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HISTORY AND APPLICATIONS OF NONLINEAR EIGENVALUE PROBLEMS

DRAGANA GARDAŠEVIĆ [1]

Abstract. Matrix analysis is a powerful tool in the study of modern phenomena and in the application of mathematics in science, engineering, medicine, economics, physics and other scientific disciplines. The eigenvalue analysis, due to its numerous applications, represents an attractive area of applied mathematics. During the 20th century, various researches on the topic of eigenvalue theory were carried out. Hence the idea of this paper is to give an overview of the history of the origin of this type of problem, as well as its applications, primarily in engineering, which would make this paper useful as a starting point for further studies of these specific problems.

1. Introduction

In the study of various phenomena in science and engineering, matrix analysis enables a quick and simple mode for getting answer to the questions about the position of the eigenvalues in the complex plane (i.e. their values), system stability, etc., without excessive computational costs. The eigenvalue analysis, as an important part of the matrix theory, represents an attractive area of applied mathematics. Any regular pattern that can be seen in nature probably has something to do with an eigenvalue problem (EVP): giant tides come from resonance fed by pumping at a frequency near an eigenvalue; musical instruments are built upon eigenvalues, cloud stripes in the sky, ripples on the sand, droplets in a splash etc. [1]. These problems can be written in the form

$$T(z)v = 0,$$

where, $T(z)$ is a matrix valued function defined on an open set $\Omega \subseteq C^{n,n}$, that defines the observed problem, and $v \neq 0$ is corresponding eigenvector for the eigenvalue $z \in C^n$ of the observed problem. The set of all eigenvalues is called the spectrum of a matrix valued function $T(z)$. EVPs are divided into several categories, according to the form of the matrix function they are defined with,

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and, therefore, different methods are applied for their solving. The EVPs are classified as:

- Standard EVP, or SEVP, defined with matrix function $T(z) = Az - I$, that depends on $A \in C^{n,n}$,
- Generalized EVP, or GEVP, defined with matrix function $T(z) = Az - B$, for $A, B \in C^{n,n}$,
- Linear EVP, or LEVP, defined with linear matrix function $T(z)$ that linearly depends on z (SEVP and GEVP are LEVPs),
- Quadratic EVP, or QEVP, defined with matrix function $T(z) = Az^2 + Bz + C$, for $A, B, C \in C^{n,n}$,
- Polynomial EVP, or PEVP (of the degree higher than two), defined with matrix polynomial of the n-th order,

  $$T(z) = A_n z^n + A_{n-1} z^{n-1} + \cdots + A_1 z + A_0$$

  for $A_i \in C^{n,n}$, $i = 1, 2, ..., n$,
- Rational EVP, or REVP, where $T(z)$ has rational dependance of z,
- Inverse EVP, where $T(z)$ is matrix valued function whose coefficients are inverse than those of the beginning problem,
- Nonlinear EVP, or NLEVP, where $T(z)$ has nonlinear dependance on z.

It is obvious that all beforementioned EVPs are NLEVPs, where $A, B, C, A_i \in C^{n,n}, i = 0, ..., n$ and $z \in C^n$ are called *coefficient matrices* of $T(z)$.

At the physical level, eigenvalues represent the spectrum of vibrations [2]. Their applications in science and technology are various: acoustic field simulations, computational quantum chemistry, structural dynamics, electromagnetic modeling of particle accelerators, vibrations of fluid–solid structures, or stability analysis of time–delay systems, time–dependent boundary conditions, time–dependent material parameters, or the use of special basis functions [3, 4]. Boundary–element discretizations, on the other hand, can lead to NLEVPs even if the underlying operator eigenvalue problem is linear [5]. The process of finding the principal stress and vibration frequency and finding the principal axis of inertia are also EVPs, as well as stable distribution and convergence of Markov chains, and estimation of matrix operations errors [2]. QEVPs represent most of the models of mechanical systems: vibration analysis and fluid dynamics; exponential problems that are used to model delay–differential equations, for example, in electronic devices. In the last few decades researchers have made great improvements in study of LEVPs, PEVPs and REVPs. Besides, a special case of EVPs are the ones that are nonlinear in the eigenvalue parameter. Such problems arise, e.g., from boundary integral formulations of elliptic PDE eigenvalue problems and typically exclude the use of established nonlinear eigenvalue solvers [6].

