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DESINGULARIZATION FORMULAS FOR THE STOCKWELL AND DIRECTIONAL STOCKWELL TRANSFORMS

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Abstract. We establish a connection between the Stockwell transform, the Radon transform, and the directional Stockwell transform. We also provide desingularization formulas for the Stockwell transform and the directional Stockwell transform.

1. Introduction

The Stockwell transform (ST), introduced by Stockwell in [17], is a combination of the best features of the short-time Fourier transform and the wavelet transform. It can also be considered as a phase correction of the wavelet transform [16]. The authors of [3] generalized the ST and established continuous and discrete reconsutruction formulas for a signal from its ST. The multi-dimensional ST was introduced by Riba et al. in [13], which also includes the multi-dimensional Gabor transforms as a special case, while the authors of [14] developed the distributional framework for the ST.

The ridgelet transform, introduced by Candès, is a powerful tool for analyzing higher-dimensional phenomena, particularly those involving singularities along curves and hyperplanes. It is a composition of the Radon transform with the one-dimensional wavelet transform [1, 2]. The idea of this transform is to use the Radon transform to project the hyperplane singularity to a point singularity and then apply the one-dimensional wavelet transform. The authors of [11] extended ridgelet theory to the space of Lizorkin distributions. Following Candes's idea, the directional sensitive variants of the short-time Fourier transform were defined by Grafakos et al. in [6], and later by Giv in [5].

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The directional sensitive variant of the ST, called the directional Stockwell transform (DST), and its synthesis operator have been introduced and analysed in [4]. The authors proved the Parseval's identity and the reconstruction formula, and then extended the DST theory to the space of Lizorkin distributions. Since the kernel of this transform may not be in the Lizorkin space of test function $\mathcal{S}_0(\mathbb{R}^n)$, the direct approach does not work for the Lizorkin distributions. The largest distribution space where the direct approach works is $D'_{L^1}(\mathbb{R}^n)$ (see (2.15) below). The DST may also be considered as a composition of the Radon transform with the one-dimensional ST [4]. It should be noted that Lone et al. in [12] provide a slightly different definition of the directional sensitive variant of the multidimensional Stockwell transform in $L^2(\mathbb{R}^n)$.

In the present paper, we aim to provide an intrinsic connection between the Radon transform, the ST, and the DST, as well as desingularization formulas for the ST and the DST.

The paper is organized as follows. In Section 2 we introduce important notations, describe the spaces of test functions and distributions in which we work, and provide definitions and known results for the Radon transform, the ST, and the DST of Lizorkin distributions. In Section 3, in a very natural way, we introduce the ST on $\mathcal{S}(\mathbb{S}^{n-1}\times\mathbb{R})$, and following the duality approach, on $\mathcal{S}'(\mathbb{S}^{n-1}\times\mathbb{R})$ (Here $\mathbb{S}^{n-1}$ stands for the unit sphere in $\mathbb{R}^n$). On the other hand, using the theory of tensor products of topological vector spaces, we alternatively define the ST of a distribution $G\in\mathcal{S}_0'(\mathbb{S}^{n-1}\times\mathbb{R})$ with respect to a window ψ as a smooth function from $\mathbb{R}\times\mathbb{R}^\times$ to $\mathcal{D}'(\mathbb{S}^{n-1})$ of slow growth in the variables $(b,a)\in\mathbb{R}\times\mathbb{R}^\times$, given by

$$S_\psi G(\mathbf{u},b,a):=\langle G(\mathbf{u},p),\overline{\psi_{b,a}(p)}\,\rangle_p,\quad \mathbf{u}\in\mathbb{S}^{n-1}.$$

Additionally, a desingularization formula for the ST and a characterization of the bounded subsets in $\mathcal{S}_0'(\mathbb{S}^{n-1}\times\mathbb{R})$ by the ST are provided. In Section 4, a relation between the Radon transform, the ST and the DST is presented and a desingularization formula for the DST is obtained.

2. Preliminaries

2.1. **Notations and spaces.** We use the standard notations of n -dimensional calculus. The Fourier transform $\mathcal{F}f$ of a function $f\in L^1(\mathbb{R}^n)$ is defined as $\mathcal{F}f(\boldsymbol{\xi})=\widehat{f}(\boldsymbol{\xi})=\int_{\mathbb{R}^n}f(\mathbf{x})e^{-i\mathbf{x}\cdot\boldsymbol{\xi}}d\mathbf{x},\ \ \boldsymbol{\xi}\in\mathbb{R}^n$ and it is extended to $L^2(\mathbb{R}^n)$ as usual [10]. The notation $\langle f,\varphi\rangle$ stands for the dual paring between the distribution f and the test function φ, whereas $(f,\varphi)_{L^2}$ for the L^2 inner product of f and φ. All dual spaces in this article are equipped with the strong dual topology [18].

The space $\mathcal{S}(\mathbb{R}^n)$, known as the Schwartz space of rapidly decreasing smooth functions, consists of all functions $\varphi\in C^\infty(\mathbb{R}^n)$ such that

$$\rho_\nu(\varphi)=\sup_{\mathbf{x}\in\mathbb{R}^n,\,|\boldsymbol{\alpha}|\le\nu}(1+|\mathbf{x}|)^\nu|\partial^{\boldsymbol{\alpha}}_{\mathbf{x}}\varphi(\mathbf{x})|<\infty, \tag{2.1}$$

for all $\nu\in\mathbb{N}_0$ [10, 15]. It is topologized by means of seminorms (2.1). Its dual $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions. The Lizorkin space of test functions $\mathcal{S}_0(\mathbb{R}^n)$ consists of all Schwartz test functions $\varphi\in\mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n}\mathbf{x}^{\mathbf{m}}\varphi(\mathbf{x})d\mathbf{x}=$

0, for all $\mathbf{m} \in \mathbb{N}_0^n$. It is provided with the topology inherited from $\mathcal{S}(\mathbb{R}^n)$. Its dual space $\mathcal{S}_0'(\mathbb{R}^n)$, known as the space of Lizorkin distributions, is canonically isomorphic to the quotient of $\mathcal{S}'(\mathbb{R}^n)$ by the space of polynomials [9]. In our work, we also use the space $\mathcal{S}_1(\mathbb{R}) = \{\varphi \in \mathcal{S}(\mathbb{R}) : e^{ix}\varphi(x) \in \mathcal{S}_0(\mathbb{R})\}$. The Fourier transform in an isomorphism of $\mathcal{S}(\mathbb{R}^n)$ onto itself [10]. The Fourier transform of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$, denoted as $\mathcal{F}f$, is defined by duality $\langle \mathcal{F}f, \varphi\rangle = \langle f, \mathcal{F}\varphi\rangle$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

The space $D_{L^p}(\mathbb{R}^n)$, $1 \leq p \leq \infty$, consists of all smooth functions φ such that all derivatives belong to $L^p(\mathbb{R}^n)$. It is equipped with the following topology: the sequence $(\varphi_j)_j$ converges to 0 in $D_{L^p}(\mathbb{R}^n)$, if $\partial^{\mathbf{m}}\varphi_j \to 0$ as $j \to \infty$ in $L^p(\mathbb{R}^n)$, for all $\mathbf{m} \in \mathbb{N}_0^n$ (see [15], page 199). The space $D'_{L^p}(\mathbb{R}^n)$ consists of all distributions f which can be represented as $f = \sum_{j=1}^N \partial^{\boldsymbol{\alpha}_j} f_j$, $f_j \in L^p(\mathbb{R}^n)$, for some $N \in \mathbb{N}$ and $\boldsymbol{\alpha}_j \in \mathbb{N}_0^n$ ([15], Thm. XXV, page 201).

