m}{p_j},$$

$$\beta_1 = (\beta + \frac{2n}{r})m - \frac{2n}{r}, \frac{1}{p_j} + \frac{q}{q_j} = 1, j = 1, ..., m.$$

2) Let

$$(S_{\beta_0})(f(\overrightarrow{z})) = \int_D f(w) \prod_{j=1}^{m} K_{\tilde{\beta}_0}(z_j, w) d\nu_{\beta_1}(w),$$

$\overrightarrow{z} = (z_1, ..., z_m), z_j \in \Lambda, j = 1, ..., m$. *Let* $\beta > \tilde{\beta}_0^*, j = 1, ..., m$, *for some fixed large enough* $\tilde{\beta}_0^*$, $0 < p_j < \infty$. *Then, the operator* $(S_{\overrightarrow{\beta}})$ *maps* $A_\nu^{p_m}(\Lambda)$ *or* $L_\nu^{p_m}$ *into* $A_{\overrightarrow{\nu}}^{\overrightarrow{p}}(\Lambda^m)$,

$$\nu = \sum_{j=1}^{m-1} [\nu_j + (n-1)](\frac{p_m}{p_j}) + \nu_m, 1 < p_j < \infty, \nu_j > -1, j = 1, ..., m, \nu > -1,$$

if for $F \in A_{\overrightarrow{\alpha}}^{\overrightarrow{q}}(\Lambda^m) F(w, ..., w) \in A^{q_m}(\tilde{\alpha})(\Lambda)$, *where*

$$\tilde{\alpha} = \beta_0 - \frac{\tau q_m}{p_m}, \tau = \alpha_m + \sum_{j=1}^{m-1} (\alpha_j + n + 1) \frac{p_m}{p_j},$$

$\tilde{\beta}_0 = \beta_0 + n + 1, \beta_1 = (\beta_0 + n + 1)m - (n + 1), \frac{1}{p_j} + \frac{1}{q_j} = 1, j = 1, ..., m.$

In the following theorem, we show the boundedness of certain Herz-type integral operators (which are analogous to Bergman-type operators) acting on function spaces of different dimensions in tube domains in $\mathbb{C}^n$. The proof of this theorem is purely technical and was given in papers by the first author and S. Kurilenko. We refer the reader to those papers for the definition of the function spaces mentioned in the theorem. To the best of our knowledge, these results are new even in the simplest cases of the unit disk and the polydisk.

Theorem 18. [3] *For* $1 \le p < \infty$, *the following estimate holds*

$$\|T_{\vec{\alpha},\beta,\Gamma}(g)\|_{(L^p_{\vec{\nu}})_1(T^m_\Omega)} \le c\|g\|_{(L^p_\tau)(T_\Omega)},$$

where $\frac{1}{p} + \frac{1}{p'} = 1, \beta > -1, p'\Gamma > -1$,

$$\alpha_j > \frac{\beta}{m} + \frac{2n}{rm} + \frac{3n}{rp} - \frac{1}{p} + \frac{3n}{rp'm} - \frac{1}{p'm} + \frac{\Gamma}{m}, j = 1, ..., m,$$

and

$$\tau = p\beta + \frac{2nm}{r} - p\sum_{j=1}^{m}(\alpha_j - \frac{\nu_j}{p}) + p\Gamma + \frac{4np}{rp'} - \frac{nm}{rp}.$$

Let $\alpha_j > 0, b_j > -1, \Gamma_j > -1, j = 1, ..., m$. Now we define another new integral Herz-type operator

$$[T^1_{\vec{\alpha},\vec{\beta},\vec{\Gamma}}(g)](z_1, ..., z_m) =$$

$$= \int_{T_\Omega} ... \int_{T_\Omega} \int_{B(\tilde{w}_1,r)} ... \int_{B(\tilde{w}_m,r)} g(w_1, ..., w_m) \frac{\prod_{j=1}^m [\Delta^{\beta_j}(Im w_j)] dv(w_1)...dv(w_m)}{[\prod_{j=1}^m \Delta^{\alpha_j}(\frac{z_j - \bar{w}_j}{i})]} \times .$$

$$\times \prod_{j=1}^{m} \Delta^{\Gamma_j}(Im \tilde{w}_j) dv(\tilde{w}_1)...dv(\tilde{w}_m),$$

where $z_j \in T_\Omega, j = 1, ..., m$ for a function g from L^1 class on product of tubes. These Bergman-type integral operators are new even in one dimensional case.

In the following two theorems, we present new results on Bergman-type integral operators in tube domains, which were obtained in joint work by the first author and S. Kurilenko. These results may also be valid in bounded strongly pseudoconvex domains. Although the proofs involve lengthy technical calculations, the results themselves are new, even in the one-dimensional case. In particular, these Bergman-type (Herz-type) integral operators are new already for the unit disk.

