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COMPARISON OF NUMERICAL AND EXACT SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH CONSTANT DELAY

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Abstract. Since it is well known that the analytical solution of fractional differential equations with delays is challenging, if not impossible, we have compared the exact solutions of two such equations using four numerical methods: Fractional Euler Method (FEM), Backward Euler Method (BEM), Weighted Difference Method (WDM), and Fractional Adams Method (FAM) in this study. The efficiency of each numerical approach is evaluated based on the relative difference between the exact and approximate solutions at predetermined time points, presented graphically and in tabular form. To provide a comprehensive comparison, the execution time is calculated by taking the average of over 10 executions for each method. From the results, we conclude that FAM has the lowest relative errors, is more accurate, and has a lower computational cost. FAM is the best alternative for accurate solutions and short execution times over extended timescales, FEM offers satisfactory accuracy and efficiency at intermediate timescales, WDM shows progress over time but has higher computational costs, and BEM consistently exhibits significant error rates and the variability of the execution time in the two examples.

Keywords: Fractional differential equations with delay, Fractional Euler methods, Fractional Adams method.

1. INTRODUCTION

Fractional delay differential equations (FDDEs) are a class of dynamical systems that merge the concepts of fractional calculus (which includes derivatives and integrals of non-integer orders) and delay differential equations (which include delays in the formulation of equations) [1], [2]. Thus, it includes fractional

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derivatives and time delays, making the equations suitable for accurately modeling real-world phenomena in various fields, such as physics, bioengineering, and control systems [3].

Solving FDDEs can be difficult because they are more convoluted than ordinary differential equations (ODEs). Therefore, we use two approaches towards finding the solutions of FDDEs: *numerical methods* - which use computers to approximate answers, and *analytical methods* - which try to find exact solutions using formulas [2]. The paper [4] provides comprehensive results regarding the existence, uniqueness, and stability of solutions to fractional differential equations with delay, supported by numerical experiments and theoretical analysis. These discoveries enhance the comprehension and utilization of fractional calculus in many domains. The exact solutions of FDDEs, particularly those involving the Caputo derivative, provide a benchmark for evaluating the accuracy of numerical methods. For instance, we derive the exact solution for a linear FDDE with the Caputo derivative to illustrate the impact of initial functions on the solution and the nature of the fractional derivative. However, finding explicit solutions for nonlinear FDDEs is generally not feasible, necessitating using numerical methods [5]. Various numerical methods have developed to solve FDDEs, including the Adomian Decomposition Method (ADM), Runge-Kutta-type methods (RKM), and the simplified reproducing kernel method (SRKM) [6], [7]. For instance, the ADM expresses the nonlinear part of the FDDE using Adomian polynomials, allowing for a recursive solution process that can compared with exact solutions to evaluate accuracy [6]. Numerical examples demonstrate that methods like Fractional Multistep Differential Transformation Method (FMDTM), Fractional Adams-Bashforth Method (FAB), and Fractional Adams-Bashforth-Moulton Method (FABM) agree with exact solutions, particularly for specific fractional orders and time intervals [8]. The exact integration and approximation curves for these methods reveal that while some may diverge over time, others maintain a consistent behavior, highlighting the importance of choosing the appropriate numerical method based on the specific FDDE under investigation [8]. Additionally, implementing equispaced grids and product-integration rules for the Caputo derivative further aids in obtaining accurate numerical approximations, which can be directly compared with exact solutions to assess their reliability [5].

The comparison of exact solutions with numerical methods reveals that methods like the L1 algorithm, used for evaluating fractional derivatives, and the CPM method, which has shown reasonable convergence rates, provide accurate numerical approximations that closely match the exact solutions [3], [7]. For example, the CPM method's solutions for specific FDDE problems have minimal errors compared to the exact solutions, as demonstrated in various numerical applications [7]. Overall, while exact solutions serve as a critical reference, the development and refinement of numerical methods continue to play a crucial role in solving FDDEs with high accuracy and efficiency, as evidenced by the diverse approaches and their comparative analyses in the literature [9], [5], [6], [3], [7].