The space $\mathcal{D}(\mathbb{S}^{n-1})$ consists of all smooth function on the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$. If $\mathcal{A}$ is a locally convex space of smooth functions on $\mathbb{R}$, then we write $\mathcal{A}(\mathbb{S}^{n-1} \times \mathbb{R})$ for the space of functions $\varrho(\mathbf{u}, p)$ being smooth in $\mathbf{u} \in \mathbb{S}^{n-1}$ and having the properties of $\mathcal{A}$ in the variable $p \in \mathbb{R}$.

Let $\mathbb{Y}^{n+1} = \mathbb{S}^{n-1} \times \mathbb{R} \times \mathbb{R}^\times$, where $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ and $n \geq 2$. The space $\mathcal{S}(\mathbb{Y}^{n+1})$ consists of all smooth functions $\Phi \in C^\infty(\mathbb{Y}^{n+1})$ such that

$$\rho_{s,r}^{l,m,k}(\Phi) = \sup_{(\mathbf{u},b,a)\in\mathbb{Y}^{n+1}} (1+|b|^2)^{r/2}(|a|^s+|a|^{-s})|\partial_a^l \partial_b^m \Delta_{\mathbf{u}}^k \Phi(\mathbf{u}, b, a)| < \infty, \quad (2.2)$$

for all $s, r, l, m, k \in \mathbb{N}_0$, where $\Delta_{\mathbf{u}}$ stands for the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^{n-1}$. It is topologized by family of seminorms (2.2). Its dual space is denoted by $\mathcal{S}'(\mathbb{Y}^{n+1})$. We take $|a|^{n-2} dbdad\mathbf{u}$ as a standard measure on $\mathbb{Y}^{n+1}$, where $d\mathbf{u}$ stands for the surface measure on the sphere $\mathbb{S}^{n-1}$. If F is a locally integrable function of slow growth on $\mathbb{Y}^{n+1}$, i.e. if there exist $C > 0$ and $s \in \mathbb{N}_0$ such that

$$|F(\mathbf{u}, b, a)| \leq C\left(|a|^s+|a|^{-s}\right)(1+|b|)^s, \quad (\mathbf{u}, b, a) \in \mathbb{Y}^{n+1},$$

then F will be identified with an element of $\mathcal{S}'(\mathbb{Y}^{n+1})$ via the action

$$\langle F, \Phi\rangle := \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}^\times} \int_{\mathbb{R}} F(\mathbf{u}, b, a)\Phi(\mathbf{u}, b, a)|a|^{n-2} dbdad\mathbf{u}, \quad \Phi \in \mathcal{S}(\mathbb{Y}^{n+1}).$$

We recall the space $\mathcal{S}(\mathbb{R} \times \mathbb{R}^\times)$ of highly localized test function over $\mathbb{R} \times \mathbb{R}^\times$ consists of all smooth function Ψ on $\mathbb{R} \times \mathbb{R}^\times$ for which

$$\sup_{(b,a)\in\mathbb{R}\times\mathbb{R}^\times} (1+|b|^2)^{r/2}(|a|^s+|a|^{-s})|\partial_a^l \partial_b^m \Psi(b, a)| < \infty,$$

for all $s, r, l, m \in \mathbb{N}_0$ [14]. It is topologized in the usual way. We take $|a|^{-1} dbda$ as a standard measure on $\mathbb{R} \times \mathbb{R}^\times$. Furthermore, any locally integrable function F on $\mathbb{R} \times \mathbb{R}^\times$ that satisfies

$$|F(b, a)| \leq C\left(|a|^s+|a|^{-s}\right)(1+|b|)^s, \quad (b, a) \in \mathbb{R} \times \mathbb{R}^\times$$

for some $C > 0$ and $s \in \mathbb{N}_0$, is identified with an element of $\mathcal{S}'(\mathbb{R} \times \mathbb{R}^\times)$ via the action

$$\langle F, \Psi \rangle := \int_{\mathbb{R}^\times} \int_{\mathbb{R}} F(b,a)\Psi(b,a)|a|^{-1} db da, \quad \Psi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^\times). \tag{2.3}$$

The nuclearity of the Schwartz spaces (the Schwartz kernel theorem) yields the following equalities, $\mathcal{S}(\mathbb{Y}^{n+1}) = \mathcal{S}(\mathbb{R} \times \mathbb{R}^\times)\hat{\otimes}\mathcal{D}(\mathbb{S}^{n-1})$ and $\mathcal{S}_0(\mathbb{S}^{n-1} \times \mathbb{R}) = \mathcal{S}_0(\mathbb{R})\hat{\otimes}\mathcal{D}(\mathbb{S}^{n-1})$, where $X\hat{\otimes}Y$ is the topological tensor product space obtained as the completion of $X \otimes Y$ in the π-topology or, equivalently in these cases, the ϵ-topology [8, 18]. It also leads to the following isomorphisms

$$\mathcal{S}'(\mathbb{Y}^{n+1}) \cong \mathcal{S}'(\mathbb{R} \times \mathbb{R}^\times, \mathcal{D}'(\mathbb{S}^{n-1})) \cong \mathcal{D}'(\mathbb{S}^{n-1}, \mathcal{S}'(\mathbb{R} \times \mathbb{R}^\times)),$$

the very last two spaces are spaces of vector-valued distributions, see, for instance [18]. These three spaces can be identified via the standard identification

$$\langle F, \varphi \otimes \Psi \rangle = \langle \langle F, \Psi \rangle, \varphi \rangle = \langle \langle F, \varphi \rangle, \Psi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{S}^{n-1}),\ \Psi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^\times). \tag{2.4}$$

In a similar manner, we may obtain

$$\mathcal{S}_0'(\mathbb{S}^{n-1} \times \mathbb{R}) \cong \mathcal{S}_0'(\mathbb{R}, \mathcal{D}'(\mathbb{S}^{n-1})) \cong \mathcal{D}'(\mathbb{S}^{n-1}, \mathcal{S}_0'(\mathbb{R})),$$

which is also being realized via the standard identification

$$\langle F, \varphi \otimes \Psi \rangle = \langle \langle F, \Psi \rangle, \varphi \rangle = \langle \langle F, \varphi \rangle, \Psi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{S}^{n-1}),\ \Psi \in \mathcal{S}_0(\mathbb{R}). \tag{2.5}$$

2.2. **The Radon transform.** The Radon transform plays a crucial role in our analysis. Therefore, in this subsection, we mention some of the basic results of it [7, 8, 11].

For an integrable function $f \in L^1(\mathbb{R}^n)$, the Radon transform is defined as

$$Rf_{\mathbf{u}}(p) = Rf(\mathbf{u}, p) = \int_{\mathbf{x}\cdot\mathbf{u}=p} f(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^n} f(\mathbf{x})\delta(p - \mathbf{x}\cdot\mathbf{u}) d\mathbf{x},$$

where $\mathbf{u} \in \mathbb{S}^{n-1}$, $p \in \mathbb{R}$ and δ is the Dirac-delta function [8]. Using the Fubini's theorem, one can show that $Rf \in L^1(\mathbb{S}^{n-1} \times \mathbb{R})$ for $f \in L^1(\mathbb{R}^n)$. The dual Radon transform R^* of a function $\varrho \in L^\infty(\mathbb{S}^{n-1} \times \mathbb{R})$ is given by

$$R^*\varrho(\mathbf{x}) = \int_{\mathbb{S}^{n-1}} \varrho(\mathbf{u}, \mathbf{x}\cdot\mathbf{u}) d\mathbf{u}, \tag{2.6}$$

where $\mathbf{x} \in \mathbb{R}^n$, [8].

