

## CHAIN CONNECTEDNESS – HOMOTOPY AND SHAPE INVARIANT

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**Abstract.** The characterization of connectedness by coverings i.e. chain connectedness led to some new concepts. For example chain components coincide with quasicomponents. We present some important results on chain connectedness and on chain components.

### 1. INTRODUCTION

In the second half of the 19th century Cantor defined connectedness in the Euclidean spaces. In the beginning of 20th century F. Riesz gave the today well-known definition of connectedness:  $X$  is connected if it cannot be represented as a union of two nonempty disjoint open sets. F. Hausdorff, after he introduced the definition of topological spaces, gave the same definition for topological spaces. The disadvantage of this definition is that it is given by a negative sentence. In [4] is given a definition for connectedness of topological spaces using chains and coverings.

Throughout, we will take  $X$  to be a topological space and  $\mathcal{U}$  an open cover of  $X$ .

Let  $\mathcal{F}$  be a family of subsets in  $X$  and let  $x, y \in X$ . A finite sequence  $F_1, F_2, \dots, F_n$  in  $\mathcal{F}$  is a chain in  $\mathcal{F}$  from  $x$  to  $y$  (or a chain joining  $x$  and  $y$ ) if  $F_i \cap F_{i+1} \neq \emptyset$ ,  $i = 1, 2, \dots, n - 1$ . We say that  $x$  and  $y$  are chainable in  $\mathcal{F}$  if there exists a chain in  $\mathcal{F}$  such that  $x \in F_1, y \in F_n$ .

For two points  $x, y \in X$  we say that  $x$  and  $y$  are chainable in  $X$  (or chain connected in  $X$ ) if for any open cover  $\mathcal{U}$  of  $X$ ,  $x$  and  $y$  are chainable in  $\mathcal{U}$ . We say that  $x$  and  $y$  are chainable for  $x$  and  $y$  are chainable in  $X$ , if the space  $X$  is known. Now we are able to define connectedness of the space using coverings, as given in [4].

**Definition 1.** *The topological space  $X$  is chain connected if any two points in  $X$  are chainable .*

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The next theorem shows that the notions of connectedness and chain connectedness are equivalent.

**Theorem 1.** [4] *The topological space  $X$  is connected if and only if it is chain connected.*

## 2. PROPERTIES OF CHAIN CONNECTEDNESS

We repeat the standard definition of connectedness: Two sets  $A, B \subseteq X$  are separated in  $X$  if  $\overline{A} \cap B = \emptyset$  and  $A \cap \overline{B} = \emptyset$ . A set  $C \subseteq X$  is connected in  $X$  if it cannot be expressed as a union of two nonempty separated subsets.

The definition of connectedness of  $C$  does not depend on the space where  $C$  is embedded, i.e. if  $C \subseteq Y \subseteq X$ , then  $C$  is connected in  $Y$  if and only if  $C$  is connected in  $X$ .

We present the definition of chain connectedness of a set in a topological space.

**Definition 2.** We say that the set  $C \subseteq X$  is *chain connected* in  $X$  if for every  $x, y \in C$ ,  $x$  and  $y$  are chainable in  $X$ . The space  $X$  is *chain connected* if it is chain connected in  $X$ .

Let  $x \in C \subseteq X$ . Chain component of  $x$  of  $C$  in  $X$  is the largest chain connected subset of  $C$  in  $X$  that contains  $x$ . The definition of chain connectedness of a set in a topological space differs from the definition of connectedness of a set in a topological space, [3, 8]. For example, the chain components of the set in the space are union of quasicomponents of the set.

**Theorem 2.** [3, 8] *Let  $X$  be a topological space and  $C \subseteq Y \subseteq X$ . If  $C$  is chain connected in  $Y$ , then it is chain connected in  $X$ .*

The converse does not hold in general.