The comparison of exact and numerical solutions validates the numerical methods and provides insights into their computational effectiveness and applicability to real-world problems modeled by FDDEs [7]. Therefore, the continuous development and comparison of numerical methods against exact solutions remain critical research area to ensure the accurate modeling and simulation of complex dynamical systems described by FDDEs.

The choice of method and its implementation can significantly affect the accuracy and computational efficiency of the solution, with some methods being more suitable for specific types of FDDEs or problem domains [9]. Overall, FDDEs provide a robust framework for modeling complex systems with memory and delay effects, and ongoing research continues to enhance the methods for solving these equations efficiently and accurately. This paper will compare the exact and approximate solutions of fractional differential equations with delays, which is crucial for validating the accuracy and efficiency of numerical methods. The numerical methods used in this paper are *Fractional Euler method* (Forward, Backward and Weighted Difference) and *Fractional Adams method*, processed in the *Mathematica* 14.0 software.

The stability analysis of FDDEs is crucial, as it helps in understanding the behavior of solutions over time. Researchers have developed various methods to study the stability of both linear and nonlinear FDDEs, including using characteristic equations, Laplace transforms, and numerical algorithms [10].

2. BASIC DEFINITION

The fractional derivatives in FDDEs can be defined in various ways, such as the Grunwald-Letnikov, Caputo, or Riemann-Liouville sense, each requiring specific initial conditions to initialize the problem [5]. The fractional derivative in FDDEs is often defined in the Caputo sense, which is more convenient for practical applications [9]. For $\tau > 0$ [3], the general form of an FDDE can expressed as:

$$D^{\alpha} y(t) = f(t, y(t), y(t-\tau)), \quad t \in [0, T], \quad 0 < \alpha < 1, \tag{2.1}$$

$$y(t) = \phi(t), \quad t \in [-\tau, 0]. \tag{2.2}$$

where D^{α} denotes the fractional derivative of order α, $y(t)$ is the unknown function, and τ is the delay term, and f is a given function [9]. The initial conditions for FDDEs are typically specified over an interval rather than at a single point, reflecting past states' influence on the system's current behavior [9].

Definition 1.1. [11] The Euler Gamma function $\Gamma(z)$ is defined by the so-called Euler integral of the second kind:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, dt \quad (\Re(z) > 0), \tag{2.3}$$

where $t^{z-1} = e^{(z-1)\log(t)}$. This integral is convergent for all complex $z \in C$ with $\Re(z) > 0$.

From [2], [11], [12] we recall some definitions and notations.

Definition 1.2. The (left) Grünwald-Letnikov definition is a generalization of the traditional derivative. It is defined as:

$$^{GL}D^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\left\lfloor \frac{t-a}{h} \right\rfloor} (-1)^k \binom{\alpha}{k} f(t - kh), \tag{2.4}$$

where, α is the order of the derivative, h is the step size, and $\binom{\alpha}{k}$ is the binomial coefficient.

Definition 1.3. The (left) Riemann-Liouville fractional derivative is defined using an integral operator. For a function $f(t)$, it is given by:

$$^{RL}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau \tag{2.5}$$

where, $n = \lceil \alpha \rceil$ (n is the smallest integer greater than α), $\Gamma(\cdot)$ is the gamma function, and a is the lower limit of integral.

Definition 1.4. The (left) Caputo fractional derivative, is defined as:

$$^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau \tag{2.6}$$

where, $n = \lceil \alpha \rceil$ (n is the smallest integer greater than α), $\Gamma(\cdot)$ is the gamma function, and a is the lower limit of integral.

3. NUMERICAL METHODS

We shall use numerical approaches to solve the initial value problem (2.2) for the fractional derivative (2.1).

3.1. **Fractional Euler Method.**

The Euler Method (in all versions) is commonly used to solve ordinary differential equations. Whereas, in the book [12], this method is used for fractional ordinary differential equations. This section will apply numerical solutions to the fractional delay differential equation.