Interest in this type of problem has increased in the last few decades, when various papers revealed their wide range of applications. Therefore, the idea of this paper is to present an overview of the origin and development of EVP, as well as the methods developed for solving them over time.

Therefore, the paper is organized in three sections: the first section gives information about the origin of the EVPs, the second section reviews the known methods for solving EVPs, along with its generalizations, while the third section gives more information about applications.

2. The origin and history of NLEVPs

Interest for the study of EVPs has started in the middle of the 18th century, in the study of quadratic forms and differential equations, with Euler (who has dealt with the rationing movement of the rigid body and has revealed the significance of the principal axes) and Lagrange (who realized that the principal axes are the eigenvectors of the inertia matrix) [7]. In the next few decades Cauchy noticed that eigenvalues could be used for the classification of the quadric surfaces. Well-known mathematicians Fourier, Sturm, Hermite, Brioschi, Clebsch, Weierstrass, Liouville, Poencaré and Hilbert, have continued to work on the theory of eigenvalues [8]. In the first half of the 20th century, Hilbert studied operators and its eigenvalues via infinite matrices [7]. Later on, Schmidt and Hammerstein continued to work on eigenvalue analysis, inspired by Hilbert and Landau, while, on the other side, Golomb worked on parameter-dependent nonlinear integral equations. In the 1930s, Von Mises established a numerical procedure for computing eigenvalues and eigenvectors by which for every linear change of eigenvector, there exists a corresponding scalar value named *eigenvalue* [9]. In the 1940s, Miranda and Harazov studied rational EVP-integral equations and operators in Hilbert space. Both authors obtained results for LEVP related to the existence of eigenvalues, successive extremal characterization, expansion theorems, and solvability of inhomogeneous problems [10, 11].

Another direction of studying the theory of eigenvalues is to include damping and perturbations. In 1914, Faber published paper on vibration with damping and made a fairly complete study of the existence and asymptotic behavior of eigenvalues and eigenfunctions, the Green's function, and expansion properties [12].

The former Soviet Union scientists Krasnoselski and Gohberg have strongly impacted on the interest for NLEVPs in the Eastern Europe [13]. It's wide range of applications made this area of mathematics still attractive for research.

The origin of the term *eigenvalue* dates from 1904, when David Hilbert introduced the term, derived from German word *eigen*, meaning "own", "peculiar to", "characteristic", or "individual". For some time, the standard term in English was *proper value*, but the more distinctive term *eigenvalue* is standard today. Cauchy created the word *Racine Characteristic* or *Characteristic Root*, which is named *Eigenvalue* and this is still in use [8].

A nice review of the history of the theory of matrices in the 19th century is [14]. In this paper, a spectral theory originated from the theory of linear substitutions and bilinear forms in the period 1826-1876 is given.

3. Computing methods for NLEVPs

Numerical algorithms for computing eigenvalues appeared in 1929, with Von Mises's power method [15, 16]. Next well-known problem, the QR factorization, appeared in 1961 by G. F. Frances and V. Kublanovskaya, simultaneously [17]. During next few decades of the 20th century, various methods for solving EVPs are created. The expansion of methods for solving EVPs started in the middle of the 20th century, along with the development of computers. Since then, it is developed permanently, thanks to it's actuality. There are several papers written during the last twenty years in which an overview of existing methods is given (see, for example, [13, 18, 19, 20]). In 2003, an interesting workshop of reviewing results on NLEVPs was held [21]. Here, we give an updated review of the latest results in this area.

Methods of solving EVPs have been developed in two directions: in Numerical Analysis and Matrix Analysis at one side, and in Nonlinear Functional Analysis, that deals, for example, with the applications to differential equations (as the basis of, among others, Bifurcation Theory [13]) and asymptotic numerical modeling [22], at the other side.