Next, we list some theorems about chain connectedness. Most of them are proved in [3] and [8]:

**Theorem 3.** [8] *Let  $C$  be a chain connected set in  $X$  and  $C \subseteq D \subseteq \overline{C}$ . Then  $D$  is chain connected in  $X$ .*

**Theorem 4.** *Let  $C_i, i \in I$  be chain connected sets in  $X$ . If for all  $i, i' \in I, C_i \cap \overline{C_{i'}} \neq \emptyset$  or  $C_i' \cap \overline{C_i} \neq \emptyset$ , then  $\bigcup_{i \in I} C_i$  is chain connected in  $X$ .*

**Corollary 4.1.** *Let  $C_i, i \in I$  be chain connected sets in  $X$ . If for all  $i, i' \in I, C_i \cap C_{i'} \neq \emptyset$ , then  $\bigcup_{i \in I} C_i$  is chain connected in  $X$ .*

**Corollary 4.2.** *Let  $C_i, i \in I$  be chain connected sets in  $X$  such that  $x \in \bigcap_{i \in I} C_i$ , for a point  $x \in X$ . Then  $\bigcup_{i \in I} C_i$  is chain connected in  $X$ .*

*Proof.* From Corollary 4.1 we conclude that  $\bigcup_{i \in I} C_i$  is chain connected in  $X$ .  $\square$

**Theorem 5.** [3], [8] *Let  $C$  be a chain connected set in  $X$  and  $f : X \rightarrow Y$  is a continuous function. Then  $f(C)$  is a chain connected set in  $Y$ .*

**Theorem 6.** A product  $\prod_{a \in A} X_a$  of chain connected spaces  $X_a$ ,  $a \in A$ , is chain connected.

*Proof.* Let  $X_a$ ,  $a \in A$ , be chain connected spaces and  $(x_a|a \in A)$  and  $(y_a|a \in A)$  be points of the product  $\prod_{a \in A} X_a$ , and  $\mathcal{W}$  be an arbitrary covering of the product.

There is a member  $W$  of the covering containing the point  $y$  and there is a base neighborhood  $B$  of  $y$ , such that  $B \subseteq W$ . Since  $B$  is a base neighborhood, by definition of the product topology,  $B = \prod_{a \in A} U_a$  where  $U_a$  is an open set in  $X_a$  for a finite number of indices  $a \in \{a_1, \dots, a_n\}$  and  $U_a = X_a$  for all other indices.

Consider the point  $x^1 = (x_a^1|a \in A)$  where  $x_a^1 = x_a$  for all  $a \neq a_1$  and  $x_{a_1}^1 = y_{a_1}$ . Then points  $x = (x_a|a \in A)$  and  $x^1$  belong to the subspace  $\{x^1\} \times X_{a_1}$ , where  $x' = (x_a|a \in A \setminus \{a_1\})$ , which is connected and so they are chainable in  $\mathcal{W}$ . If the point  $x^2 = (x_a^2|a \in A)$  has a coordinate  $x_{a_2}^2 = y_{a_2}$  and all other coordinates are the same with the point  $x^1$ , then  $x^1$  and  $x^2$  belong to a subspace homeomorphic to  $X_{a_2}$  and so they are chainable in  $\mathcal{W}$ . Proceeding in this way we obtain that  $x^{n-1}$  and  $x^n$  are chainable in  $\mathcal{W}$ . Thus  $x$  and  $x^n$  are chainable in  $\mathcal{W}$ . Since  $x^n$  has coordinates  $x_{a_1}^n = y_{a_1}$ ,  $x_{a_2}^n = y_{a_2}$ , ...,  $x_{a_n}^n = y_{a_n}$  it follows that  $x^n \in B \subseteq W$ .  $W$  is a neighbourhood of  $y$  and we obtained that two points  $x$  and  $y$  are chainable in an arbitrary covering  $\mathcal{W}$ .  $\square$

### 3. CHAIN COMPONENTS

Note that  $x$  and  $y$  may be chainable although they are not in the same connected components. The relation two points in  $X$  to be chainable, defines an equivalence relation in  $X$ , as defined in [3]. We denote by  $x \sim y$ , if  $x$  and  $y$  are chainable. This relation splits the space into equivalence classes  $Ch(x)$ ,  $x \in X$ .

**Definition 3.** Let  $x \in X$ . We call the set  $Ch(x)$  the *chain component* of  $x$  in  $X$ .

In other words, the chain component of  $x$  is the largest chain connected set in  $X$  that contains  $x$ . Hence, every chain connected subset is contained only in one of the chain components.

From the properties of equivalence relations, we can conclude that for  $x, y \in X$  either  $Ch(x) = Ch(y)$  or  $Ch(x) \cap Ch(y) = \emptyset$  and  $\bigcup_{x \in X} Ch(x) = X$ .

For a given  $x \in X$  the intersection of all clopen (closed and open) sets which contain  $x$  is a *quasicomponent* of  $x$ . We denote it by  $Q(x)$  There are not many books dealing with quasicomponents ([1], [2]).