3.1.1. *Fractional Forward Euler Method (FEM).* FEM uses a forward difference approximation for $D^{\alpha} y(t)$. The left fractional rectangular formula approximates the fractional derivative and can be expressed by:

$$y_{n+1} = \sum_{j=0}^{m-1} \frac{t_{n+1}^{j}}{j!} y_0^{(j)} + \Delta t^{\alpha} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_j, y(t_j - \tau)) \tag{3.1}$$

where, $b_{j,n+1} = \frac{1}{\Gamma(\alpha+1)} \left[(n-j+1)^{\alpha} - (n-j)^{\alpha} \right]$.

3.1.2. *Fractional Backward Euler Method (BEM).* BEM uses a backward difference approximation for $D^\alpha y(t)$. The right fractional rectangular formula approximates the fractional derivative and can be expressed by:

$$y_{n+1} = \sum_{j=0}^{m-1} \frac{t_{n+1}^{j}}{j!} y_0^{(j)} + \Delta t^\alpha \sum_{j=0}^{n} b_{j,n+1} f(t_{j+1}, y_{j+1}, y(t_{j+1} - \tau)) \tag{3.2}$$

where, $b_{j,n+1} = \frac{1}{\Gamma(\alpha+1)} \left[(n-j+1)^\alpha - (n-j)^\alpha\right]$.

3.1.3. *Fractional Weighted Difference Method (WDM).* WDM is a combination of forward and backward differences. The weighted fractional rectangular formula approximates the fractional derivative and can be expressed by:

$$y_{n+1} = \sum_{j=0}^{m-1} \frac{t_{n+1}^{j}}{j!} y_0^{(j)} +$$

$$+\Delta t^\alpha \sum_{j=0}^{n} b_{j,n+1} \left[\theta f(t_j, y_j, y(t_j - \tau)) + (1-\theta) f(t_{j+1}, y_{j+1}, y(t_{j+1} - \tau))\right] \tag{3.3}$$

where, $b_{j,n+1} = \frac{1}{\Gamma(\alpha+1)} \left[(n-j+1)^\alpha - (n-j)^\alpha\right]$.

3.2. **Fractional Adams Method (FAM).** FAM employs a predictor-corrector method with fractional derivatives. For solving FDDEs, FAM utilize the product trapezoidal quadrature formula [12], [13], [14]:

$$y_{n+1} = \sum_{k=0}^{\lceil\alpha\rceil-1} \varphi_k(t) \frac{t_{n+1}^{k}}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n+1} a_{j,n+1} f_1(t_j, y_j, y(t_j - \tau)) \tag{3.4}$$

$$\text{where, } a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & j = 0 \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & 1 \le j \le n \\ 1, & j = n+1 \end{cases}$$

The implicit nature of the scheme (3.4) necessitates extensive computation, prompting the employment of some approximation techniques, thereby yielding the fractional Adams method:

$$\begin{cases} y_{n+1} = \displaystyle\sum_{k=0}^{\lceil\alpha\rceil-1} \varphi_k(t) \frac{t_{n+1}^{k}}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n+1} b_{j,n+1} f_1(t_j, y_j, y(t_j - \tau)) \\ y_{n+1} = \displaystyle\sum_{k=0}^{\lceil\alpha\rceil-1} \varphi_k(0) \frac{t_{n+1}^{k}}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f_1(t_j, y_j, y(t_j - \tau)) \\ \qquad + \displaystyle\frac{h^\alpha}{\Gamma(\alpha+2)} f_1(t_{n+1}, y_{n+1}, y(t_{n+1} - \tau)) \end{cases}$$

where, y_{n+1} and $y(t_{n+1})$ represent the approximate and the exact solutions, respectively.

The deviation between the curves representing the approximation and the curve representing the exact solution is so minimal that it is difficult to discern any notable difference when observing the figure presented. So, from [15] we describe the relative difference parameter $R(t)$ by definition and the formula:

Definition 3.1 The relative difference parameter $R(t)$ is the ratio of the absolute difference between the exact (y_{exact}) and the approximate (y_{approx}) values of the $y(t)$ at the fixed time t to the maximum of their absolute values, multiplied by 100.

$$R_y(t) = \frac{|y_{\text{exact}}(t) - y_{\text{approx}}(t)|}{\max\left(|y_{\text{exact}}(t)|, |y_{\text{approx}}(t)|\right)} \times 100.$$