The Radon transform $R : \mathcal{S}_0(\mathbb{R}^n) \to \mathcal{S}_0(\mathbb{S}^{n-1} \times \mathbb{R})$ and its dual $R^* : \mathcal{S}_0(\mathbb{S}^{n-1} \times \mathbb{R}) \to \mathcal{S}_0(\mathbb{R}^n)$ are continuous linear maps ([11], Cor. 6.1). The Radon transform is extended and studied in the space of Lizorkin distributions [11]. It is proven that the Radon transform $R : \mathcal{S}_0'(\mathbb{R}^n) \to \mathcal{S}_0'(\mathbb{S}^{n-1} \times \mathbb{R})$, defined as

$$\langle Rf, \varrho \rangle = \langle f, R^*\varrho \rangle, \quad f \in \mathcal{S}_0'(\mathbb{R}^n),\ \varrho \in \mathcal{S}_0(\mathbb{S}^{n-1} \times \mathbb{R}), \tag{2.7}$$

is continuous on $\mathcal{S}_0'(\mathbb{R}^n)$ ([11], Cor. 6.3).

The Fourier slice theorem provides a relation between the Fourier transform and the Radon transform, which states that for $f \in L^1(\mathbb{R}^n)$, the following holds

$$\widehat{Rf_{\mathbf{u}}}(p) = \widehat{f}(p\mathbf{u}),$$

where $\mathbf{u} \in \mathbb{S}^{n-1}$ and $p \in \mathbb{R}$, [8].

2.3. **One-dimensional Stockwell transform (ST).** The ST of $f \in L^2(\mathbb{R})$ with respect to a window $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is defined as

$$S_\psi f(b,a) = \frac{|a|}{\sqrt{2\pi}} \int_{\mathbb{R}} f(p)\overline{\psi}(a(p-b))e^{-ipa}dp, \tag{2.8}$$

for $b \in \mathbb{R}$ and $a \in \mathbb{R}^\times$ [3, 16, 17]. Since we work with the ST of distributions, for $f \in \mathcal{S}'(\mathbb{R})$ and $\psi \in \mathcal{S}(\mathbb{R})$, one can replace (2.8) by

$$S_\psi f(b,a) = \frac{1}{\sqrt{2\pi}} \langle f(p), |a|\,\overline{\psi}(a(p-b))e^{-ipa}\rangle, \;\; b \in \mathbb{R}, a \in \mathbb{R}^\times.$$

For more details about the distributional ST we refer to [14].

2.4. **The directional Stockwell transform (DST).** In this subsection, we give the definition and several properties of the DST from [4].

Let $\psi \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$. A function $\eta \in \mathcal{S}(\mathbb{R})$ is called a reconstruction window for ψ if

$$C_{\psi,\eta} = \frac{1}{\pi} \int_{\mathbb{R}} \overline{\widehat{\psi}(\xi-1)}\widehat{\eta}(\xi-1)|\xi|^{-n}d\xi \tag{2.9}$$

is nonzero and finite. One can show that every non-trivial window $\psi \in \mathcal{S}(\mathbb{R})$ has a reconstruction window η which may be chosen from the space $\mathcal{S}_1(\mathbb{R})$.

Let $\psi \in \mathcal{S}(\mathbb{R})$. The DST of $f \in L^1(\mathbb{R}^n)$ is defined as

$$DS_\psi f(\mathbf{u},b,a) = (f(\mathbf{x}), \psi_{\mathbf{u},b,a}(\mathbf{x}))_{L^2} = \frac{|a|}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\mathbf{x})\overline{\psi}(a(\mathbf{x}\cdot\mathbf{u}-b))e^{-ia(\mathbf{x}\cdot\mathbf{u})}d\mathbf{x}, \tag{2.10}$$

and the directional Stockwell synthesis transform of $\Phi \in \mathcal{S}(\mathbb{Y}^{n+1})$ with respect to ψ is defined as

$$DS_\psi^*\Phi(\mathbf{x}) := \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}^\times} \int_{\mathbb{R}} \Phi(\mathbf{u},b,a)\psi_{\mathbf{u},b,a}(\mathbf{x})|a|^{n-2}dbdad\mathbf{u}, \;\; \mathbf{x} \in \mathbb{R}^n,$$

where

$$\psi_{\mathbf{u},b,a}(\mathbf{x}) = \frac{|a|}{(2\pi)^{n/2}}\psi(a(\mathbf{x}\cdot\mathbf{u}-b))e^{ia(\mathbf{x}\cdot\mathbf{u})}, \; (\mathbf{u},b,a) \in \mathbb{Y}^{n+1}, \; \mathbf{x} \in \mathbb{R}^n.$$

An immediate result is the following relation between the Radon, directional Stockwell and one-dimensional Stockwell transforms,

$$DS_\psi f(\mathbf{u},b,a) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} S_\psi(Rf_{\mathbf{u}})(b,a), \;\; (\mathbf{u},b,a) \in \mathbb{Y}^{n+1}, \tag{2.11}$$

(see relation (3.2) in [4]). It is proven in ([4], Thm 4.1 and Thm. 4.2) that the DST $DS_\psi : \mathcal{S}_0(\mathbb{R}^n) \to \mathcal{S}(\mathbb{Y}^{n+1})$ and the directional Stockwell synthesis operator $DS_\psi^* : \mathcal{S}(\mathbb{Y}^{n+1}) \to \mathcal{S}_0(\mathbb{R}^n)$ are continuous linear maps, provided $\psi \in \mathcal{S}_1(\mathbb{R})$. These results allow defining the DST of a distribution $f \in \mathcal{S}_0'(\mathbb{R}^n)$ with respect to $\psi \in \mathcal{S}_1(\mathbb{R})$ as the element $DS_\psi f \in \mathcal{S}'(\mathbb{Y}^{n+1})$ whose action on test functions is given by duality,

$$\langle DS_\psi f, \Phi\rangle := \langle f, \overline{DS_\psi^*(\overline{\Phi})}\,\rangle, \;\; \Phi \in \mathcal{S}(\mathbb{Y}^{n+1}), \tag{2.12}$$

([4], Def. 5.1). Moreover, the directional Stockwell synthesis transform of $F \in \mathcal{S}'(\mathbb{Y}^{n+1})$ with respect to $\psi \in \mathcal{S}_1(\mathbb{R})$ is defined by

$$\langle DS^*_\psi F, \varphi\rangle := \langle F, \overline{DS_\psi(\overline{\varphi})}\rangle, \quad \varphi \in \mathcal{S}_0(\mathbb{R}^n), \tag{2.13}$$

([4], Def. 5.2). The mappings $DS_\psi : \mathcal{S}'_0(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{Y}^{n+1})$ and $DS^*_\psi : \mathcal{S}'(\mathbb{Y}^{n+1}) \to \mathcal{S}'_0(\mathbb{R}^n)$ are continuous linear operators for every $\psi \in \mathcal{S}_1(\mathbb{R})$ ([4], Prop. 5.3).