Numerical methods for solving NLEVPs are created according to the structure of the coefficient matrices that define the observed eigenvalue problem. For dense problems, the application of methods depends on the available storage capacities. The standard approach for the numerical solution for SEVP and GEVP is to reduce the matrices involved to some simpler form that reveals the eigenvalues, for instance, the Schur form for a single matrix A and the generalized Schur form for coefficient matrices A, B of the GEVP. Unfortunately, these canonical forms cannot be generalized to coefficient matrices of $T(z)$ for EVPs of degree greater than 1. Most of the numerical methods that deal directly with QEP are variants of Newton's method. They compute one eigenpair in each step and converge if the starting guess is close enough to the solution, but in practice, even for a good initial guess, there is no guarantee that the method will converge to the desired eigenvalue [23]. Iterative projection methods for large sparse problems use the sparsity of the matrix structure for matrix vector multiplication with the coefficient matrices plus possibly sparse factorizations of matrices, when shift-and-invert is used to obtain the position of eigenvalues in the interior of the spectrum. The available storage sets the limit for the system sizes that can be dealt with [18, 24].

The NLEVP solvers are often divided in two categories: direct and iterative [20, 25]:

- direct methods, such as QR factorization, divide-and-conquer method, which serves for accurate finding of all eigenvalues of relatively small matrices (of the order of 10^3). These methods work on an iterative mode, and they (almost in all cases) converge in a fixed number of steps.
- iterative methods, such as subspace methods, project the matrix of a problem onto a low–dimensional subspace and then solve that problem by

direct methods. These methods are suitable for large sparse matrices, because they give approximate eigenpairs from a low-dimensional subspace.

For large sparse problems, the choice of method may depend of certain properties of the matrix (structure, size), the data of interest (what, to which accuracy), the available operations (transpose of the matrix, preconditioner) and the machine architecture [26]. The most common approach of solving the n-th degree NLEVPs is via linearization, a technique developed in 1956, independently by Miller at one side, and Miranda and Harazov at the other side [12]. It defines the embedding of NLEVP into a larger LEVP, in which the k-th order $n \times n$ matrix polynomial embedds into larger $kn \times kn$ LEVP [27, 28, 29]. This technique transforms the observed problem to an equivalent LEVP $(zE - A)v = 0$ with the same eigenvalues. In this way, the initial problem becomes much bigger and the resulting LEVP may be much more sensitive to perturbations than the original problem (see [34] for details). Additionally, the inherent symmetry structure of the problem is usually destroyed [18]. This procedure is usually performed via iterative projection methods. There are several types of these methods, that differ up to the preservation of the structure of the initial system (iterative projection methods for LEVPs, stucture preserving iterative projection methods for linearized problems, iterative projection methods for nonlinear problems, Jacobi–Davidson type methods, Rational Krylov method) [18, 20].

Iterative projection methods for sparse LEVPs: Lanczos, Arnoldi, rational Krylov (shift-and-invert methods) and Jacobi–Davidson method are based on the construction of a Krylov subspace (called search space) and the projection of the problem into this subspace, obtaining a small dense system, and then handling it by a dense solver. The eigenvalues of the projected problem are used as approximations to the eigenvalues of the large sparse problem. Iterative projection methods avoid matrix factorizations as much as possible and the search space is usually generated via operationally cheap iterative procedure that is based on matrix vector products. If more than one eigenvalue is observed, a deflation technique is based on partial Schur decomposition of the matrix A. Many of the ideas in these projection methods can be generalized also to NLEVPs [30, 31, 32, 33].

Structure preserving iterative projection methods use structure preserving linearizations and symmetries in the generation of the search space, producing small problems of the same structure (see, for example [29, 35, 37, 38]). Each of these methods requires a structured GEVP to be reduced to a Hamiltonian or symplectic matrix. In the standard Arnoldi method, if the structure is violated by roundoff errors, additional orthogonalization step is added.

Iterative projection methods for NLEVPs demand the expansion of the search spaces by directions that have a high approximation potential for the next desired eigenvector. In that way, nonlinear Arnoldi method and rational Krylov method are generalized [3, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51]. The limitation in this process is revealed in the approximation of more than one eigenvalue, when it is needed to inhibit the method to converge to the same eigenvalue repeatedly. For LEVPs this is done by using Schur forms or generalized Schur forms for the projected problem and then locking or purging certain eigenvalues, but, for

NLEVPs, such Schur forms doesn't exist and this presents one of the most difficult tasks in achieving good convergence. So, different approaches are developed to solve this problem (see, for example, [50, 52].

Jacobi–Davidson type methods are efficient in situations when the convergence of the preconditioner deteriorates considerably. Then a natural generalization of the Jacobi–Davidson method for linear eigenproblems is applied for NLEVPs [53, 54, 55].