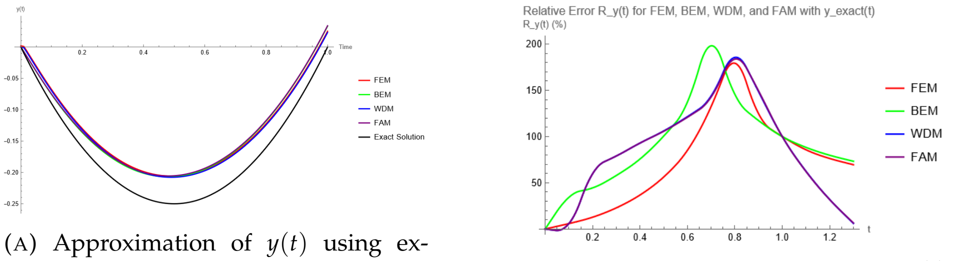
4. NUMERICAL EXAMPLES

In this section, we employ the above discretization methods to conduct four numerical analyses of fractional-order delay differential equations. All numerical results are derived using *Mathematica* version 14.0.

Example 4.1. [3], [16], [17], [18], [19], [20], [21] Consider the fractional order equation with delay in terms of the Caputo:

$$D^{\alpha}y(t) = \frac{2}{\Gamma(3-\alpha)}t^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)}t^{1-\alpha} + 2\tau t - \tau - \tau^2 + y(t-\tau) - y(t), \quad \alpha \in (0,1) \tag{4.1}$$

and $y(t) = 0, t \leq 0$. The exact solution is $y(t) = t^2 - t$ for $\alpha = 1$.



(A) Approximation of $y(t)$ using exact solution (black line) and numerical methods such as FEM (red line), BEM (green line), WDM (blue line), and FAM (purple line).

(B) Relative difference parameter $R(t)$ corresponding to FEM (red line), BEM (green line), WDM (blue line), and FAM (purple line).

FIGURE 1. The fractional-order $\alpha = 0.9$, the delay $\tau = 0.1$, integration step-size $h = 0.01$, initial condition $y(0) = 0$ and $t \in [0,1]$.

FDDEs manifest an elevated degree of complexity and pose multifaceted challenges for approximation. As a result, it becomes increasingly difficult for any numerical method to maintain accuracy over prolonged temporal intervals. For time $t \in [0,1]$, the errors that occur for every one of the numerical approaches used in this paper are minimal and comparable in magnitude, approximately ranging from 0.34 to 0.37. On the other hand, the errors dramatically rise during time $t \in [0, 10]$, reaching values in the $10^{17} - 10^{18}$ range, suggesting a notable increase in error over a more extended time. Since with increasing time t, the error intensifies and the divergence of the curves becomes evident, in this paper, we concentrated solely on the temporal interval t constrained within the limits of [0,1], fractional order $\alpha = 0.9$, the delay $\tau = 0.1$ and step-size $h = 0.01$. Figure 1 displays the performance of numerical approaches in approximating $y(t)$ and the related relative difference $R(t)$ dynamics as time increases. Figure 1A plots various numerical approximations of the function $y(t)$ and the exact solution throughout the interval $t \in [0, 1]$. The numerical methods FEM, BEM, WDM, and FAM used to approximate $y(t)$ are represented by the colored red, green, blue, and purple curves. In contrast, the black curve represents the exact solution. The curves diverge from the exact solution with increasing time, suggesting that the approaches' accuracy varies. The relative error $R_y(t)$ is a parameter used to determine the precision of numerical approximation methods by comparing them with the exact solution at the accurate time t. This calculation quantifies the departure of an approximation from the exact value, allowing us to compare the performance of methods. Figure 1B plots the relative error $R_y(t)$ as a function of time, demonstrating the performance of numerical approaches (FEM, BEM, WDM, and FAM) over time. The plot shows that the error evolves dynamically, with every method initially increasing, peaking at a specific point, and decreasing as time passes. Based on Figure 1 and Table 1, we conclude that FAM and FEM are generally more accurate, with fewer relative errors across more time points. FAM performs better where time increases ($t > 1$), whereas FEM is more accurate where the time t is close to 1. BEM is the least accurate, with significant initial errors, making it less dependable, whereas WDM begins with higher errors but improves over time.