One can show that if $\eta \in \mathcal{S}_1(\mathbb{R})$ is a reconstruction window for a non-trivial window $\psi \in \mathcal{S}_1(\mathbb{R})$, then

$$\frac{1}{C_{\psi,\eta}} DS^*_\eta \circ DS_\psi = Id_{\mathcal{S}'_0(\mathbb{R}^n)}, \tag{2.14}$$

([4], Prop. 5.4). It is also proven ([4], Prop. 6.1) that the DST of $f \in D'_{L^1}(\mathbb{R}^n)$ with respect to $\psi \in \mathcal{S}_1(\mathbb{R})$ is given by the function (2.10), i.e.

$$\langle DS_\psi f, \Phi\rangle = \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}^\times} \int_{\mathbb{R}} DS_\psi f(\mathbf{u}, b, a)\Phi(\mathbf{u}, b, a)|a|^{n-2} db da d\mathbf{u}, \quad \Phi \in \mathcal{S}(\mathbb{Y}^{n+1}). \tag{2.15}$$

3. The Stockwell transform on $\mathcal{S}(\mathbb{S}^{n-1} \times \mathbb{R})$ and $\mathcal{S}'_0(\mathbb{S}^{n-1} \times \mathbb{R})$

To achieve the main research objectives of the paper, we first introduce and analyze the ST on $\mathcal{S}(\mathbb{S}^{n-1} \times \mathbb{R})$ and $\mathcal{S}'_0(\mathbb{S}^{n-1} \times \mathbb{R})$.

3.1. The Stockwell transform on $\mathcal{S}(\mathbb{S}^{n-1} \times \mathbb{R})$.

In this section, we use the following definition of reconstruction window.

Definition 3.1. *Let $\psi \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$. A function $\eta \in \mathcal{S}(\mathbb{R})$ is called a reconstruction window for ψ if*

$$D_{\psi,\eta} = \frac{1}{2\pi} \int_{\mathbb{R}} \overline{\widehat{\psi}(\xi - 1)}\widehat{\eta}(\xi - 1)|\xi|^{-1} d\xi$$

is nonzero and finite.

One can show that every non-trivial window $\psi \in \mathcal{S}(\mathbb{R})$ has a reconstruction window η which may be chosen from the space $\mathcal{S}_1(\mathbb{R})$.

Let $\psi \in \mathcal{S}(\mathbb{R})$. The ST of a function $\varrho \in \mathcal{S}(\mathbb{S}^{n-1} \times \mathbb{R})$ with respect to ψ is defined as

$$S_\psi \varrho(\mathbf{u}, b, a) := \int_{\mathbb{R}} \varrho(\mathbf{u}, p)\overline{\psi_{b,a}(p)} dp = \langle \varrho(\mathbf{u}, p), \overline{\psi_{b,a}(p)}\rangle_p, \tag{3.1}$$

where

$$\psi_{b,a}(p) = \frac{|a|}{\sqrt{2\pi}} \psi(a(p - b))e^{iap}, \ (\mathbf{u}, b, a) \in \mathbb{Y}^{n+1}, \ p \in \mathbb{R}.$$

One can show, similarly as for the ST on $\mathcal{S}_0(\mathbb{R})$ ([14], Prop. 3.2), that for a non-trivial window $\psi \in \mathcal{S}(\mathbb{R})$ with reconstruction window $\eta \in \mathcal{S}(\mathbb{R})$, the following Parseval's formula holds,

$$\int_{\mathbb{R}} \varrho(\mathbf{u}, p)\overline{\vartheta(\mathbf{u}, p)} dp = \frac{1}{D_{\psi,\eta}} \int_{\mathbb{R}^\times} \int_{\mathbb{R}} S_\psi \varrho(\{\mathbf{u}, b, a)\overline{S_\eta \vartheta(\mathbf{u}, b, a)}|a|^{-1} db da,$$

for $\varrho, \vartheta \in \mathcal{S}(\mathbb{S}^{n-1} \times \mathbb{R})$. Furthermore, if $\varrho \in \mathcal{S}(\mathbb{S}^{n-1} \times \mathbb{R})$, then the following reconstruction formula holds pointwisely,

$$\varrho(\mathbf{u}, p) = \frac{1}{D_{\psi,\eta}} \int_{\mathbb{R}^\times} \int_{\mathbb{R}} S_\psi \varrho(\mathbf{u}, b, a) \eta_{b,a}(p) |a|^{-1} db da, \quad (\mathbf{u}, p) \in \mathbb{S}^{n-1} \times \mathbb{R}, \qquad (3.2)$$

where $\eta \in \mathcal{S}(\mathbb{R})$ is a reconstruction window for $\psi \in \mathcal{S}(\mathbb{R}) \setminus \{0\}$.

The reconstruction formula (3.2) suggests us to define the Stockwell synthesis operator that maps the function on $\mathbb{Y}^{n+1}$ to a function on $\mathbb{S}^{n-1} \times \mathbb{R}$. Given $\psi \in \mathcal{S}(\mathbb{R})$, we define the Stockwell synthesis operator as

$$S^*_\psi \Phi(\mathbf{u}, p) := \int_{\mathbb{R}^\times} \int_{\mathbb{R}} \Phi(\mathbf{u}, b, a) \psi_{b,a}(p) |a|^{-1} db da, \quad (\mathbf{u}, p) \in \mathbb{S}^{n-1} \times \mathbb{R}, \qquad (3.3)$$

where $\Phi \in \mathcal{S}(\mathbb{Y}^{n+1})$.

It can be proved, similar to ([4], Prop. 3.6), that the Stockwell synthesis operator (3.3) is in fact the transpose of the ST (3.1) in the following sense: Let $\psi \in \mathcal{S}(\mathbb{R})$. If $\varrho \in \mathcal{S}(\mathbb{S}^{n-1} \times \mathbb{R})$ and $\Phi \in \mathcal{S}(\mathbb{Y}^{n+1})$, then

$$\int_{\mathbb{R}} \varrho(\mathbf{u}, p) \overline{S^*_\psi(\overline{\Phi})}(\mathbf{u}, p) dp = \int_{\mathbb{R}^\times} \int_{\mathbb{R}} S_\psi \varrho(\mathbf{u}, b, a) \Phi(\mathbf{u}, b, a) |a|^{-1} db da.$$

Under the standard identification (2.3), we can write the last relation as

$$\langle \varrho(\mathbf{u}, p), \overline{S^*_\psi(\overline{\Phi})}(\mathbf{u}, p) \rangle_p = \langle S_\psi \varrho(\mathbf{u}, b, a), \Phi(\mathbf{u}, b, a) \rangle_{b,a},$$

which will serve as our model for defining the distributional ST on $\mathcal{S}'_0(\mathbb{S}^{n-1} \times \mathbb{R})$ as in [4] .

One can readily note, similar as in ([4], Thm. 4.1 and Thm. 4.2), that the bilinear maps $S : \mathcal{S}_0(\mathbb{S}^{n-1} \times \mathbb{R}) \times \mathcal{S}_1(\mathbb{R}) \to \mathcal{S}(\mathbb{Y}^{n+1})$ defined as $(\varrho, \psi) \to S_\psi \varrho$ and $S^* : \mathcal{S}(\mathbb{Y}^{n+1}) \times \mathcal{S}_1(\mathbb{R}) \to \mathcal{S}_0(\mathbb{S}^{n-1} \times \mathbb{R})$ defined as $(\Phi, \psi) \to S^*_\psi \Phi$ are continuous.