The next theorem is proved in [4]:

**Theorem 7.** The chain components in  $X$  coincide with quasicomponents in  $X$ .

From this fact we conclude that all properties of quasicomponents can be described by chain connectedness. We list some of them.

**Theorem 8.** [3, 8] The chain components are closed sets.

**Theorem 9.** If  $X \subseteq Y$  and  $V_Y$  is a chain component in  $Y$ , then there exists a chain component  $V_X$  in  $X$  such that  $V_Y \subseteq V_X$ .

*Proof.* Since  $V_Y$  is chain connected in  $Y$ ,  $V_Y$  is chain connected in  $X$  also, hence there exists a chain component  $V_X$  in  $X$  such that  $V_Y \subseteq V_X$ .  $\square$

**Definition 4.** We say that  $x$  and  $y$  are functionally separated if there exists a map  $f : X \rightarrow \{0, 1\}$  such that  $f(x) = 0$  and  $f(y) = 1$ , where  $\{0, 1\}$  has a discrete topology.

**Theorem 10.** [3] *Let  $x, y \in X$ . Elements  $x$  and  $y$  cannot be functionally separated if and only if  $x$  and  $y$  are chain connected in  $X$ .*

**Corollary 10.1.** *If  $V_1(x) = \{y \in X \mid x \text{ and } y \text{ are chain connected in } X\}$  and  $V_2(x) = \{y \in X \mid x \text{ and } y \text{ cannot be functionally separated}\}$ , then  $V_1(x) = V_2(x)$ , for all  $x \in X$ .*

**Theorem 11.** *Let  $V_1$  and  $V_2$  be two different chain components in  $X$ . Then there exist two clopen sets  $E_1, E_2 \subseteq X$  such that  $V_1 \subseteq E_1$ ,  $V_2 \subseteq E_2$  and  $E_1 \cap E_2 = \emptyset$ .*

*Proof.* Let  $x \in V_1$  and  $y \in V_2$ . Then  $V_1 = Ch(x)$ ,  $V_2 = Ch(y)$  and  $x$  and  $y$  are not chain connected in  $X$ , i.e. there exists a cover  $\mathcal{U}$  of  $X$  for which there is no chain from  $x$  to  $y$ .

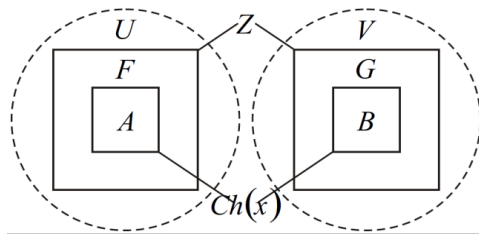
Let  $E_1 = \{z \in X \mid \text{there is a chain in } \mathcal{U} \text{ from } x \text{ to } z\}$  and  $E_2 = \{z \in X \mid \text{there is a chain in } \mathcal{U} \text{ from } y \text{ to } z\}$ . So,  $E_1$  and  $E_2$  are clopen sets (see theorem 2.2 in [3]) such that  $E_1 \cap E_2 = \emptyset$ ,  $V_1 \subseteq E_1$ ,  $V_2 \subseteq E_2$ .  $\square$

For  $x \in X$ , the maximal connected set in  $X$  containing  $x$  is called a component of  $x$ . We denote it by  $C(x)$ . In [3] and [8] it is proven that, since components are also chain connected sets, each component is contained in one of the chain components.

The next two theorems tell us when components and chain components coincide.

**Theorem 12.** *If  $X$  is a compact Hausdorff space, then  $C(x) = Ch(x)$ ,  $x \in X$*

*Proof.* Let  $x \in X$ . Since  $C(x) \subseteq Ch(x)$  it is enough to prove that  $Ch(x)$  is connected. Suppose  $Ch(x) = A \cup B$  where  $A$  and  $B$  are disjoint closed sets in  $Ch(x)$  and let  $x \in A$ . Since  $Ch(x)$  is closed in  $X$ ,  $A$  and  $B$  are also closed in  $X$ .  $X$  is a normal space, as a compact Hausdorff space, hence there exist open sets  $U$  and  $V$  in  $X$ , such that  $A \subseteq U$ ,  $B \subseteq V$  and  $U \cap V = \emptyset$ .



Consider the set  $A_X(x, \mathcal{U})$  constructed by all the points  $y \in X$  for which there is a chain in the cover  $\mathcal{U}$  of  $X$ , from  $x$  to  $y$ .  $A_X(x, \mathcal{U})$  is a clopen nonempty set (see Theorem 2.2 [3]) and  $Ch(x) = \bigcap_{\mathcal{U}} A_X(x, \mathcal{U})$ .