For $\alpha = 0.9$ and $t = 0, 0.1, ..., 1.3$ we have:

t	y_{exact}	y_{FEM}	y_{BEM}	y_{WDM}	y_{FAM}	R_{FEM}	R_{BEM}	R_{WDM}	R_{FAM}
0	0	0	0	0	0	0	0	0	0
0.1	-0.09	-0.0847432	-0.05920263	-0.0847432	-0.0847432	5.84089	34.2193	5.84089	5.84089
0.2	-0.16	-0.139597	-0.0887741	-0.139597	-0.138446	12.7519	44.5156	19.7115	19.7115
0.3	-0.21	-0.162006	-0.08850803	-0.04441162	-0.04441162	22.8543	57.8533	78.9804	78.9804
0.4	-0.24	-0.151754	-0.05827871	-0.0169604	-0.0169604	36.7692	75.7172	92.9332	92.9332
0.5	-0.25	-0.109379	0.0013277	0.0151134	0.0143793	56.2484	100.531	106.045	105.752
0.6	-0.24	-0.0357096	0.0896959	0.0511642	0.0506062	85.121	137.373	151.825	120.805
0.7	-0.21	0.0683203	0.206158	0.0923061	0.0899351	132.533	198.17	143.955	142.82
0.8	-0.16	0.201757	0.350035	0.136908	0.133976	179.303	145.71	185.568	183.73
0.9	-0.09	0.363672	0.520664	0.1821789	0.1821789	124.748	117.286	148.566	149.402
1.0	0	0.553814	0.744178	0.234564	0.234564	100	100	100	100
1.1	0.11	0.76947	0.939665	0.291164	0.291164	85.7044	88.2937	62.4796	62.2206
1.2	0.24	1.01177	1.18687	0.352507	0.35202	76.2792	79.7787	31.9162	31.8221
1.3	0.39	1.27938	1.45849	0.415382	0.417178	69.5165	73.26	6.11052	6.51473

TABLE 1. Table of numerical values between the exact solution, numerical methods, and relative error at the accurate time t, for $\alpha = 0.9$ and $t = 0, 0.1, ..., 1.3$.

To support the results obtained from Table 1 and Figure 1, we present the time measurement for each method in Table 2 and Figure 2. As we can see, the execution times (s) are multiplied by 10^6 to convert them into microseconds (μs) for better readability. The execution time of each method was taken as the average of over 10 runs rather than just one. The results indicate that FEM, BEM, and FAM have the same computational cost, while WDM has a slightly higher computational cost. Since FAM executes for a brief time and has high accuracy, it remains an optimal choice for long-term simulations if a balance between accuracy and execution time is required. On the other hand, knowing that FEM is accurate for simulations with short time periods (close to 1) and that the execution time is low, this can make it a beneficial option for simulations, but it should be used with caution because FEM does not maintain accuracy for large values. Meanwhile, BEM, although it is among the methods with a quick execution time, has a high cost in accuracy. This makes it a less reliable method for cases where precision is required. And as a method with the highest execution time, WDM, since over time there is continuous improvement in accuracy, can be considered as a valid alternative for cases whose simulations have long time periods.

Method	**Execution Time (s)**
FEM	0.00000004 (0.04 μs)
BEM	0.00000004 (0.04 μs)
WDM	0.00000007 (0.07 μs)
FAM	0.00000004 (0.04 μs)

TABLE 2. Execution time to compute the solution for each numerical method.

