Математички Билтен
48 (LXXIV) 1 (2024), 65–71

UDC: 510.22:515.122/.123
DOI: 10.37560/matbil2448165m

CHAIN RELATION IN A FAMILY OF SETS. CHAIN FUNCTION

ZORAN MISAJLESKI AND EMIN DURMISHI

Abstract. In this paper we define connectedness of a family of sets, by defining the terms finite chain in the family and chain relation between two sets. Furthermore, we define a chain function between two families of sets. Some properties of connected families and chain functions are obtained.

1. Introduction

In [6] a characterization of connectedness of a topological space is given by using the notion of a finite chain in a covering, while in [5] it is generalized into the notion of a chain connected set in a topological space. In [5] the notion of a chain connected set in a space consisting of a set and a family of coverings is introduced.

The topological space X is connected if for every open covering $\mathcal{F}$ and every two points $x, y \in X$, there exists a finite sequence $F_1, F_2, \ldots, F_n$ of elements of $\mathcal{F}$ such that x belongs to the first element of the sequence, y belongs to the last element and the intersection of every two consecutive elements is nonempty (see [6]). If this is satisfied, we say that for every open covering $\mathcal{F}$ and every two points $x, y \in X$ there exists a chain in $\mathcal{F}$ that connects x and y. The advantage of this criterion of connectedness is that it is suitable for generalizations.

In papers [5, 7, 10, 8, 1, 2, 4, 9] this notion is generalized to a chain connected set in a topological space. The set C is chain connected in a topological space X if for every $x, y \in C$ and for every open covering $\mathcal{F}$ of X, there exists a chain in $\mathcal{F}$ that connects x and y. It follows that the topological space is connected if and only if it is chain connected in X. If this statement is true for a particular covering $\mathcal{F}$, we say that C is $\mathcal{F}$-chain connected in X.

In the paper [5], topological spaces are generalized by considering arbitrary families of sets instead of open coverings, and the concept of chain connectedness is further extended to a space more general than a topological space.

2010 *Mathematics Subject Classification.* Primary: 03E20 Secondary: 54D05.
Key words and phrases. Chain, Chain connectedness, Chain function.

In this article an arbitrary family $\mathcal{F}$ of sets is considered and the notions of $\mathcal{F}$-chain relation between two members of that family and $\mathcal{F}$-chain relation between two elements of $\bigcup_{F\in\mathcal{F}} F$ are introduced.

2. $\mathcal{F}$-Chain connected sets in a family of sets

Let $\mathcal{F}$ be a family of sets and let $A, B \in \mathcal{F}$.

Definition 2.1. *A* ***chain*** *$F_1, F_2, \ldots, F_n$* ***in*** *$\mathcal{F}$ is a finite sequence $F_1, F_2, \ldots, F_n$ such that*

$$F_i \in \mathcal{F},\ i = 1, 2, \ldots, n;\ \text{and}\ F_i \cap F_{i+1} \neq \emptyset,\ i = 1, 2, \ldots, n-1.$$

Definition 2.2. *A* ***chain in*** *$\mathcal{F}$* ***from*** *A* ***to*** *B is a chain*

$$F_1, F_2, \ldots, F_n$$

in $\mathcal{F}$ such that $A \cap F_1 \neq \emptyset$ and $B \cap F_n \neq \emptyset$. In that case we say that A is in a ***$\mathcal{F}$-chain relation*** *with B and we use notation $A \underset{\mathcal{F}}{\sim} B$.*

Proposition 2.1. ***$\mathcal{F}$-chain relation*** *is an equivalence relation.*