Rational Krylov method [49, 56, 57] combines linearization and the Arnoldi method and in that way generalizes the rational Krylov method for LEVPs, reducting the large dimension of the system to a much smaller one and finding the solution of a NLEVP. It is suitable for sparse NLEVPs.

A disadvantage of the shift-and-invert Arnoldi methods is that a change of the shift parameter requires a new Krylov subspace to be built. Another approach is a direct solution obtained by means of the Jacobi–Davidson method, although this method has been investigated far less extensively [28].

Another approach of solving EVPs is based on Newton′ iterative procedure. Numerical methods based on Newton′s method compute eigenvalues of NLEVPs by choosing for an initial guess the only crucial parameter of a Newton–type method. Various algorithms for this type methods have been developed and applied from the 1950s to 2000s: the Newton–QR iteration of Kublanovskaya (1970) and its variant in Garrett, Bai and Li (2016), the Newton-trace iteration of Lancaster (1966), nonlinear inverse iteration (Unger 1950), residual inverse iteration (Neumaier 1985), Rayleigh functional iterations (Lancaster 1961, Schreiber 2008), the block Newton method of Kressner (2009), and for large sparse NLEVPs, Jacobi–Davidson-type methods (Betcke and Voss 2004, Sleijpen, Booten, Fokkema and van der Vorst 1996). Two types of Newton–type method can be recognized: the application to a scalar equation $f(z) = 0$ whose roots correspond to the eigenvalues of $T(z)$ and the application directly to the vector problem $T(z)v = 0$ together with some normalization condition on v [19].

Solution approaches based on approximation or interpolation of the NLEVP are well studied in the literature. The boundary element method (BEM) results are reviewed in [58, 59], while the linearization methods for a given matrix polynomial or rational matrix-valued function $T(z)$ are discussed in details and reviewed nicely in [26].

Diagonal dominance (DD)-Geršgorin method–In the past few decades, the actuality of the NLEVPs has motivated scientists to develop a new method of solving EVPS. This new localization method is based on the property of diagonal dominance of coefficient matrices and famous Geršgorin theorem. Although the Geršgorin theorem is older than a century, and a diagonal dominance property is known and studied almost the same as much, the localization method based on a relation between them is developed in the last two to three decades. The idea of this method is to localize not the exact eigenvalues individually, but to localize the spectra, reducing the computational cost in that way. Richard Varga, in his book *Geršgorin and his circles*, has related these facts and developed a solution method for LEVP [60]. His work has inspired V. Kostić and Lj. Cvetković to generalize these results for GEVP (see, for example [61, 62]). Later on, these results have been generalized to NLEVP, especially PEVP [63, 64, 65, 66]. The advantage of this method is the simplicity of calculations based on the diagonal dominance property and relation with the Geršgorin localization theorem and its generalizations, which results in reducing computational costs.

Methods based on contour integral represent a new group of methods that are based on the Keldysh's theorem [19]. These methods are based on the generalization of the Sakurai-Sugiura (SS) method for LEVPs, suitable for modern parallel computers, that find certain eigenvalues in a given domain and solve non-Hermitian systems; and also in a series of works by Asakura et al., Beyn and Yokota and Sakurai [19]. This method reduces the dimension of the problem by focusing on only the eigenvalues of physical interest. The generalized method for PEVPs, given in [28], enables calculating the eigenvalues of matrix polynomial by solving the GEVP, which is derived by solving in parallel independent systems of linear equations. The theory here is based on the Smith form (the canonical form) for polynomial matrix. Recent model of a contour-based eigensolver is the Contour Integral Spectrum Slicing (CISS) method that can solve generalized linear and nonlinear eigenvalue problems [19, 68].

As PEVPs are the most common in applications, it can be noted that most of the papers with the topic on PEVPs are dealing with regular matrix polynomials. Other problems, for example those that arise in automatic modeling of overdetermined systems with redundant equations [83], can often be reduced to regular PEVPs by using appropriate reduction procedures [69]. The limitation of these procedures is reflected to the applicability on large scale problems and the huge number of needed numerical operations even for small scale problems. For the problems where the leading coefficient matrix is singular (such as constraint multibody systems [70], circuit simulation [71, 72], optical waveguide design [73] etc.), not all linearizations properly reflect the multiplicities of the eigenvalue infinity [69] and therefore strong linearizations must be used [23]. A new numerical method for the exact solution of nonlinear eigenvalue problems, that can be applied to large-scale problems, combines homotopy and perturbation techniques (see, for example, [74]).