$X \setminus (U \cup V)$  is a compact as a closed subset of a compact space  $X$  and since

$$\bigcup_{\mathcal{U}} X \setminus A_X(x, \mathcal{U}) = X \setminus \bigcap_{\mathcal{U}} A_X(x, \mathcal{U}) = X \setminus Ch(x) \supseteq X \setminus (U \cup V)$$

the family of sets  $\{X \setminus A_X(x, \mathcal{U}) \mid \mathcal{U} \text{ is a covering of } X\}$  is a cover for  $X \setminus (U \cup V)$ , hence there exists a subfamily  $X \setminus A_X(x, \mathcal{U}_i)$ ,  $i = 1, 2, \dots, n$ , which covers  $X \setminus (U \cup V)$ . Then  $Ch(x) \subseteq \bigcap_{i=1}^n A_X(x, \mathcal{U}_i) \subseteq U \cup V$ .

Consider  $Z = \bigcap_{i=1}^n A_X(x, \mathcal{U}_i)$ .  $Z$  is clopen, as a finite intersection of clopen sets.

The sets  $F = Z \cap U$  and  $G = Z \cap V$  are open as intersections of two open sets, and closed since  $\bar{F} = \bar{Z} \cap \bar{U} \subseteq Z \cap \bar{U} = Z \cap (U \cup V) \cap \bar{U} = Z \cap U = F$  and  $\bar{G} = \bar{Z} \cap \bar{V} \subseteq Z \cap \bar{V} = Z \cap (U \cup V) \cap \bar{V} = Z \cap V = G$ . Then  $B = \emptyset$ , since  $Ch(x)$  is chain component. Otherwise, if there exists an  $y \in B$ , then there would not be a chain in  $\mathcal{U} = \{F, X \setminus F\}$  joining  $x$  and  $y$ . So,  $Ch(x)$  cannot be expressed as a union of two separated sets. Hence  $Ch(x)$  is connected. Since  $C(x)$  is the maximal connected set containing  $x$ , we conclude that  $Ch(x) \subseteq C(x)$ .  $\square$

**Theorem 13.** *If  $X$  is a locally connected space, then  $Ch(x) = C(x)$ , for all  $x \in X$ .*

To prove this theorem, we need the next lemma.

**Lemma 1.** *If the chain component is open in  $X$ , then it is connected.*

*Proof.* Let  $Ch(x)$  be open. Since  $Ch(x)$  is also closed, it is a clopen set. If  $Ch(x)$  was not connected, then there would exist a covering  $\mathcal{U}$  of  $Ch(x)$  and two points  $x$  and  $y$  of  $Ch(x)$  that are not chainable in  $\mathcal{U}$ . Since  $Ch(x)$  is open in  $X$  it follows that elements of  $Ch(x)$  are also open in  $X$ . Then  $x$  and  $y$  are not chainable in the covering  $\mathcal{U} \cup \{X \setminus Ch(x)\}$  of  $X$ , that contradicts the chain connectedness of  $Ch(x)$ .  $\square$

*Proof.* (of Theorem 13) From the previous lemma, it is enough to prove that  $Ch(x)$  is open in  $X$ . Let  $y \in Ch(x)$ . Then for any neighborhood  $U$  of  $y$  there is a connected neighborhood  $V$  of  $y$  such that  $V \subseteq U$ . This implies that for any  $z \in V$ ,  $y$  and  $z$  are chainable in  $X$ . Since  $x$  and  $y$  are also chainable in  $X$ , it follows that  $x$  and  $z$  are chainable in  $X$ . This implies  $V \subseteq Ch(x)$ .  $\square$

#### 4. CHAIN CONNECTEDNESS IS A HOMOTOPY AND SHAPE INVARIANT

**Theorem 14.** *If  $f : X \rightarrow Y$  is a continuous function, then  $f(Ch(x)) \subseteq Ch(f(x))$ , for all  $x \in X$ .*

*Proof.* Since  $Ch(x)$  is chain connected in  $X$  and  $f$  is continuous, from Theorem 13 in [8],  $f(Ch(x))$  is chain connected in  $Y$ . Then there is a chain component  $V$  in  $Y$  such that  $f(Ch(x)) \subseteq V$ . From  $x \in Ch(x)$  it follows  $f(x) \in f(Ch(x)) \subseteq V$ , hence  $Ch(f(x)) = V$  so  $f(Ch(x)) \subseteq Ch(f(x))$ .  $\square$

So, if we denote by  $QX$  set of quasicomponents of  $X$  then a continuous function  $f : X \rightarrow Y$  induces a unique function  $f_{\#} : QX \rightarrow QY$ .