Proof. Clearly $\mathcal{F}$-chain relation is reflexive. Let $A \underset{\mathcal{F}}{\sim} B$. Then there exists a sequence $F_1, F_2, \ldots, F_n$ in $\mathcal{F}$ such that $F_i \cap F_{i+1} \neq \emptyset,\ i = 1, 2, \ldots, n-1,\ A \cap F_1 \neq \emptyset$ and $B \cap F_n \neq \emptyset$, i.e., $F_n, F_{n-1}, \ldots, F_1$ is a sequence in $\mathcal{F}$ such that $F_i \cap F_{i+1} \neq \emptyset,\ i = 1, 2, \ldots, n-1,\ B \cap F_n \neq \emptyset$ and $A \cap F_1 \neq \emptyset$, i.e., $B \underset{\mathcal{F}}{\sim} A$. Let $A \underset{\mathcal{F}}{\sim} B$ and $B \underset{\mathcal{F}}{\sim} C$. Firstly it follows that there exists a sequence $F_1, F_2, \ldots, F_n$ in $\mathcal{F}$ such that $F_i \cap F_{i+1} \neq \emptyset,\ i = 1, 2, \ldots, n-1,\ A \cap F_1 \neq \emptyset$ and $B \cap F_n \neq \emptyset$; and a sequence $G_1, G_2, \ldots, G_m$ in $\mathcal{F}$ such that $G_i \cap G_{i+1} \neq \emptyset,\ i = 1, 2, \ldots, m-1,\ B \cap G_1 \neq \emptyset$ and $C \cap G_m \neq \emptyset$ and secondly a sequence $F_1, F_2, \ldots, F_n, B, G_1, G_2, \ldots, G_m$ such that every two consecutive elements have nonempty intersection, $A \cap F_1 \neq \emptyset$ and $C \cap G_m \neq \emptyset$. Hence $A \underset{\mathcal{F}}{\sim} C$. □

Because of the Proposition 2.1, instead of a chain in $\mathcal{F}$ from A to B, we can say: a chain in $\mathcal{F}$ that connects A and B and instead A is in $\mathcal{F}$-chain relation with B we say: A and B are in $\mathcal{F}$-chain relation.

Since $\mathcal{F}$-chain relation is an equivalence relation, it follows that the set $\mathcal{F}$ can be divided into classes of equivalence $[A] = \left\{ B \in \mathcal{F} \mid A \underset{\mathcal{F}}{\sim} B \right\}, \forall A \in \mathcal{F}$.

Let $\mathcal{F}$ be a family of sets, let $A, B \in \mathcal{F}$ and let $x, y \in \bigcup_{F\in\mathcal{F}} F$.

Definition 2.3. *The element x is in $\mathcal{F}$-chain relation with y, denoted by $x \underset{\mathcal{F}}{\sim} y$, if there exists a chain $F_1, F_2, \ldots, F_n$ in $\mathcal{F}$ such that $x \in F_1$ and $y \in F_n$.*

The relation is an equivalence relation in $\bigcup_{F\in\mathcal{F}} F$. If $x \underset{\mathcal{F}}{\sim} y$ we say that there exists a chain $F_1, F_2, \ldots, F_n$ in $\mathcal{F}$ that connects x and y.

Proposition 2.2. *The sets A and B are in $\mathcal{F}$- chain relation if and only if there exists a pair $(x, y) \in A \times B$ such that $x \underset{\mathcal{F}}{\sim} y$.*

Proof. If A and B are in $\mathcal{F}$-chain relation, then there exists a chain $F_1, F_2, \dots, F_n$ in $\mathcal{F}$ such that $F_i \cap F_{i+1} \neq \emptyset$, $i = 1, 2, \dots, n-1$, $A \cap F_1 \neq \emptyset$ and $B \cap F_n \neq \emptyset$. It follows that there exist $x \in A \cap F_1$ and $y \in B \cap F_n$. Thus $(x, y) \in A \times B$ and $F_1, F_2, \dots, F_n$ is a chain in $\mathcal{F}$ that connects x and y. The converse direction is analogous. □

Let $\mathcal{F}$ and $\mathcal{G}$ be two families of sets. We say that $\mathcal{F}$ is refinement of $\mathcal{G}$, denoted by $\mathcal{F} \prec \mathcal{G}$, if for every $A \in \mathcal{F}$ there exists $B \in \mathcal{G}$ such that $A \subseteq B$.