Theorem 19. [3] *For $1 \leq p < \infty$, the following estimate holds*

$$\|T^1_{\vec{\alpha},\vec{\beta},\vec{\Gamma}}(g)\|_{(L^p_{\vec{\nu}})_2(T^m_\Omega)} \leq c\|g\|_{(L^p_{\vec{\tau}})(T^m_\Omega)},$$

where $\frac{1}{p} + \frac{1}{p'} = 1, \beta_j > -1, p'\Gamma_j > -1$,

$$\alpha_j > \beta_j + \Gamma_j + \frac{s_j}{p} - 1 + \frac{3n}{r} + \frac{n}{p} + \frac{2n}{rp'}, j = 1, ..., m,$$

and

$$\tau_j = p\beta_j + \frac{2np}{r} + s_j - p\alpha_j + \nu_j + p\Gamma_j + \frac{2np}{rp'}, j = 1, ..., m.$$

Let

$$(T_\beta h)(\vec{w}) = \underbrace{\int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n}}_{m} \int_\Omega \frac{h(x_1 + iy, ..., x_m + iy)\Delta^\beta(y)}{\prod_{j=1}^m \Delta^{\beta_j + \frac{n/r+mn/r}{m}}(\frac{\bar{x}_j + i\bar{y} - w_j}{i})} dx dy_1 ... dy_m,$$

where $\vec{w} = \{w_1, ..., w_m\} = \{\zeta_1 + i\eta, ..., \zeta_m + i\eta\} \in T_\Omega$,

$$(\tilde{T}_{\vec{\beta}}h)(\vec{w}) = \int_{\Omega^m} \int_{\mathbb{R}^n} \frac{h(x + iy_1, ..., x + iy_m)\Delta^{\beta_1}(y_1)...\Delta^{\beta_m}(y_m)}{\prod_{j=1}^m \Delta^{\beta_j + \frac{n/r+mn/r}{m}}(\frac{\bar{x} + i\bar{y}_j - w_j}{i})} dx dy_1 ... dy_m,$$

$\vec{w} = \{w_1, ..., w_m\} = \{\zeta + i\eta_1, ..., \zeta + i\eta_m\} \in T_\Omega$, and $\vec{\beta} = \{\beta_1, ..., \beta_m\}, \beta > \frac{n}{r} - 1, \beta_j > \frac{n}{r} - 1, j = 1, ..., m, h \in L^1(T^m_\Omega)$.

Theorem 20. [4] *Let $\beta_j > \frac{n}{r} - 1, j = 1, ..., m, 1 < p < \infty, \beta_j > 2 + \frac{n}{r} - \frac{2n}{pr} - \frac{3}{m} - \frac{2}{p} + \frac{n}{rmp} + \frac{2}{mp}, \tau_j > \frac{n}{r}p - p - \frac{n}{r}, \tau_j - p\beta_j < 1 - \frac{n}{r}, j = 1, ..., m$. Then,*

$$\int_{\Omega^m} \int_{\mathbb{R}^n} |(\tilde{T}_{\vec{\beta}}h)(\vec{w})|^p \Delta^{\tau_1}(\eta_1) \times ... \times \Delta^{\tau_m}(\eta_m) d\eta d\zeta_1 ... d\zeta_m \leq$$

$$\leq c \int_{\Omega^m} \int_{\mathbb{R}^n} |h(\vec{z})|^p \Delta^{\tau_1}(y_1) \times ... \times \Delta^{\tau_m}(y_m) dx dy_1 ... dy_m,$$

where $\vec{w} = \{\zeta + i\eta_1, ..., \zeta + i\eta_m\}, \vec{z} = \{z_1, ..., z_m\} = \{x + iy_1, ...x + iy_m\}, \vec{z} \in T^m_\Omega, \vec{w} \in T^m_\Omega$.

Next, let $1 \leq p < \infty, f = f(z_1, ...z_m)$. We consider analytic subspaces of $H(T_\Omega^m)$, where $T_\Omega^m = T_\Omega \times ... \times T_\Omega, \nu_j > \frac{n}{r} - 1, \nu > \frac{n}{r} - 1, j = 1, ..., m$. These are the spaces $(A_\nu^p)_1, (A_\nu^p)_2, (A_\nu^p)_3$ defined by the following norms:

$$\|f\|^p_{(A^p_{\vec{\nu}})_1} = \int\limits_{T_\Omega} .... \int\limits_{T_\Omega} |f(x_1 + iy_1, ..., x_m + iy_m)|^p \prod_{j=1}^m \Delta^{\nu_j - \frac{n}{r}}(y_j) dx_j dy_j < \infty,$$