Similarly to the case of the DST ([4], Prop. 4.3), if $\eta \in \mathcal{S}_1(\mathbb{R})$ is a reconstruction window for $\psi \in \mathcal{S}_1(\mathbb{R}) \setminus \{0\}$, then it holds

$$\frac{1}{D_{\psi,\eta}} S^*_\eta \circ S_\psi = Id_{\mathcal{S}_0(\mathbb{S}^{n-1} \times \mathbb{R})}. \qquad (3.4)$$

3.2. **The Stockwell transform on $\mathcal{S}'_0(\mathbb{S}^{n-1} \times \mathbb{R})$.** The above-mentioned continuity results for the ST and the Stockwell synthesis operator allow us to define the ST of $G \in \mathcal{S}'_0(\mathbb{S}^{n-1} \times \mathbb{R})$ with respect to $\psi \in \mathcal{S}_1(\mathbb{R})$ as the element $S_\psi G \in \mathcal{S}'(\mathbb{Y}^{n+1})$ whose action on test functions is given by

$$\langle S_\psi G, \Phi \rangle := \langle G, \overline{S^*_\psi(\overline{\Phi})} \rangle, \quad \Phi \in \mathcal{S}(\mathbb{Y}^{n+1}), \qquad (3.5)$$

as well as the appropriate Stockwell synthesis operator $S^*_\psi : \mathcal{S}'(\mathbb{Y}^{n+1}) \to \mathcal{S}'_0(\mathbb{S}^{n-1} \times \mathbb{R})$ by

$$\langle S^*_\psi F, \varrho \rangle := \langle F, \overline{S_\psi(\overline{\varrho})} \rangle, \quad F \in \mathcal{S}'(\mathbb{Y}^{n+1}), \quad \varrho \in \mathcal{S}_0(\mathbb{S}^{n-1} \times \mathbb{R}). \qquad (3.6)$$

By taking the transposes in $S_\psi : \mathcal{S}_0(\mathbb{S}^{n-1} \times \mathbb{R}) \to \mathcal{S}(\mathbb{Y}^{n+1})$ and $S^*_\psi : \mathcal{S}(\mathbb{Y}^{n+1}) \to \mathcal{S}_0(\mathbb{S}^{n-1} \times \mathbb{R})$, for $\psi \in \mathcal{S}_1(\mathbb{R})$, and using the Corollary of Prop. 19.5 of [18], we obtain the following result:

Proposition 3.1. *Let $\psi \in \mathcal{S}_1(\mathbb{R})$. The mappings $S_\psi : \mathcal{S}_0'(\mathbb{S}^{n-1} \times \mathbb{R}) \to \mathcal{S}'(\mathbb{Y}^{n+1})$ and $S_\psi^* : \mathcal{S}'(\mathbb{Y}^{n+1}) \to \mathcal{S}_0'(\mathbb{S}^{n-1} \times \mathbb{R})$ are continuous linear maps.*

One can show a generalization of the reconstruction formula (3.4) for the space of Lizorkin distributions, namely,

$$\frac{1}{D_{\psi,\eta}} S_\eta^* \circ S_\psi = Id_{\mathcal{S}_0'(\mathbb{S}^{n-1}\times\mathbb{R})}, \tag{3.7}$$

where $\eta \in \mathcal{S}_1(\mathbb{R})$ is a reconstruction window for $\psi \in \mathcal{S}_1(\mathbb{R}) \setminus \{0\}$.

Since $\mathcal{S}_0'(\mathbb{S}^{n-1} \times \mathbb{R}) \cong \mathcal{S}_0'(\mathbb{R}, \mathcal{D}'(\mathbb{S}^{n-1}))$ and $\mathcal{S}'(\mathbb{Y}^{n+1}) \cong \mathcal{S}'(\mathbb{R} \times \mathbb{R}^\times, \mathcal{D}'(\mathbb{S}^{n-1}))$, for $\psi \in \mathcal{S}_1(\mathbb{R})$, we may alternatively define the ST

$$S_\psi : \mathcal{S}_0'(\mathbb{S}^{n-1} \times \mathbb{R}) \cong \mathcal{S}_0'(\mathbb{R}, \mathcal{D}'(\mathbb{S}^{n-1})) \to \mathcal{S}'(\mathbb{Y}^{n+1}) \cong \mathcal{S}'(\mathbb{R} \times \mathbb{R}^\times, \mathcal{D}'(\mathbb{S}^{n-1})),$$

as a smooth function from $\mathbb{R} \times \mathbb{R}^\times \to \mathcal{D}'(\mathbb{S}^{n-1})$ of slow growth in the variables $(b,a) \in \mathbb{R} \times \mathbb{R}^\times$, given by

$$S_\psi G(\mathbf{u}, b, a) := \langle G(\mathbf{u}, p), \overline{\psi_{b,a}(p)} \rangle_p, \quad G \in \mathcal{S}_0'(\mathbb{S}^{n-1} \times \mathbb{R}). \tag{3.8}$$

Proposition 3.2. *Let $G \in \mathcal{S}_0'(\mathbb{S}^{n-1}\times\mathbb{R})$ and $\psi \in \mathcal{S}_1(\mathbb{R})$. Then, the function $S_\psi G(\mathbf{u}, b, a)$ defined by*

$$S_\psi G(\mathbf{u}, b, a) := \langle G(\mathbf{u}, p), \overline{\psi_{b,a}(p)} \rangle_p,$$

is a smooth function from $\mathbb{R} \times \mathbb{R}^\times \to \mathcal{D}'(\mathbb{S}^{n-1})$ of slow growth in the variables $(b,a) \in \mathbb{R} \times \mathbb{R}^\times$. Furthermore, for $\Phi \in \mathcal{S}(\mathbb{Y}^{n+1})$, we obtain

$$\langle S_\psi G, \Phi \rangle = \int_{\mathbb{R}^\times} \int_{\mathbb{R}} \langle S_\psi G(\mathbf{u}, b, a), \Phi(\mathbf{u}, b, a) \rangle_{\mathbf{u}} |a|^{-1} dbda. \tag{3.9}$$

Proof. Since $G \in \mathcal{S}_0'(\mathbb{S}^{n-1} \times \mathbb{R})$, for every $\varphi \in \mathcal{D}(\mathbb{S}^{n-1})$ one can easily show that $\langle S_\psi G(\mathbf{u}, b, a), \varphi(\mathbf{u}) \rangle_{\mathbf{u}} = \langle G(\mathbf{u}, p), \varphi(\mathbf{u}) \overline{\psi_{b,a}}(p) \rangle_{\mathbf{u},p}$ is a smooth function in the variable $(b,a) \in \mathbb{R} \times \mathbb{R}^\times$ and of slow growth, i.e.

$$|\langle S_\psi G(\mathbf{u}, b, a), \varphi(\mathbf{u}) \rangle_{\mathbf{u}}| \leq C \left(|a|^s + |a|^{-s} \right) (1 + |b|)^s,$$

for some $C = C_\varphi > 0$ and $s = s_\varphi \in \mathbb{N}_0$.