Theorem 1. *If for the sets A and B holds $A \underset{\mathcal{F}}{\sim} B$ and there exist $C, D \in \mathcal{F}$ such that $A \subseteq C$ and $B \subseteq D$, then $C \underset{\mathcal{F}}{\sim} D$.*

Proof. Let $A \underset{\mathcal{F}}{\sim} B$, i.e., there exists a sequence $F_1, F_2, \dots, F_n$ in $\mathcal{F}$ such that $F_i \cap F_{i+1} \neq \emptyset$, $i = 1, 2, \dots, n-1$, $A \cap F_1 \neq \emptyset$ and $B \cap F_n \neq \emptyset$. Since $A \subseteq C$ and $B \subseteq D$ it follows that $C \cap F_1 \neq \emptyset$ and $D \cap F_n \neq \emptyset$, i.e., $F_1, F_2, \dots, F_n$ is a chain in $\mathcal{F}$ that connects C and D. □

Theorem 2. *If for the sets A and B holds $A \underset{\mathcal{F}}{\sim} B$ and $\mathcal{F} \prec \mathcal{G}$, then there exist $C, D \in \mathcal{G}$ such that $A \subseteq C$, $B \subseteq D$ and $C \underset{\mathcal{G}}{\sim} D$.*

Proof. Let $A \underset{\mathcal{F}}{\sim} B$, i.e., there exists a sequence $F_1, F_2, \dots, F_n$ in $\mathcal{F}$ such that $F_i \cap F_{i+1} \neq \emptyset$, $i = 1, 2, \dots, n-1$, $A \cap F_1 \neq \emptyset$ and $B \cap F_n \neq \emptyset$. Let $A \subseteq C$ and $B \subseteq D$. Since $\mathcal{F} \prec \mathcal{G}$ it follows firstly that there exists $G_i \in \mathcal{G}$ such that $F_i \subseteq G_i$, $i = 1, 2, \dots, n$ and secondly that the sequence $G_1, G_2, \dots, G_n$ is a chain in $\mathcal{G}$ that connects C and D. Thus $C \underset{\mathcal{G}}{\sim} D$. □

Let $\mathcal{F}$ be a family of sets.

Definition 2.4. *If any two elements of the family $\mathcal{F}$ are in a $\mathcal{F}$-chain relation, then $\mathcal{F}$ is called **a chain family**.*

As a direct consequence of the theorems 1 and 2 we have the following result.

Corollary 2.1. *If $\mathcal{F}$ is a chain family and $\mathcal{F} \prec \mathcal{G}$, then $\mathcal{G}$ is a chain family.*

Let X be a topological space and $A \subseteq X$.

Theorem 3. *The set A is chain connected in X if and only if for every open covering $\mathcal{U}$ of X, every two members of $\mathcal{U}$ which have nonempty intersection with A, are in $\mathcal{U}$-chain relation.*

Proof. Let A be chain connected in X, let $\mathcal{U}$ be an arbitrary open covering of X, and let $U, V \in \mathcal{U}$ be such that $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$. Let $x \in U \cap A$ and $y \in V \cap A$. Then there exists a chain $U_1, U_2, \dots, U_n$ in $\mathcal{U}$ that connects x and y. That same chain connects U and V.

Conversely, suppose that for any open covering $\mathcal{U}$ of X, every two members of $\mathcal{U}$ which have nonempty intersection with A, are in $\mathcal{U}$-chain relation. Let $\mathcal{U}$ be an open covering of X and $U, V \in \mathcal{U}$ such that $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$. It follows that U and V are in $\mathcal{U}$-chain relation. Let $x, y \in A$. Then there exist $U, V \in \mathcal{U}$ such that $x \in U$ and $y \in V$. It follows firstly that $x \in U \cap A$ and $y \in V \cap A$, and secondly that there exists a chain in $\mathcal{U}$ that connects x and y. Since this can be done for any open covering $\mathcal{U}$ of X and for arbitrary $x, y \in A$, then A is chain connected in X. □

If a chain family $\mathcal{F}$ is a covering of the set X, then $\mathcal{F}$ is called a **chain covering**.

A direct consequence of the previous theorem and criterion for connectedness by chain is the following corollary.