**Theorem 15.** *If  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$  are homotopic, then  $Ch(f(x)) = Ch(g(x))$ , for all  $x \in X$ .*

*Proof.* Let  $x \in X$ . It is enough to prove that  $f(x)$  and  $g(x)$  are chainable in  $Y$ . Since  $f$  and  $g$  are homotopic there is a mapping  $H : X \times I \rightarrow Y$  such that  $H(x, 0) = f(x)$  and  $H(x, 1) = g(x)$ . The set  $\{x\} \times I$  is chain connected (it is connected as a product of connected sets, hence chain connected). From the continuity of  $H$ , the set  $H(\{x\} \times I)$  must be chain connected in  $Y$ , hence  $H(x, 0) = f(x)$  and  $H(x, 1) = g(x)$  are chainable in  $Y$ .  $\square$

**Corollary 15.1.** *If  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$  are homotopic and  $A$  is a chain connected set in  $X$ , then  $f(A)$  and  $g(A)$  are contained in the same chain component of  $Y$ .*

*Proof.* Let  $x \in A$ . Then  $A \subseteq Ch(x)$ , therefore  $f(A) \subseteq f(Ch(x)) \subseteq Ch(f(x))$ . Similarly  $g(A) \subseteq Ch(g(x))$ . From the previous theorem we have that  $Ch(f(x)) = Ch(g(x))$  thus  $f(A)$  and  $g(A)$  must be contained in the same chain component of  $Y$ .  $\square$

**Corollary 15.2.** *If  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$  are homotopic then  $f_{\#} : QX \rightarrow QY$  coincide with  $g_{\#} : QX \rightarrow QY$ . So, if we denote by  $[f]$  the homotopy class, then  $[f]$  induces a unique function  $[f]_{\#} : QX \rightarrow QY$ .*

Homotopy theory is not an appropriate tool for study of spaces with bad local behavior, for example not locally connected. Usually for such spaces shape theory is more applicable. We present a short description of shape by intrinsic approach i.e. intrinsic shape.

**Definition 5.** A function  $f : X \rightarrow Y$  is  $\mathcal{V}$ -continuous at point  $x \in X$ , if there exists a neighborhood  $U_x$  of  $x$ , and  $V \in \mathcal{V}$ , such that  $f(U_x) \subseteq V$ .

A function  $f : X \rightarrow Y$  is  $\mathcal{V}$ -continuous, if it is  $\mathcal{V}$ -continuous at every point  $x \in X$ . In this case, the family of all  $U_x$ , form a covering of  $X$ . By this,  $f : X \rightarrow Y$  is  $\mathcal{V}$ -continuous if there exists a covering  $\mathcal{U}$  of  $X$ , such that for any  $x \in X$ , there exists a neighborhood  $U$  of  $x$ , and  $V \in \mathcal{V}$  such that  $f(U) \subseteq V$ . We denote shortly  $f(\mathcal{U}) \prec \mathcal{V}$ .

If  $f : X \rightarrow Y$  is  $\mathcal{V}$ -continuous, then  $f : X \rightarrow Y$  is  $\mathcal{W}$ -continuous, for any  $\mathcal{W}$  such that  $\mathcal{V} \prec \mathcal{W}$ .

If  $\mathcal{V}$  is a covering of  $Y$  and  $V \in \mathcal{V}$ , the star of  $\mathcal{V}$ , is the open set  $st(V) = \{\cup W \mid W \in \mathcal{V}, W \cap V \neq \emptyset\}$ . We form a new covering of  $Y$ ,  $st(\mathcal{V}) = \{st(V) \mid v \in \mathcal{V}\}$ . We say that  $W$  is from  $st(V)$  if  $W \in \mathcal{V}$  and  $W \subseteq st(V)$ .

**Definition 6.** The functions  $f, g : X \rightarrow Y$  are  $\mathcal{V}$ -homotopic, if there exists a function  $F : X \times I \rightarrow Y$  such that:

- 1)  $F : X \times I \rightarrow Y$  is  $st(\mathcal{V})$ -continuous;
- 2)  $F : X \times I \rightarrow Y$  is  $\mathcal{V}$ -continuous at all points of  $X \times \partial I$ ;
- 3)  $F(x, 0) = f(x)$ ,  $F(x, 1) = g(x)$ .