Now we aim to prove relation (3.9). Since $\mathcal{S}(\mathbb{Y}^{n+1}) = \mathcal{S}(\mathbb{R} \times \mathbb{R}^\times) \hat{\otimes} \mathcal{D}(\mathbb{S}^{n-1})$, it is enough to prove relation (3.9) for $\Phi(\mathbf{u}, b, a) = \varphi(\mathbf{u}) \Psi(b, a)$, where $\varphi \in \mathcal{D}(\mathbb{S}^{n-1})$ and $\Psi \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^\times)$. Since $G \in \mathcal{S}_0'(\mathbb{S}^{n-1}\times\mathbb{R}) \cong \mathcal{D}'(\mathbb{S}^{n-1}, \mathcal{S}_0'(\mathbb{R}))$, then $\langle G(\mathbf{u}, p), \varphi(\mathbf{u}) \rangle_{\mathbf{u}} \in \mathcal{S}_0'(\mathbb{R})$. So,

$$\langle G(\mathbf{u}, p), \varphi(\mathbf{u}) \rangle_{\mathbf{u}} = \partial^\alpha h(p) + r(p), \tag{3.10}$$

where $r = r_\varphi$ is a polynomial on $\mathbb{R}$ and $h = h_\varphi$ is a continuous function of at most polynomial growth on $\mathbb{R}$ ([15], Thm. VI, page 239). Then, under the standard identification (2.5), relation (3.10), and the Fubini's theorem, we have

$$\begin{aligned}
&\langle S_\psi G, \Phi \rangle = \langle G(\mathbf{u}, p), \overline{S_\psi^*(\overline{\Phi})}(\mathbf{u}, p) \rangle_{\mathbf{u},p} \\
&= \langle G(\mathbf{u}, p), \int_{\mathbb{R}^\times} \int_{\mathbb{R}} \varphi(\mathbf{u}) \Psi(b, a) \overline{\psi_{b,a}(p)} |a|^{-1} dbda \rangle_{\mathbf{u},p} \\
&= \langle \langle G(\mathbf{u}, p), \varphi(\mathbf{u}) \rangle_{\mathbf{u}}, \int_{\mathbb{R}^\times} \int_{\mathbb{R}} \Psi(b, a) \overline{\psi_{b,a}(p)} |a|^{-1} dbda \rangle_p
\end{aligned}$$

$$
\begin{aligned}
&= \langle \partial^{\alpha} h(p) + r(p), \int_{\mathbb{R}^\times}\int_{\mathbb{R}} \Psi(b,a)\overline{\psi_{b,a}(p)}\,|a|^{-1}dbda\,\rangle_p \\
&= (-1)^{\alpha}\langle h(p), \int_{\mathbb{R}^\times}\int_{\mathbb{R}} \Psi(b,a)\partial_p^{\alpha}(\overline{\psi_{b,a}(p)})\,|a|^{-1}dbda\rangle_p \\
&+ \langle r(p), \int_{\mathbb{R}^\times}\int_{\mathbb{R}} \Psi(b,a)\overline{\psi_{b,a}(p)}|a|^{-1}dbda\rangle_p \\
&= (-1)^{\alpha}\int_{\mathbb{R}} h(p)dp\int_{\mathbb{R}^\times}\int_{\mathbb{R}} \Psi(b,a)\partial_p^{\alpha}(\overline{\psi_{b,a}(p)})\,|a|^{-1}dbda \\
&+ \int_{\mathbb{R}} r(p)dp\int_{\mathbb{R}^\times}\int_{\mathbb{R}} \Psi(b,a)\overline{\psi_{b,a}(p)}|a|^{-1}dbda \\
&= \int_{\mathbb{R}^\times}\int_{\mathbb{R}} \langle \partial^{\alpha} h(p), \overline{\psi_{b,a}(p)}\,\rangle_p \Psi(b,a)|a|^{-1}dbda \\
&+ \int_{\mathbb{R}^\times}\int_{\mathbb{R}} \langle r(p), \overline{\psi_{b,a}(p)}\,\rangle_p \Psi(b,a)|a|^{-1}dbda \\
&= \int_{\mathbb{R}^\times}\int_{\mathbb{R}} \langle \partial^{\alpha} h(p) + r(p), \overline{\psi_{b,a}(p)}\,\rangle_p \Psi(b,a)|a|^{-1}dbda \\
&= \int_{\mathbb{R}^\times}\int_{\mathbb{R}} \langle\langle G(\mathbf{u},p), \varphi(\mathbf{u})\rangle_{\mathbf{u}}, \overline{\psi_{b,a}(p)}\,\rangle_p \Psi(b,a)|a|^{-1}dbda \\
&= \int_{\mathbb{R}^\times}\int_{\mathbb{R}} \langle\langle G(\mathbf{u},p), \overline{\psi_{b,a}(p)}\rangle_p, \varphi(\mathbf{u})\,\rangle_{\mathbf{u}} \Psi(b,a)|a|^{-1}dbda \\
&= \int_{\mathbb{R}^\times}\int_{\mathbb{R}} \langle S_{\psi}G(\mathbf{u},b,a), \varphi(\mathbf{u})\rangle_{\mathbf{u}} \Psi(b,a)|a|^{-1}dbda \\
&= \int_{\mathbb{R}^\times}\int_{\mathbb{R}} \langle S_{\psi}G(\mathbf{u},b,a), \Phi(\mathbf{u},b,a)\rangle_{\mathbf{u}} |a|^{-1}dbda.
\end{aligned}
$$

□

3.3. **Stockwell desingularization in $\mathcal{S}_0'(\mathbb{S}^{n-1}\times\mathbb{R})$.** The next proposition provides a desingularization formula for the ST. It generalizes the extended Parseval's relation (7) obtained in [14].

Proposition 3.3. (Stockwell desingularization) *Let $G \in \mathcal{S}_0'(\mathbb{S}^{n-1}\times\mathbb{R})$ and $\psi \in \mathcal{S}_1(\mathbb{R})$ be a non-trivial window. If $\eta \in \mathcal{S}_1(\mathbb{R})$ is a reconstruction window for ψ, then*

$$
\langle G, \varrho\rangle = \frac{1}{D_{\psi,\eta}}\int_{\mathbb{R}^\times}\int_{\mathbb{R}} \langle S_{\psi}G(\mathbf{u},b,a), \overline{S_{\eta}\overline{\varrho}(\mathbf{u},b,a)}\rangle_{\mathbf{u}}|a|^{-1}dbda, \quad \varrho \in \mathcal{S}_0(\mathbb{S}^{n-1}\times\mathbb{R}). \tag{3.11}
$$

Proof. By (3.7), (3.6) and Prop. 3.2, we have

$$
\begin{aligned}
\langle G, \varrho\rangle &= \frac{1}{D_{\psi,\eta}}\langle S_{\eta}^{*}(S_{\psi}G), \varrho\rangle = \frac{1}{D_{\psi,\eta}}\langle S_{\psi}G, \overline{S_{\eta}\overline{\varrho}}\,\rangle \\
&= \frac{1}{D_{\psi,\eta}}\int_{\mathbb{R}^\times}\int_{\mathbb{R}} \langle S_{\psi}G(\mathbf{u},b,a), \overline{S_{\eta}\overline{\varrho}(\mathbf{u},b,a)}\,\rangle_{\mathbf{u}}|a|^{-1}dbda.
\end{aligned}
$$

□

3.4. **Stockwell characterization of bounded subsets of $\mathcal{S}_0'(\mathbb{S}^{n-1}\times\mathbb{R})$.** The next proposition gives a characterization of the bounded subsets of $\mathcal{S}_0'(\mathbb{S}^{n-1}\times\mathbb{R})$ via the ST.