Corollary 3.1. *The topological space is connected if and only if every open covering of X is chain covering.*

Let X be a topological space. Let $\mathcal{F}$ be a family of subsets in X. Previously, when we considered a chain in $\mathcal{F}$ between two sets A and B, we assumed that $A, B \in \mathcal{F}$. We can generalize the notion of $\mathcal{F}$-chain relatedness between arbitrary sets A and B. The sets A and B are $\mathcal{F}$-chain related if there exists a chain in $\mathcal{F}$ that connects A and B. Now, we consider two families of sets: the family $\mathcal{F}$ and the family that consists of $\mathcal{F}$ and the sets A and B. In this case, most of the properties are analogues, but there are some exceptions. If $A, B \in \mathcal{P}(X)$ where $\mathcal{P}(X)$ is the power set of an arbitrary set X, then the $\mathcal{F}$-chain relation does not have to be reflexive and transitive. However, we can obtain new properties. For example: Consider X to be a topological space. By $\bar{A}$ we mean the closure of A. If A and B are in $\mathcal{F}$-chain relation, then $\bar{A}$ and $\bar{B}$ are also in $\mathcal{F}$-chain relation. The next theorem shows under which conditions the converse statement is true.

Theorem 4. *Let $\mathcal{F}$ be a family of open sets in a topological space X. If $\bar{A} \underset{\mathcal{F}}{\sim} \bar{B}$ then $A \underset{\mathcal{F}}{\sim} B$.*

Proof. Let $\bar{A} \underset{\mathcal{F}}{\sim} \bar{B}$, i.e., there exists a chain $F_1, F_2, \ldots, F_n$ in $\mathcal{F}$ that connects $\overline{A}$ and $\overline{B}$ such that $x \in \overline{A} \cap F_1$ and $y \in \overline{B} \cap F_n$ for some elements x and y. Since F_1 and F_n are open sets that contain x and y, respectively, it follows that $A \cap F_1 \neq \emptyset$ and $B \cap F_n \neq \emptyset$, i.e., $F_1, F_2, \ldots, F_n$ is a chain in $\mathcal{F}$ that connects A and B. □

3. Chain function

Let $\mathcal{F}$ and $\mathcal{G}$ be two families of sets.

Definition 3.1. *The function $F : \mathcal{F} \to \mathcal{G}$ is a **chain function** if for every $A, B \in \mathcal{F}$ such that $A \cap B \neq \emptyset$, it follows that $F(A) \cap F(B) \neq \emptyset$.*

Proposition 3.1. *If $F : \mathcal{F} \to \mathcal{G}$ is a chain function, then for every chain $F_1, F_2, \ldots, F_n$ in $\mathcal{F}$, $F(F_1), F(F_2), \ldots, F(F_n)$ is a chain in $\mathcal{G}$.*

Proof. Let $F : \mathcal{F} \to \mathcal{G}$ be a chain function and let $F_1, F_2, \ldots, F_n$ be a chain in $\mathcal{F}$, i.e., $F_i \in \mathcal{F}$, $i = 1, 2, \ldots, n$; and $F_i \cap F_{i+1} \neq \emptyset$, $i = 1, 2, \ldots, n-1$. Since F is a chain function, it follows that $F(F_i) \in \mathcal{G}$, $i = 1, 2, \ldots, n$ and $F(F_i) \cap F(F_{i+1}) \neq \emptyset$, $i = 1, 2, \ldots, n-1$. Hence

$$F(F_1), F(F_2), \ldots, F(F_n)$$

is a chain in $\mathcal{G}$. □

Proposition 3.2. *If $F : \mathcal{F} \to \mathcal{G}$ is a chain function, then for every $A, B \in \mathcal{F}$ with $A \underset{\mathcal{F}}{\sim} B$ it follows that $F(A) \underset{\mathcal{G}}{\sim} F(B)$.*

Proof. Let $A \underset{\mathcal{F}}{\sim} B$, i.e., there exists a chain $F_1, F_2, \ldots, F_n$ in $\mathcal{F}$ such that $A \cap F_1 \neq \emptyset$ and $B \cap F_n \neq \emptyset$. Since F is a chain function, the sequence $F(F_1), F(F_2), \ldots, F(F_n)$ is a chain in $\mathcal{G}$ such that $F(A) \cap F(F_1) \neq 0$ and $F(B) \cap F(F_n) \neq 0$. Thus $F(A) \underset{\mathcal{G}}{\sim} F(B)$. □

The next example shows that the converse claim of the previous proposition does not hold in general.