$$\|f\|^p_{(A^p_{\vec{\nu}})_2} = \int\limits_{\mathbb{R}^n} .... \int\limits_{\mathbb{R}^n} \int\limits_{\Omega} |f(x_1 + iy_1, ..., x_m + iy_m)|^p \Delta^{\nu - \frac{n}{r}}(y) (\prod_{j=1}^m dx_j) dy < \infty,$$

$$\|f\|^p_{(A^p_{\vec{\nu}})_3} = \int\limits_{\mathbb{R}^n} \int\limits_{\Omega} .... \int\limits_{\Omega} |f(x_1 + iy_1, ..., x_m + iy_m)|^p \prod_{j=1}^m \Delta^{\nu_j - \frac{n}{r}}(y_j) dx dy_j < \infty.$$

Let also for $g \in L^1(T_\Omega^m)$,

$$(V_{\vec{\alpha},\vec{\beta}} g)(\vec{w}) = \underbrace{\int\limits_{\Omega} ... \int\limits_{\Omega}}_{m} \int\limits_{\mathbb{R}^n} \frac{g(x + iy_1, ..., x + iy_m)(\Delta y_1)^{\beta_1} \times ... \times (\Delta y_m)^{\beta_m}}{\prod_{j=1}^m \Delta^{\alpha_j}(\frac{\bar{x} + i\bar{y}_j - w_j}{i})} dx dy_1 ... dy_m,$$

$$(U_{\vec{\alpha},\vec{\beta}} g)(\vec{w}) = \underbrace{\int\limits_{\mathbb{R}^n} ... \int\limits_{\mathbb{R}^n}}_{m} \int\limits_{\Omega} \frac{g(x_1 + iy, ..., x_m + iy)(\Delta y)^{\beta}}{\prod_{j=1}^m \Delta^{\alpha_j}(\frac{\bar{x} + i\bar{y}_j - w_j}{i})} dy dx_1 ... dx_m,$$

$w = (w_1, ..., w_m), w_j \in T_\Omega, j = 1, ..., m, \beta_j > \frac{n}{r} - 1, \beta > \frac{n}{r} - 1, \alpha_j > 0, j = 1, ..., m, \vec{\beta} = (\beta_1, ..., \beta_m)$ or $\vec{\beta} = (\beta, ..., \beta)$.

Let also for $g \in L^1(T_\Omega^m)$,

$$[G_{\vec{\alpha},\beta}(g)](x + iy_1, ..., x + iy_m) = \int\limits_{T_\Omega} \frac{g(w)[\Delta^{\beta}(Imw)] dv(w)}{|\prod_{j=1}^m \Delta^{\alpha_j}(\frac{w - (\bar{x} + \bar{y}_j)}{i})|},$$

where $x \in \mathbb{R}^n, y_j \in \Omega, j = 1, ..., m, \alpha_j > 0, \beta > \frac{n}{r} - 1, j = 1, ..., m$;

$$[G_{\vec{\alpha},\vec{\beta}}(g)](x_1 + iy, ..., x_m + iy) = \int\limits_{T_\Omega} \frac{g(w)[\Delta^{\beta}(Imw)] dv(w)}{|\prod_{j=1}^m \Delta^{\alpha_j}(\frac{w - (\bar{x} + i\bar{y}_j)}{i})|},$$

where $x_j \in \mathbb{R}^n, j = 1, ..., m, y \in \Omega, \beta > \frac{n}{r} - 1, \alpha_j > 0, j = 1, ..., m$.

In what follows, we define various quasinorms on products of tubular domains and on semiproducts of such domains. These quasinorms extend the classical quasinorm of the analytic Bergman space $A^p_\alpha(T_\Omega)$ on tube domains, which has been studied by Sehba and many other authors. In theorem 21, we present new results on the action of certain Bergman-type integral operators on analytic spaces in tube domains equipped with these quasinorms, thereby extending previously known results obtained by B. Sehba and his coauthors.

Theorem 21. [4] *For $1 \le p < \infty$, some $\alpha_j \in (\alpha_0, \alpha_0^{'})$; $\beta_j \in (\beta_0, \beta_0^{'})$; $\beta \in (\tilde{\beta}_0, \tilde{\beta}_0^{'})$, $\nu_j > \frac{n}{r} - 1$, $\tau_j > \frac{n}{r} - 1$, $j = 1, ..., m$ for some fixed positive $\alpha_0^j, (\alpha_0^j)', \beta_0^j, (\beta_0^j)', j = 1, ..., m$. The following estimates hold:*