Proposition 3.4. *Let $\psi\in\mathcal{S}_1(\mathbb{R})\setminus\{0\}$. A subset $\mathcal{B}\subset\mathcal{S}_0'(\mathbb{S}^{n-1}\times\mathbb{R})$ is weakly (strongly) bounded in $\mathcal{S}_0'(\mathbb{S}^{n-1}\times\mathbb{R})$ if and only if there exists $m=m_{\mathcal{B}}\in\mathbb{N}_0$ and $l=l_{\mathcal{B}}\in\mathbb{N}_0$ such that for every $\varphi\in\mathcal{D}(\mathbb{S}^{n-1})$ one can find $C=C_{\varphi,\mathcal{B}}>0$ with*

$$\Big|\langle S_\psi G(\mathbf{u},b,a),\varphi(\mathbf{u})\rangle_{\mathbf{u}}\Big|\le C(|a|+|a|^{-1})^l(1+|b|)^m, \tag{3.12}$$

for all $G\in\mathcal{B}$ and $(b,a)\in\mathbb{R}\times\mathbb{R}^\times$.

Proof. For seminorms in $\mathcal{S}_0(\mathbb{S}^{n-1}\times\mathbb{R})$, we make the choice

$$\dot\rho_{N,q,k}(\varrho)=\sup_{(\mathbf{u},p)\in\mathbb{S}^{n-1}\times\mathbb{R}}(1+|p|)^N\Big|\frac{d^q}{dp^q}(\Delta_{\mathbf{u}}^k\varrho)(\mathbf{u},p)\Big|,\quad N,q,k\in\mathbb{N}_0,$$

(see Rel. 7 in [7]). Note that weak boundedness is equivalent to strong boundedness, due to the Banach–Steinhaus theorem [18].

Suppose that $\mathcal{B}$ is weakly bounded subset of $\mathcal{S}_0'(\mathbb{S}^{n-1}\times\mathbb{R})$ then, by the Banach-Steinhaus theorem, it is equicontinuous ([18], Thm. 33.2), i.e., there exist $C'>0$ and $N_i,q_i,k_i\in\mathbb{N}_0$ $(i=1,2,...,r,$ for some $r\in\mathbb{N})$, that depend on $\mathcal{B}$, such that,

$$|\langle G,\varrho\rangle|\le C'\sum_{i-1}^{r}\dot\rho_{N_i,q_i,k_i}(\varrho), \tag{3.13}$$

for all $G\in\mathcal{B}$ and $\varrho\in\mathcal{S}_0(\mathbb{S}^{n-1}\times\mathbb{R})$.

Using relations (3.8) and (3.13), for $\varphi\in\mathcal{D}(\mathbb{S}^{n-1})$, we obtain

$$\begin{aligned}\Big|\langle S_\psi G(\mathbf{u},b,a),\varphi(\mathbf{u})\rangle_{\mathbf{u}}\Big|&=\Big|\langle G(\mathbf{u},p),\varphi(\mathbf{u})\overline{\psi_{b,a}(p)}\rangle_{\mathbf{u},p}\Big|\\&\le C'\sum_{i-1}^{r}\dot\rho_{N_i,q_i,k_i}(\varphi(\mathbf{u})\overline{\psi_{b,a}}(p)).\end{aligned}$$

Now the proof comes true by Lemma 6.1 of [14].

The converse, using relations (3.9) and (3.12), for $\varphi\in\mathcal{D}(\mathbb{S}^{n-1})$ and $\Psi\in\mathcal{S}(\mathbb{R}\times\mathbb{R}^\times)$, we get

$$\begin{aligned}\Big|\langle S_\psi G,\varphi\Psi\rangle\Big|&=\Big|\int_{\mathbb{R}^\times}\int_{\mathbb{R}}\langle S_\psi G(\mathbf{u},b,a),\varphi(\mathbf{u})\rangle_{\mathbf{u}}\Psi(b,a)|a|^{-1}dbda\Big|\\&\le C\int_{\mathbb{R}^\times}\int_{\mathbb{R}}(|a|+|a|^{-1})^l(1+|b|)^m|\Psi(b,a)||a|^{-1}dbda,\end{aligned}$$

for all $G\in\mathcal{B}$. Now since $L_b(\mathcal{S}(\mathbb{R}\times\mathbb{R}^\times),\mathcal{D}'(\mathbb{S}^{n-1}))=:\mathcal{S}'(\mathbb{R}\times\mathbb{R}^\times,\mathcal{D}'(\mathbb{S}^{n-1}))\cong\mathcal{S}'(\mathbb{Y}^{n+1})$, where $L_b(X;Y)$ stands for the space $L(X;Y)$ of all continuous linear maps of X into Y equipped with the topology of bounded convergence ([18], page 337), by double application of the Banach-Steinhaus theorem, we may conclude that $\{S_\psi G: G\in\mathcal{B}\}$ is a weakly bounded in $\mathcal{S}'(\mathbb{Y}^{n+1})$ and by the inversion formula (3.7), it follows that $\mathcal{B}$ is weakly bounded in $\mathcal{S}_0'(\mathbb{S}^{n-1}\times\mathbb{R})$. □

4. Desingularization formula for the DST

Let $f \in \mathcal{S}_0(\mathbb{R}^n)$ and $\psi \in \mathcal{S}_1(\mathbb{R})$. By relations (2.11), (2.8) and (3.1), we obtain

$$\begin{aligned} DS_\psi f(\mathbf{u}, b, a) &= \frac{1}{(2\pi)^{\frac{n-1}{2}}} S_\psi(Rf_{\mathbf{u}})(b, a) \\ &= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \frac{|a|}{\sqrt{2\pi}} \int_{\mathbb{R}} Rf_{\mathbf{u}}(p)\overline{\psi}(a(p-b))e^{-ipa}dp \\ &= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \frac{|a|}{\sqrt{2\pi}} \int_{\mathbb{R}} Rf(\mathbf{u}, p)\overline{\psi}(a(p-b))e^{-ipa}dp \\ &= \frac{1}{(2\pi)^{\frac{n-1}{2}}} S_\psi(Rf)(\mathbf{u}, b, a). \end{aligned} \tag{4.1}$$

Now, using (2.15), (4.1) and the Fubini's theorem, for $\Phi \in \mathcal{S}(\mathbb{Y}^{n+1})$ we have

$$\begin{aligned} \langle DS_\psi f, \Phi\rangle &= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^\times} \int_{\mathbb{R}} |a|^{n-2} dbda \int_{\mathbb{S}^{n-1}} S_\psi(Rf)(\mathbf{u}, b, a)\Phi(\mathbf{u}, b, a)d\mathbf{u} \\ &= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^\times} \int_{\mathbb{R}} \langle S_\psi(Rf)(\mathbf{u}, b, a), \Phi(\mathbf{u}, b, a)\rangle_{\mathbf{u}} |a|^{n-2} dbda. \end{aligned}$$

The last relation motivates us to the next proposition, which states a relation between the Radon transform, the ST and the DST.

Proposition 4.1. *Let $f \in \mathcal{S}_0'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}_1(\mathbb{R})$. Then*

$$\langle DS_\psi f, \Phi\rangle = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^\times} \int_{\mathbb{R}} \langle S_\psi(Rf)(\mathbf{u}, b, a), \Phi(\mathbf{u}, b, a)\rangle_{\mathbf{u}} |a|^{n-2} dbda, \tag{4.2}$$

$\Phi \in \mathcal{S}(\mathbb{Y}^{n+1})$. Furthermore, $DS_\psi f \in \mathbb{C}^\infty(\mathbb{R} \times \mathbb{R}^\times, \mathcal{D}'(\mathbb{S}^{n-1}))$ and it is of slow growth on $\mathbb{R} \times \mathbb{R}^\times$.