Example 1. *Let*

$$\mathcal{F} = \{[-1,0], [0,1], \{2\}\}, \ \mathcal{G} = \{(-1,0), (-1,1), (0,1)\}$$

and $F : \mathcal{F} \to \mathcal{G}$ be defined by

$$F([-1,0]) = (-1,0), \ F([0,1]) = (0,1) \text{ and } F(\{2\}) = (-1,1).$$

Then the only pair $A, B \in \mathcal{F}$ for which $A \underset{\mathcal{F}}{\sim} B$, is $[-1,0], [0,1] \in \mathcal{F}$ and for that pair holds $F([-1,0]) \underset{\mathcal{G}}{\sim} F([0,1])$, but F is not a chain function since

$$[-1,0] \cap [0,1] \neq \emptyset \text{ and } F([-1,0]) \cap F([0,1]) = \emptyset.$$

Proposition 3.3. *Every chain function $F : \mathcal{F} \to \mathcal{G}$ induces a function between the equivalence classes $F_\# : [\mathcal{F}] \to [\mathcal{G}]$.*

Proof. Let $[A] \in [\mathcal{F}]$, $[A] = \left\{ B \in \mathcal{F} | \, A \underset{\mathcal{F}}{\sim} B \right\}$. Define $F_\# : [\mathcal{F}] \to [\mathcal{G}]$ by

$$F_\#([A]) = [F(A)]$$

Using Proposition 3.2 it follows that the function $F_\#$ is well defined. □

The following example shows that $F_\#$ does not have to be a bijection, even when F is a bijection.

Example 2. *Let $\mathcal{F} = \{\{-1\}, \{1\}\}$, $\mathcal{G} = \{(-1,0], [0,1)\}$ and $F : \mathcal{F} \to \mathcal{G}$ be defined by*

$$F(\{-1\}) = (-1,0] \text{ and } F(\{1\}) = [0,1).$$

Then $[\{-1\}] \neq [\{1\}]$ but

$$F_\#([\{-1\}]) = F_\#([\{1\}]) = \{(-1,0], [0,1)\}.$$

The following example shows that if F is a bijective chain function, then F^{-1} does not have to be a chain function.

Example 3. *Let* $\mathcal{F} = \{(-1,1), (0,2), (2,3)\}$, $\mathcal{G} = \{(-1,1), (0,2), (1,3)\}$ *and* $F : \mathcal{F} \to \mathcal{G}$ *be defined by*

$$F((-1,1)) = (-1,1), \ F((0,2)) = (0,2), \ F((2,3)) = (1,3).$$

Then $F(\mathcal{F}) = \mathcal{G}$ *and* F *is a bijective chain function. A nonempty intersection have only the sets* $(-1,1)$ *and* $(0,2)$, *for which* $F((-1,1)) \cap F((0,2)) \neq \emptyset$, *but* F^{-1} *is not a chain function. Namely*

$$(0,2) \cap (1,3) \neq \emptyset \text{ but } F^{-1}((0,2)) \cap F^{-1}((1,3)) = \emptyset.$$

Theorem 5. *Let* X *and* Y *be two sets and let* $\mathcal{U}$ *be an arbitrary covering of* X. *Then every function* $f : X \to Y$ *induces a chain function* $F : \mathcal{U} \to f(\mathcal{U})$, $f(\mathcal{U}) = \{f(U) \,|U \in \mathcal{U}\}$ *defined by*

$$F(U) = f(U), \ \forall U \in \mathcal{U}.$$

Moreover, if f *is a bijection, then* F *is a bijection and* F^{-1} *is a chain function.*

Proof. Let $f : X \to Y$ be a given function. Define $F : \mathcal{U} \to f(\mathcal{U})$ by

$$F(U) = f(U), \text{ where } f(U) = \{f(x) |\, x \in U\}.$$

Let $U_1, U_2 \in \mathcal{U}$ such that $U_1 \cap U_2 \neq \emptyset$. Then, for $x \in U_1 \cap U_2$ the following holds:

$$f(x) \in f(U_1) \cap f(U_2), \text{ i.e., } F(U_1) \cap F(U_2) \neq \emptyset.$$

Thus, F is a chain function.