The relation of  $\mathcal{V}$ -homotopy is denoted by  $f \underset{\mathcal{V}}{\sim} g$ . This is an equivalence relation.

A proximate net  $f_{\mathcal{V}} : X \rightarrow Y$  is a net of  $\mathcal{V}$ -continuous functions indexed by all coverings of  $Y$ , such that, if  $\mathcal{W}$  is finer than  $\mathcal{V}$ , then  $f_{\mathcal{V}}$  and  $f_{\mathcal{W}}$  are  $\mathcal{V}$ -homotopic. Two proximate nets  $(f_{\mathcal{V}})$  and  $(f_{\mathcal{V}'})$  are homotopic if  $f_{\mathcal{V}}$  and  $f_{\mathcal{V}'}$  are  $\mathcal{V}$ -homotopic for all coverings of  $Y$ . The category with objects paracompact spaces and morphisms homotopy classes of proximate nets is well defined and isomorphic objects in this category have the same shape ([5], [7]).

A continuous map  $f : X \rightarrow Y$  can be considered as a proximate net  $(f_{\mathcal{V}}) : X \rightarrow Y$  putting  $f_{\mathcal{V}} = f$ . If  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$  are homotopic then the associated proximate nets  $(f_{\mathcal{V}}) : X \rightarrow Y$  and  $(g_{\mathcal{V}}) : X \rightarrow Y$  are also homotopic and they define the same shape morphism.

**Lemma 2.** *The points  $x, y \in Y$  are  $st(\mathcal{V})$ -chainable if and only if they are  $\mathcal{V}$ -chainable.*

*Proof.* Let  $x, y \in Y$  are  $st(\mathcal{V})$ -chainable. Then there exists a chain  $st(V_1), st(V_2), \dots, st(V_n)$  in  $st(\mathcal{V})$  that connects  $x$  and  $y$ . Since for  $V \in \mathcal{V}$  we have that  $st(V) = \{\cup W \mid W \in \mathcal{V}, W \cap V \neq \emptyset\}$ , then for any two consecutive members  $V_i, V_{i+1}$ ,  $i = 1, 2, \dots, n-1$ , there exist  $U_{i+1}$  from  $st(V_{i+1})$  and  $W_i$  from  $st(V_i)$  such that  $W_i \cap U_{i+1} \neq \emptyset$  and there exist  $U_1$  from  $st(V_1)$  such that  $x \in U_1$  and  $W_n$  from  $st(V_n)$  such that  $y \in W_n$ . Then  $U_1, V_1, W_1, U_2, V_2, W_2, \dots, U_n, V_n, W_n$ , represents a chain in  $\mathcal{V}$  that connects  $x$  and  $y$ .

Conversely, if  $x, y \in Y$  are  $\mathcal{V}$ -chainable, since  $\mathcal{V} \prec st(\mathcal{V})$ , then they are  $st(\mathcal{V})$ -chainable.  $\square$

**Lemma 3.** *If  $F : X \times I \rightarrow Y$  is  $\mathcal{V}$ -continuous function, then for every  $x \in X$  there exists a minimal covering  $I_1, I_2, \dots, I_n$  of open intervals in  $I = [0, 1]$  (which depends on  $x$ ) such that*

$$0 \in I_1, I_k \cap I_{k+1} \neq \emptyset \text{ for every } k = 1, 2, \dots, n-1, \text{ and } 1 \in I_n;$$

and  $V_k \in \mathcal{V}, k = 1, 2, \dots, n$  such that  $F(\{x\} \times I_k) \subseteq V_k$  for every  $k = 1, 2, \dots, n$ .