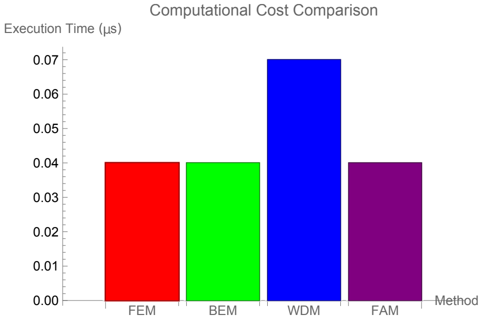


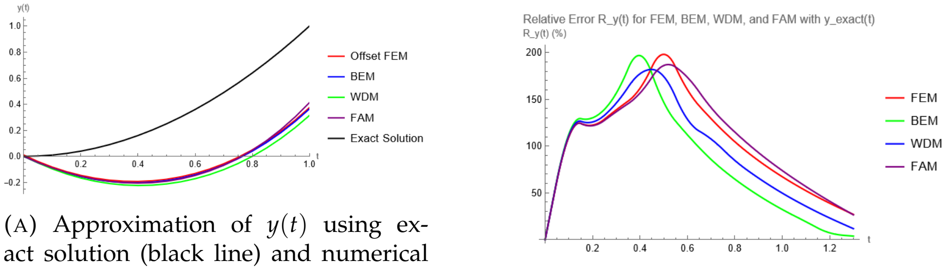
FIGURE 2. Comparison of computational cost (execution time) for the four numerical methods: FEM, BEM, WDM, and FAM.

Example 4.2. [22] Consider the fractional delay differential equation of the form:

$$D^{1/2}y(t) = y(t-1) - y(t) + 2t - 1 + \frac{8t^{3/2}}{3\sqrt{\pi}}, \quad t > 0 \tag{4.2}$$

and $y(t) = t^2, t \in [0,1]$. The exact solution is $y(t) = t^2$.

In this example, the fractional order is $\alpha = 0.5$, the step of size $h = 0.05$, the initial condition $y(0) = 0$, the delay $\tau = 1$, and time as in previous example $t \in [0,1]$ (except for the relative difference goes up to 1.3). Similar to the previous example, the correct solution curve is shown in black, FEM in red (Offset Fem-



(A) Approximation of $y(t)$ using exact solution (black line) and numerical methods such as Offset FEM- Slightly offset FEM $+0.01$ for better visibility (red line), BEM (green line), WDM (blue line) and FAM (purple line).

(B) Relative difference parameter $R(t)$ corresponding to FEM (red line), BEM (green line), WDM (blue line) and FAM (purple line).

FIGURE 3. The fractional-order $\alpha = 0.5$, the delay $\tau = 1$, integration step-size $h = 0.05$, initial condition $y(0) = 0$ and $t \in [0, 1]$.

Slightly offset FEM $+0.01$ for better visibility), BEM in green, WDM in blue, and FAM in purple. The most accurate methods for approximating $y(t)$ at subsequent periods are FAM and WDM, which exhibit minor relative errors, particularly after $t = 0.9$. FAM is more accurate than the other approaches, with a minimum error of 26.324% at $t = 1.3$ and an overall error of 2.0011, with the most significant reduction in error at later periods. FEM is dependable for intermediate and later periods since it exhibits a notable improvement after the error peaks at $t = 0.5$. On the other hand, the least accurate is BEM, which has more significant errors that increase early on and stay high up to 171.372% at $t = 0.5$. On the other hand, WDM starts with more considerable faults but gets much better with time, eventually attaining a total error of 2.1688. BEM continues to be the least efficient method for precisely estimating $y(t)$, but FAM is generally the most appropriate for precision over longer time intervals. For $\alpha = 0.5$ and $t = 0, 0.1, ..., 1.3$ we have:

Much like in the first example, to support our findings on the accuracy of numerical methods, we calculated the execution time, which is taken as the average of over 10 calculations and not just one. For better readability, the execution time results (s) have been multiplied by 10^6 to convert them into microseconds (μs). As can be seen in Table 4 and Figure 4, the method with the shortest execution time is the FEM method, while the method with slightly higher computational costs is the BEM method. Since the accuracy approaches over time, the WDM method certainly requires more execution time than the FEM and FAM. While the latter (FAM) remains the optimal method for long-term simulations, offering consistent accuracy and low execution time. Therefore, to select the right method, one must base both the accuracy and the execution time that these methods exhibit.