1) $\|G_{\vec{\alpha},\beta}(g)\|_{(A^p_{\vec{\nu}})_3} \le c_1 \|g\|_{(A^p_{\tau})(T_\Omega)}$; *for some values $\vec{\nu}$ and τ,*
2) $\|\tilde{G}_{\vec{\alpha},\beta}(g)\|_{(A^p_{\nu})_2} \le c_2 \|g\|_{(A^p_{\vec{\tau}})(T_\Omega)}$; *for some values ν and τ,*
3) $\|V_{\vec{\alpha},\vec{\beta}}(g)\|_{(A^p_{\vec{\nu}})_1} \le c_3 \|g\|_{(A^p_{\vec{\tau}})_3(T_\Omega)}$; *for some values ν and $\vec{\tau}$,*
4) $\|U_{\vec{\alpha},\vec{\beta}}(g)\|_{(A^p_{\vec{\nu}})_1} \le c_4 \|g\|_{(A^p_{\vec{\tau}})_2(T_\Omega)}$; *for some values ν and $\vec{\tau}$,*

where $\alpha_0^j, ..., (\beta_0^j)'$ depend on $\nu_j, \tau_j, p, n, \nu, \tau, j = 1, ..., m$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

In the following two theorems, we provide complete descriptions of the traces of certain BMOA-type spaces of several variables on tube domains. The same results, with very similar proofs, are valid in bounded pseudoconvex domains with smooth boundary. Detailed proofs in the context of the unit ball can be found in earlier papers by the first author. All these sharp results on traces are heavily based on new projection theorems for the Bergman-type operators $S_{a,b}$ defined above and acting between BMOA-type spaces of different dimensions.

Theorem 22. [36] *Let $p > 1, \tau \in (0, \infty), r_j \in \mathbb{N}, s_j > -1, j = 1, ..., m$. If $t = (m+1)(2n/r) + \sum_{j=1}^{m} s_j$, then for $r = \sum_{j=1}^{m} r_j$, we have*

$$Trace(M^p_{r_1,...,r_m,\tau,s_1,...,s_m}(\Omega^m)) = M^p_{r,p,\tau m}(\Omega),$$

for all $n, n|r > n_0$, where $n_0 = n_0(p, \tau, r_1, ..., r_m, m)$.

Theorem 23. [36] *Let $p \le 1, \tau \in (0, \infty), r_j \in \mathbb{N}, s_j > -1, j = 1, ..., m$. If $r_j|p \in \mathbb{N}, j = 1, ..., m$ and $t = (m-1)(2n|r) + \sum_{j=1}^{m} s_j$, then for $r = \sum_{j=1}^{m} r_j$ we have*

$$Trace(M^p_{r_1,...,r_m,\tau,s_1,...,s_m}(\Omega^m)) = M^p_{r,p,\tau m}(\Omega),$$

for all $n, n|r > n_0$, where $n_0 = n_0(p, \tau, r_1, ..., r_m, m)$.

Some interesting new results on Bergman-type projections between Bloch-type spaces of different dimensions were obtained earlier in paper by the first author and in joint work with S. Kurilenko; we refer the reader to the references for details.

In [11],an extension of a classical result on the Bergman projection from the unit disk to pseudoconvex domains and tubular domains over symmetric cones was provided. The proofs there are based on arguments developed earlier in [11] and [12] for the less general setting of the unit ball. These new results complement similar recent results on Bergman-type projections obtained in [1]-cite1 for pseudoconvex domains and in [6]-[9] for tubular domains. Over the past two decades, many authors have obtained interesting results on the Bergman projection in analytic function spaces of one and several variables with Muckenhoupt weights, Bekolle weights, etc. These topics are not considered in the present paper. Another group of interesting new results concerns Bergman projection theorems in analytic spaces of one and several variables under various boundary conditions on the domain in $\mathbb{C}$ or $\mathbb{C}^n$; we refer the interested readers to the work of E. Stein and his coauthors. All these results can be considered as extensions of old classical results on Bergman projections in classical function spaces of one and several variables in the unit disk, the polydisk and the unit ball, which were obtained by various authors in the last century.

Our results may also be valid, with very similar proofs, in a wide range of domains $\mathbb{C}^n$ with complicated structures, including Siegel domains of the second type, matrix domains, and bounded symmetric domains.

Moreover, using the arguments and approaches developed in the proofs of the theorems in this paper for tube and pseudoconvex domains, various projection theorems in harmonic function spaces on different domains (see, e.g., [38]) can likely be extended to new projection theorems between harmonic function spaces of different dimensions. We leave this as an open problem for interested readers.

Bergman projections and Bergman-type projections acting on various analytic function spaces over domains in $\mathbb{C}^n$ have been considered by many authors in recent years; see, for example, papers by Zeytuncu, Wagner, Wick, Stockdale, Li, and others. Extending these results to function spaces of different dimensions, as discussed in this paper, is an interesting new problem.

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