*Proof.* Let  $x \in X$ . Since  $F$  is  $\mathcal{V}$ -continuous it follows that for every point  $(x, y) \in X \times I, y \in I$  there exists a canonical neighborhood  $U_y \times I_y$  where  $U_y$  is a neighborhood of  $x$  and  $I_y = (a_y, b_y) \cap I$  is an interval and neighborhood of  $y$ , and  $V_y \in \mathcal{V}$  such that  $F(U_y \times I_y) \subseteq V_y$ . The collection  $\{I_y \mid y \in I\}$  forms a covering of  $I$  and since  $I$  is compact it follows that there exists a finite subcovering  $I_{y_1}, I_{y_2}, \dots, I_{y_n}$  which can be chosen to be minimal. Indices can be renamed  $I_{j_k} = I_{y_k} (V_{j_k} = V_{y_k}), k = 1, 2, \dots, n; j_k \in \{1, 2, \dots, n\}$  such that:

$$I_1 = [0, b_1], I_2 = (a_2, b_2), \dots, I_{n-1} = (a_{n-1}, b_{n-1}) \text{ and } I_n = (a_{n-1}, 1] \text{ and}$$

$$a_{i+1} < b_i, i = 1, 2, \dots, n-1.$$

Clearly  $F(\{x\} \times I_k) \subseteq V_k$ , where  $V_k \in \mathcal{V}$  for every  $k = 1, 2, \dots, n$ .  $\square$

**Theorem 16.** *If  $f, g : X \rightarrow Y$  are  $\mathcal{V}$ -homotopic mappings, then  $f(x) \underset{\mathcal{V}}{\sim} g(x)$ , for all  $x \in X$*

*Proof.* Let  $x \in X$ . Since  $f$  and  $g$  are  $\mathcal{V}$ -homotopic, there exists  $st(\mathcal{V})$ -continuous function  $F : X \times I \rightarrow Y$  such that  $F(x, 0) = f(x), F(x, 1) = g(x)$ .

Since  $[0,1]$  is connected and compact, for every covering  $\mathcal{V}$  of  $Y$ , there exists a covering  $\{J_k | k = 1, 2, \dots, n\}$  of  $[0,1]$  such that  $0 \in J_1, 1 \in J_n, J_k \cap J_{k+1} \neq \emptyset$  and  $F(\{x\} \times J_k) \subseteq V_k$  for some  $V_k \in st(\mathcal{V}), k = 1, 2, \dots, n$ .

From  $f(x) \in F(\{x\} \times J_1), g(x) \in F(\{x\} \times J_n)$  and  $F(\{x\} \times J_k) \cap F(\{x\} \times J_{k+1}) \neq \emptyset, k = 1, 2, \dots, n-1$ , since  $J_k \cap J_{k+1} \neq \emptyset, k = 1, 2, \dots, n-1$  implies  $F(\{x\} \times J_k, k = 1, 2, \dots, n-1$  are contained in elements of the  $st(\mathcal{V})$ -chain that connects  $f(x)$  and  $g(x)$ . From Lemma 2,  $f(x)$  and  $g(x)$  are  $\mathcal{V}$ -chainable.  $\square$

**Corollary 16.1.** *If  $(f_{\mathcal{V}}), (g_{\mathcal{V}}) : X \rightarrow Y$  are two homotopic proximate nets, then for any covering  $\mathcal{V}$  of  $Y$  and for any point  $x \in X$  holds  $f_{\mathcal{V}}(x) \tilde{\sim}_{\mathcal{V}} g_{\mathcal{V}}(x)$ .*

**Theorem 17.** *If  $f_{\mathcal{V}} : X \rightarrow Y$  is a  $\mathcal{V}$ -continuous function and  $x$  and  $y$  belong to the same chain component of  $X$ , then  $f_{\mathcal{V}}(x) \tilde{\sim}_{\mathcal{V}} g_{\mathcal{V}}(x)$ .*

*Proof.* Let  $\mathcal{U}$  be a covering of  $X$  such that  $f_{\mathcal{V}}(\mathcal{U}) \prec \mathcal{V}$  i.e. for every  $x \in X$  there exist  $x \in U \in \mathcal{U}$  and  $V \in \mathcal{V}$  such that  $f_{\mathcal{V}}(U) \subseteq V$ . Since  $x$  and  $y$  belong to the same chain component, there is a chain  $U_1, U_2, \dots, U_n$  in  $\mathcal{U}$  that connects  $x$  and  $y$ .