t	y_{exact}	y_{FEM}	y_{BEM}	y_{WDM}	y_{FAM}	R_{FEM}	R_{BEM}	R_{WDM}	R_{FAM}
0	0	0	0	0	0	0	0	0	0
0.1	0.01	-0.1	-0.078495	-0.089248	-0.1	110	112.74	111.205	110
0.2	0.04	-0.178495	-0.132477	-0.155486	-0.183495	122.41	130.194	125.726	121.799
0.3	0.09	-0.232477	-0.158937	-0.195707	-0.245402	138.714	156.626	145.987	136.675
0.4	0.16	-0.258937	-0.154865	-0.206901	-0.282463	161.791	196.791	177.332	156.645
0.5	0.25	-0.254865	-0.117252	-0.186059	-0.290203	198.091	146.901	174.424	186.147
0.6	0.36	-0.217252	-0.04309	-0.130171	-0.26466	160.348	111.969	128.659	173.517
0.7	0.49	-0.14309	0.070631	-0.03623	-0.201207	129.202	85.5855	39.394	141.063
0.8	0.64	-0.029369	0.226919	0.098775	-0.0954043	104.589	64.5439	84.5664	114.907
0.9	0.81	0.126919	0.428784	0.277852	0.0576746	84.331	47.0637	65.6973	92.8797
1.0	1.00	0.328784	0.679235	0.504009	0.263946	67.1216	32.0765	49.5991	73.7086
1.1	1.21	0.579235	0.98128	0.780257	0.525609	52.1293	18.9025	35.516	56.5612
1.2	1.44	0.88128	1.337929	1.109604	0.851098	38.8	7.0883	22.9442	40.8964
1.3	1.69	1.237929	1.75219	1.49506	1.2451	26.7498	3.5493	11.535	26.3254

TABLE 3. Table of numerical values between the exact solution, numerical methods, and relative error at the accurate time t, for $\alpha = 0.5$ and $t = 0, 0.1, ..., 1.3$.

Method	**Execution Time (s)**
FEM	0.00000004 (0.04 μs)
BEM	0.00000009 (0.09 μs)
WDM	0.00000007 (0.07 μs)
FAM	0.00000005 (0.05 μs)

TABLE 4. Execution time to compute the solution for each numerical method.

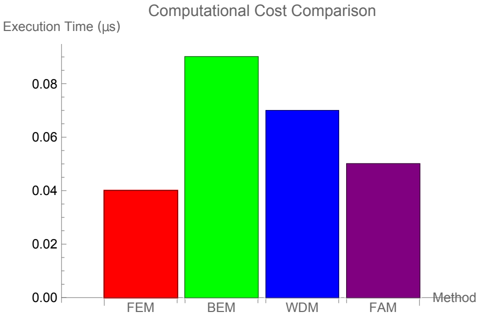


FIGURE 4. Comparison of computational cost (execution time) for the four numerical methods: FEM, BEM, WDM, and FAM.

5. CONCLUSION

In this paper, we compared four numerical methods Forward Euler Method (FEM), Backward Euler Method (BEM), Weighted Difference Method (WDM), and Fractional Adams Method (FAM), to the exact solution of FDDEs. The effectiveness of each numerical approach is evaluated based on the relative difference between the exact solution and the approximate solution at designated temporal points. Among these approaches, FAM emerges as the most precise, particularly over extended temporal intervals, yielding the lowest relative errors and concluding with an error magnitude of 2.0011. FEM exhibits satisfactory precision within intermediate periods, demonstrating significant enhancement after initial peaks in error. Despite commencing with elevated errors, WDM shows considerable progress over time, establishing itself as a feasible alternative for later phases. Conversely, BEM consistently displays elevated error rates throughout the entire time spectrum, culminating in a peak error of 171.372%, categorizing it as the least efficacious approach for approximating $y(t)$. In general, FAM is the premier option for attaining precise solutions in prolonged time intervals, whereas BEM is unreliable.

These accuracy findings are supported by the analysis of the execution time for each method and confirm that the most optimal method is FAM, balancing high accuracy and low execution time for long-term simulations. Meanwhile, for short-term simulations, the FEM method is efficient and accurate. With a longer cost and improvement in accuracy over time for long-term simulations, the WDM method can be considered. Due to the high error rate and the variability of the execution time in the two examples, the BEM remains the least reliable method. Thus, to choice of method should take into account accuracy and execution time for optimal performance.

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