Besides before mentioned techniques, there are some new techniques based on boundary element method (see, for example, [58, 59]), finite element method (see, for example, [76]), golden section search algorithm ([77]), parallel computing ([78]), etc.

Over time, a connections between mentioned methods are done: a connection with rational Krylov techniques for solving the linearized problem with Newton's method, using a link between the rational Arnoldi and Jacobi–Davidson algorithms pointed out by Ruhe ([51, 56]), Van Beeumen ([26])et al. In [26], the interpretation of a Jacobi–Davidson iteration as a Newton update, a connection between rational Krylov methods for (N)LEVPs and a basic contour-based method, etc.

4. (N)LEVP Applications

The applications of EVPs can be seen everywhere, in science and in the real life. A beautiful example is the turning of the Earth against the Moon provides the tide and a low tide, providing a natural mode of oscillation of the bay has a period of about 13 hours. Therefore, the giant tides are created from resonance fed by pumping at a frequency near an eigenvalue. In the same paper, another real-life application is given: the vibrations of the air in a flute is EVP, which explains why a note sounds musical. More detail can be read in [1].

One of the most famous EVPs, resonance, occurs when some natural frequencies of the vibrating systems become close to those of external forces. The well-known examples are: the Millennium bridge, where the weights of thousands of people who rushed to see the bridge on the first day were the external force; the Tacoma bridge in Washington, USA, where the external force was gusty wind; and the Broughton bridge in England, where the external force was the weight of the soldiers marching in the bridge. Mathematically, this problem is of reassigning the few resonant frequencies (eigenvalues) to desired locations, while keeping the remaining large number of them and corresponding eigenvectors unchanged [3, 19].

There are many applications of matrix eigenvalues in differential equations. Besides the ones given in introduction of this paper, for example, in solving a system of first-order constant coefficient equations, the focus is to find the basic solution group of the equation. When the eigenvalues of the coefficient matrix A of the equation group are all single roots, it can be transformed into finding the eigenvector corresponding to the eigenvalue. Regardless of whether the eigenvalues of the matrix are used in modeling or equations, the matrix is diagonalized, making the calculation easier. The eigenvalues and eigenvectors of the diagonalized matrix are known, and the power can be obtained by calculating the same power of the diagonal elements [12].

The EVPs is widely used in many research fields such as physics, chemistry, and biology [2, 79, 82, 83]. A neural network is a neuron model constructed by imitating biological neural networks. Since it was proposed, the application research of its typical models, such as recurrent neural networks and cellular neural networks, has become a new hot spot. With the emergence of deep neural network theory, scholars continue to combine deep neural networks to calculate matrix eigenvalues [2, 79].

In physics, Schrödinger concluded that the spectral lines in the light from stars were exactly a case of mathematicians' spectra, with each line corresponding to the difference in energy of two eigenstates. And it was these same spectral lines that led to the discovery of the red shift and the expanding universe [1]. In the area of atomic physics, the laser EVP involves light waves bouncing back and forth in a cavity, because an eigenvalue gap in the atom is tuned to an eigenvalue of the cavity that the device emits a coherent beam. Another interesting applications is in AM radio signals, where a weak signal from outside is brought into a resonant circuit and greatly amplified. This means that the change of stations is just the adjustment of the eigenvalue [1].

Magnetohydrodynamics features even knottier non-self-adjoint eigenvalue problems, which may be one of the reasons fusion power has been elusive, and may also help in explaining why the magnetic field generated by the dynamo in the core of the earth switches direction seemingly at random. It is the trace of these random reversals in core samples from ocean bottoms that led to acceptance of the theory of continental drift, a process driven by yet another eigenvalue problem: the slow radioactive decay of nuclei from their meta-stable near-eigenstates, which pumps heat into the earth's core that must escape at the surface [1].

NLEVPs occur in many areas of mathematical physics: the theory of topological degree of Leray and Schauder and the theory of critical points of Ljusternik and Schnirelmann. The applications of degree theory is related to bifurcation problems, in fluid dynamics and elasticity theory [75].