Then  $f_{\mathcal{V}}(x) \in f_{\mathcal{V}}(U_1), f_{\mathcal{V}}(y) \in f_{\mathcal{V}}(U_n)$  and there exist  $V_1, V_2, \dots, V_n$  from  $\mathcal{V}$  such that  $f_{\mathcal{V}}(U_k) \subseteq V_k, k = 1, 2, \dots, n$ , i.e.  $V_1, V_2, \dots, V_n$  is a chain in  $\mathcal{V}$  that connects  $f_{\mathcal{V}}(x)$  and  $f_{\mathcal{V}}(y)$ .  $\square$

A direct consequence of this theorem are the following results:

**Corollary 17.1.** *If  $(f_{\mathcal{V}}) : X \rightarrow Y$  is a proximate net and  $x$  and  $y$  belong to the same chain component of  $X$ , then for any covering  $\mathcal{V}$  of  $Y, f_{\mathcal{V}}(x) \tilde{\sim}_{\mathcal{V}} f_{\mathcal{V}}(y)$  holds.*

**Corollary 17.2.** *If  $(f_{\mathcal{V}}), (g_{\mathcal{V}}) : X \rightarrow Y$  are homotopic proximate nets, and  $x$  and  $y$  belong to the same chain component of  $X$ , then for any covering  $\mathcal{V}$  of  $Y, f_{\mathcal{V}}(x) \tilde{\sim}_{\mathcal{V}} g_{\mathcal{V}}(y)$  holds.*

**Theorem 18.** *Let  $(f_{\mathcal{V}}) : X \rightarrow Y$  be a proximate net and  $\mathcal{W}$  be a covering of  $Y$  consisting of disjoint open sets. Then, for each chain component  $Q$  of  $X$ , there exists  $W_Q \in \mathcal{W}$  such that for every  $\mathcal{V} \prec \mathcal{W}, f_{\mathcal{V}}(Q) \subseteq W_Q$  (and specially  $f_{\mathcal{V}}(Q) \subseteq W_Q$ ).*

*Proof.* By theorem 2.3 from [6] it follows that there exists a  $W_Q \in \mathcal{W}$  such that  $f_{\mathcal{W}}(Q) \subseteq W_Q$ . If  $\mathcal{V} \prec \mathcal{W}$ , then there exists a  $st(\mathcal{W})$ -continuous function,  $F_{\mathcal{W}\mathcal{V}} : X \times I \rightarrow Y$  connecting  $f_{\mathcal{W}}$  and  $f_{\mathcal{V}}$ . Since  $st(\mathcal{W}) = \mathcal{W}, F_{\mathcal{W}\mathcal{V}} : X \times I \rightarrow Y$  is also a  $\mathcal{W}$ -continuous function. Since  $F_{\mathcal{W}\mathcal{V}}(Q \times \{0\}) = f_{\mathcal{W}}(Q) \subseteq W_Q$  it follows that  $F_{\mathcal{W}\mathcal{V}}(Q \times I) \subseteq W_Q$ , and that  $f_{\mathcal{V}}(Q) = F_{\mathcal{W}\mathcal{V}}(Q \times \{1\}) \subseteq W_Q$ .  $\square$

Now, let  $X$  having be locally compact metric spaces,  $X$  and  $Y$  have a compact space of quasicomponents  $QX$  and  $QY$ . In [6], using Theorem 18, the following result is proved : Any proximate net  $(f_{\mathcal{V}}) : X \rightarrow Y$  induces a continuous function  $(f_{\mathcal{V}})_{\#} : QX \rightarrow QY$ .

Under condition  $QX$  and  $QY$  to be compact, replacing quasicomponents with chain components we obtain:

**Theorem 19.** *Any proximate net  $(f_{\mathcal{V}}) : X \rightarrow Y$  induces a continuous function  $(f_{\mathcal{V}})_{\#} : QX \rightarrow QY$  between spaces of chain components. If  $(f_{\mathcal{V}}), (g_{\mathcal{V}}) : X \rightarrow Y$  are homotopic proximate nets,  $(f_{\mathcal{V}}) \sim (g_{\mathcal{V}})$ , i.e. defining the same shape morphism, then:  $(f_{\mathcal{V}})_{\#} = (g_{\mathcal{V}})_{\#}$ .*

The second part of the statement is proved in the same way as in [6].

Thus, if  $(f_{\mathcal{V}})$  and  $(g_{\mathcal{V}}) : X \rightarrow Y$  are homotopic proximate nets, i.e.  $f_{\mathcal{V}}$  and  $g_{\mathcal{V}}$  are  $\mathcal{V}$ -homotopic for all coverings  $\mathcal{V}$  of  $Y$ , and consequently defining the same intrinsic shape morphism  $F$ . By this morphism to a chain component  $C$  of  $X$ , we can associated an unique chain component  $D$  of  $Y$  i.e by putting  $F_{\#}(C) = D$  is defined a map  $F_{\#} : QX \rightarrow QY$ .

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