Proof. By relations (3.9), (3.5), (2.7), (2.6), and (3.3) we have

$$\begin{aligned} &\frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^\times} \int_{\mathbb{R}} \langle S_\psi(Rf)(\mathbf{u}, b, a), \Phi(\mathbf{u}, b, a)\rangle_{\mathbf{u}} |a|^{n-2} dbda \\ &= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^\times} \int_{\mathbb{R}} \langle S_\psi(Rf)(\mathbf{u}, b, a), |a|^{n-1}\Phi(\mathbf{u}, b, a)\rangle_{\mathbf{u}} |a|^{-1} dbda \\ &= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \langle Rf, \overline{S_\psi^*(|a|^{n-1}\overline{\Phi(\mathbf{u}, b, a)})}\rangle \\ &= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \langle f(\mathbf{x}), R^*\left(\overline{S_\psi^*(|a|^{n-1}\overline{\Phi(\mathbf{u}, b, a)})}\right)(\mathbf{x})\rangle \\ &= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \langle f(\mathbf{x}), \int_{\mathbb{S}^{n-1}} \overline{S_\psi^*(|a|^{n-1}\overline{\Phi(\mathbf{u}, b, a)})}(\mathbf{u}, \mathbf{u}\cdot\mathbf{x})d\mathbf{u}\rangle \\ &= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \langle f(\mathbf{x}), \int_{\mathbb{S}^{n-1}} d\mathbf{u} \int_{\mathbb{R}^\times} \int_{\mathbb{R}} |a|^{n-1}\Phi(\mathbf{u}, b, a)\overline{\psi_{b,a}(\mathbf{u}\cdot\mathbf{x})}\, |a|^{-1} dbda\rangle \\ &= \langle f(\mathbf{x}), \int_{\mathbb{S}^{n-1}} d\mathbf{u} \int_{\mathbb{R}^\times} \int_{\mathbb{R}} |a|^{n-2}\Phi(\mathbf{u}, b, a)\overline{\psi_{\mathbf{u},b,a}(\mathbf{x})} dbda\rangle \end{aligned}$$

$$= \langle f, \overline{DS^*_\psi(\overline{\Phi})} \rangle = \langle DS_\psi f, \Phi\rangle.$$

So relation (4.2) is true.

Let $\varphi \in \mathcal{D}(\mathbb{S}^{n-1})$ and $\Psi \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^\times)$ be fixed. Under the standard identification (2.4) and relation (4.2), we obtain

$$\begin{aligned} &\langle\langle DS_\psi f(\mathbf{u},b,a), \varphi(\mathbf{u})\rangle_{\mathbf{u}}, \Psi(b,a)\rangle_{b,a} = \langle DS_\psi f, \varphi\Psi\rangle \\ &= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^\times}\int_{\mathbb{R}} \langle S_\psi(Rf)(\mathbf{u},b,a), \varphi(\mathbf{u})\rangle_{\mathbf{u}} \Psi(b,a)|a|^{n-2} dbda. \end{aligned} \tag{4.3}$$

On the other hand, from (3.8), we obtain

$$\begin{aligned} \langle S_\psi(Rf)(\mathbf{u},b,a), \varphi(\mathbf{u})\rangle_{\mathbf{u}} &= \langle\langle Rf(\mathbf{u},p), \overline{\psi_{b,a}(p)}\rangle_p, \varphi(\mathbf{u})\rangle_{\mathbf{u}} \\ &= \langle Rf(\mathbf{u},p), \varphi(\mathbf{u})\overline{\psi_{b,a}(p)}\rangle_{\mathbf{u},p}. \end{aligned} \tag{4.4}$$

Now, since $Rf \in \mathcal{S}_0'(\mathbb{S}^{n-1}\times\mathbb{R}), \psi \in \mathcal{S}_1(\mathbb{R})$ and $\varphi \in \mathcal{D}(\mathbb{S}^{n-1})$, the relation (4.4) implies that

$$(b,a) \to \langle S_\psi(Rf)(\mathbf{u},b,a), \varphi(\mathbf{u})\rangle_{\mathbf{u}}$$

is a smooth function on $\mathbb{R}\times\mathbb{R}^\times$ of slow growth in the variables $(b,a) \in \mathbb{R}\times\mathbb{R}^\times$, i.e.,

$$|\langle S_\psi(Rf)(\mathbf{u},b,a), \varphi(\mathbf{u})\rangle_{\mathbf{u}}| \le C_\varphi \left(|a|^v + |a|^{-v}\right)(1+|b|)^v,$$

for some $v = v_\varphi \in \mathbb{N}_0$ and $C_\varphi > 0$.
Now, by identification (2.3), we obtain

$$\begin{aligned} &\langle\langle S_\psi(Rf)(\mathbf{u},b,a), \varphi(\mathbf{u})\rangle_{\mathbf{u}}, \Psi(b,a)\rangle_{b,a} \\ &= \int_{\mathbb{R}^\times}\int_{\mathbb{R}} \langle S_\psi(Rf)(\mathbf{u},b,a), \varphi(\mathbf{u})\rangle_{\mathbf{u}} \Psi(b,a)|a|^{-1} dbda. \end{aligned} \tag{4.5}$$

Finally, relations (4.3) and (4.5) yield

$$\langle DS_\psi f(\mathbf{u},b,a), \varphi(\mathbf{u})\rangle_{\mathbf{u}} = \frac{1}{(2\pi)^{\frac{n-1}{2}}} |a|^{n-1} \langle S_\psi(Rf)(\mathbf{u},b,a), \varphi(\mathbf{u})\rangle_{\mathbf{u}}.$$

Then, $DS_\psi f \in \mathbb{C}^\infty(\mathbb{R}\times\mathbb{R}^\times, \mathcal{D}'(\mathbb{S}^{n-1}))$ and it is of slow growth on $\mathbb{R}\times\mathbb{R}^\times$. □

We end this article with a desingularization formula for the DST.

Proposition 4.2. (Directional Stockwell desingularization) *Let $f \in \mathcal{S}_0'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}_1(\mathbb{R})$ be a non-trivial window. If $\eta \in \mathcal{S}_1(\mathbb{R})$ is a reconstruction window for ψ, then*

$$\langle f, \varphi\rangle = \frac{1}{C_{\psi,\eta}(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^\times}\int_{\mathbb{R}} \langle S_\psi(Rf)(\mathbf{u},b,a), \overline{DS_\eta(\overline{\varphi})(\mathbf{u},b,a)}\rangle_{\mathbf{u}} |a|^{n-2} dbda, \tag{4.6}$$

$\varphi \in \mathcal{S}_0(\mathbb{R}^n)$.

Proof. By (2.14), definition (2.13) for the directional Stockwell synthesis operator and Prop. 4.1, we have

$$\langle f, \varphi\rangle = \frac{1}{C_{\psi,\eta}} \langle DS^*_\eta(DS_\psi f), \varphi\rangle = \frac{1}{C_{\psi,\eta}} \langle DS_\psi f, \overline{DS_\eta(\overline{\varphi})}\rangle$$

$$= \frac{1}{C_{\psi,\eta}(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^\times} \int_{\mathbb{R}} \langle S_\psi(Rf)(\mathbf{u},b,a), \overline{DS_\eta(\overline{\varphi})(\mathbf{u},b,a)}\,\rangle_{\mathbf{u}} |a|^{n-2} dbda.$$

□

Disclosure statement

The authors report there are no competing interests to declare.

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