Let f be a bijection, and let $U_1, U_2 \in \mathcal{U}, \ U_1 \neq U_2$. Then there exists

$$x \in (U_1 \backslash U_2) \cup (U_2 \backslash U_1), \text{ i.e., } f(x) \in (f(U_1) \backslash f(U_2)) \cup (f(U_2) \backslash f(U_1)).$$

Thus $F(U_1) \neq F(U_2)$.

Let $V \in f(\mathcal{U})$. Then there exists $U \in \mathcal{U}$ such that $V = f(U)$, i.e., $F(U) = V$.

It remains to prove that F^{-1} is a chain function. Let $U_1, U_2 \in \mathcal{U}$ be such that $f(U_1) \cap f(U_2) \neq \emptyset$. Then, since f is an injection,

$$U_1 \cap U_2 \neq \emptyset, \text{ i.e., } F^{-1}(f(U_1)) \cap F^{-1}(f(U_2)) \neq \emptyset.$$

□

Let $\mathcal{F}$ be a family of sets. If X is the union of the elements of the sets of this family, then $\mathcal{F}$ is a covering of X in X and $X = (X, \mathcal{F})$ is a one covering space, i.e., a special case of a family coverings space that consists of one covering [3, 4]. Two sets of the family are in a $\mathcal{F}$-chain relation if and only if all elements of the sets are $\mathcal{F}$-chain related. So, other properties of $\mathcal{F}$-chain connectedness in the family $\mathcal{F}$ of sets may arise from the properties of chain connectedness in a family coverings space.

The authors acknowledge professor Nikita Shekutkovski for introducing to them the idea of the chain relation in a family of sets and chain function.

REFERENCES

[1] E. Durmishi, Z. Misajleski, A. Rushiti, F. Sadiki, A. Ibraimi, *Characterisation of Isolated Points in T_1 Spaces Using Chains*, Journal of Natural Sciences and Mathematics of UT, 7 (13-14), (2022) 108–113

[2] E. Durmishi, Z. Misajleski, A. Velkoska, *Totally Weakly Chain Separated Sets in a Topological Space*, Kragujevac Journal of Mathematics 47 (7), (2023) 1075–1086

[3] Z. Misajleski, E. Durmishi, A. Velkoska, *Chain Connected Set in a Space*, Proceedings of the Codema 2022, (2022) 35–42

[4] Z. Misajleski, A. Velkoska, E. Durmishi, *Chain Separatedness in a Family Coverings Space*, Matematichki Bilten, 47 (2), (2023) 105–112

[5] Z. Misajleski, N. Shekutkovski, A. Velkoska, *Chain Connected Sets in a Topological Space*, Kragujevac Journal of Mathematics, 43 (4), (2019) 575–586

[6] N. Shekutkovski, *On the Concept of Connectedness*, Matematicki Bilten, 40 (1), (2016) 5–14

[7] N. Shekutkovski, Z. Misajleski, E. Durmishi, *Chain Connectedness*, AIP Conference Proceedings, 2183, (2019) 030015-1–030015-4

[8] N. Shekutkovski, Z. Misajleski, E. Durmishi, *Product of Chain Connected Sets in Topological Spaces*, ROMAI Journal, 17 (2), (2021) 73–80

[9] N. Shekutkovski, Z. Misajleski, E. Durmishi, *Chain Connectedness - Homotopy and Shape Invariant*, Matematichki Bilten, 47 (2), (2023) 123–131

[10] N. Shekutkovski, Z. Misajleski, A. Velkoska, E. Durmishi, *Weakly Chain Separated Sets in a Topological Space*, Mathematica Montisnigri, 52 (1), (2021) 5–16

ZORAN MISAJLESKI
SS. CYRIL AND METHODIUS UNIVERSITY,
DEPARTMENT OF MATHEMATICS, FACULTY OF CIVIL ENGINEERING,
SKOPJE,
MACEDONIA
Email address: misajleski@gf.ukim.edu.mk

EMIN DURMISHI
UNIVERSITY OF TETOVA,
DEPARTMENT OF MATHEMATICS, FACULTY OF NATURAL SCIENCES AND MATHEMATICS
TETOVO,
NORTH MACEDONIA
Email address: emin.durmishi@unite.edu.mk

Received 30.11.2024
Revised 02.03.2025
Accepted 05.03.2025