Among other practical problems is Model Updating Problem (MUP), for which preservation of symmetry and other properties is crucial. This problem arise in aerospace, manufacturing, automobile, and other vibration industries. In these industries, a theoretical finite-element generated symmetric model often needs to be updated using a few measured eigenvalues and eigenvectors, obtained from a practical structure in such a way that the updated model can be used with confidence for future designs [76].

The buckling of a plane frame occurs in structural engineering when a solid structure (column, a plane frame or a steel beam) is subjected to a heavy load, and may suddenly deform and change its shape. Leonard Euler was among the first scientists who has studied this subject. If a finite elements formulation is used, then the buckling problem can be stated as a GEVP.

Second important application in physical systems is related to time-delay systems. Every time a command is inputed in an electronic device, the electric signal takes some time to reach its destination; during a traffic jam, drivers do not immediately accelerate or brake when the car in front starts again or stops. An interesting example is given by a semiconductor laser subject to a delayed phase-conjugate feedback caused by a reflective mirror. Another interesting application concerns design of optical fibres. They are implemented everywhere in current technologies and allow data transfer at much higher bandwidth than electrical cables. They are composed by a glass core, which act as a waveguide for the light, surrounded by a transparent material, the cladding, with a lower index of refraction.

Canyon particle – Solving the electronic transport in semi–conductive material requires the study of the Poisson, Schrödinger, and transport equations. However, if the model assumes that the collisions of the electrons are negligible, then solving the single-particle Schrödinger equation with transmitting boundary conditions determines the current of the system [80, 81]. In the case of a system with contacts, the Schrödinger equation can be cast as the nonlinear eigenvalue problem [82].

The QEVP has received much attention because its formation has repeatedly arisen in many different disciplines, including applied mechanics, electrical oscillation, vibro-acoustics, fluid mechanics, and signal processing. Francois Tisseur and Karl Meerbergen, in [3], gave an excellent review of applications, mathematical properties, and a variety of numerical techniques for the QEVP. It is known that the QEVP has $2n$ finite eigenvalues over the complex field when the leading matrix coefficient A is nonsingular. The problems of this type arising in practice often entail some additional conditions for the matrices. For example, if A, B and C represent the mass, damping and stiffness matrices, respectively, in a mass-spring system, then it is required that all matrices be real-valued and symmetric, and that A and C must be positive definite and semi-definite, respectively. The process of analyzing and deriving the spectral information and, hence, inducing the dynamical behavior of a system from a priori known physical parameters such as mass, length, elasticity, inductance, capacitance, and so on is referred to as a direct problem. The inverse problem, in contrast, is to validate, determine, or estimate the parameters of the system according to its observed or expected behavior. The concern in the direct problem is to express the behavior in terms of the parameters whereas in the inverse problem the concern is to express the parameters in term of the behavior. The inverse problem is just as important as the direct problem in applications [84].

Among other interesting applications of matrix eigenvalues are: solving the general term of the Fibonacci sequence, autosomal genetic problems, and application in equations. The general term of the Fibonacci sequence is also called the golden section sequence. He introduced this term using rabbit reproduction as an example. The autosomal problem is the problem of the probability of collection and transmission of chromosome dominance and recession. Here, matrix eigenvalues and eigenvectors are used to transform the matrix into a diagonal matrix and then solve it [12].

Conclusion

Systems of communications, bridge construction, automobile stereo system design, electrical engineering, mechanical engineering, music, nature, etc. are the areas where eigenvalue analysis is applied. Eigenvalue analysis does not lose its attractiveness for decades because its main role is to improve efficiency in computationally intensive tasks. EVPs are divided into several categories, according to the form of the matrix function they are defined with, and, therefore, different methods are applied for their solving.

Numerical methods are essential tools for eigenvalue analysis, as they allow the analysis of large and complex matrices cheaper (with smaller number of calcultions than other methods). As a powerful tool in the study of modern phenomena, the eigenvalue analysis represents an attractive area of applied mathematics. During the 20th century, an expansion of researches in this area is carried out.

In this paper, various applications of NLEVP are given. The variety of these applications gives a new perspective to those who are beginners in this field and also to those who are interested in application of mathematics in engineering.

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Dragana Gardašević
Conference of Academies and High Schools of Serbia,
The Academy of Applied Technical Studies Belgrade,
Belgrade, Serbia
Email address: dgardasevic@atssb.edu.rs

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