Математички Билтен
47 (LXXIII) 2 (2024), 105–112

UDC: 515.172.7
DOI: 10.37560/matbil23472105m

CHAIN SEPARATEDNESS IN A FAMILY COVERINGS SPACE

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Abstract. The paper gives a generalization of different kind of separatedness and chain separatedness of a topological space to a family coverings space that is more general than a topological space and it consists of a set and a family of coverings of the set.

1. INTRODUCTION

The notion of separatedness using the standard definition cannot be generalized from a topological space to a more general space, without generalizing the topology of the space, since it is related to it. But separatedness defined by chain, and as well as its generalization the chain separatedness, can, such that instead of families of coverings of open sets, families of coverings of arbitrary sets will be considered. The generalizations can also be defined on even more general structures, such as a set of subsets of a given set, i.e. to define connectedness in this set. The paper gives a generalization of separatedness and chain separatedness of a space that is more general than a topological space and it consists of a set and a family of coverings of the set. In this paper we coninue to investigate the notion of chain connectedness [1–7].

The next statements of this section are from [6].

Definition 1: *F*amily coverings space $X = (X,\underline{\mathcal{U}})$ is a set X together with a family of coverings $\underline{\mathcal{U}} = \{\mathcal{U}_\alpha \,|\alpha \in I\}$ which are subsets of the set of the coverings of X in X.

In this paper by a covering $\mathcal{U}$ of X, if it is not otherwise stated, we understand a covering $\mathcal{U} \in \underline{\mathcal{U}}$ of X in X.

Let $X = (X,\underline{\mathcal{U}})$ and $Y \subseteq X$.

Date: 15.11.2023.

2010 *Mathematics Subject Classification.* Primary: 54H99.

Key words and phrases. Topology, Coverings, Space.

Definition 2: Subspace Y of the family coverings space $X = (X,\underline{\mathcal{U}})$, where $\underline{\mathcal{U}} = \{\mathcal{U}_\alpha \,|\alpha \in I\}$, is the set Y with the family of coverings

$$\underline{\mathcal{U}_Y} = \underline{\mathcal{U}} \cap Y = \{\mathcal{U}_\alpha \cap Y \,|\alpha \in I\}\,.$$

In the paper by subset $Y \subseteq X$ we understand the subspace Y of the family coverings space X.

Definition 3: Let $\mathcal{U}$ be a covering of the set X and $x, y \in X$. *A* chain in $\mathcal{U}$ that connects x and y (from x to y, from y to x) is a finite sequence of sets U_1, U_2, ..., U_n of $\mathcal{U}$ such that $x \in U_1$, $y \in U_n$ and $U_i \cap U_{i+1} \neq \emptyset$ for every $i = 1, 2, ..., n-1$.

Let X be a family coverings space and $C \subseteq X$.

Definition 4: The set C is *c*hain connected in X, if for every covering $\mathcal{U}$ of X in X and every $x, y \in C$, there exists a chain in $\mathcal{U}$ that connects x and y.

Let X be a family coverings space and $C \subseteq Y \subseteq X$.

Theorem 1. *If C is chain connected in Y, then C is chain connected in X.*

Remark 1.1: If the set C is chain connected in X, then each subset of C is chain connected in X.

Definition 5: The *c*hain component of the point x of C in X, denoted by $V_{CX}(x)$, is the maximal chain connected subset of C in X that contains x.

We denote by $V_X(C)$ or $V(C)$ in X, where C is chain connected in X, the set that consists of all elements $y \in X$, such that for every covering $\mathcal{U} \in \underline{\mathcal{U}}$ there exists a chain in $\mathcal{U}$ that connects some $x \in C$ and y.

Definition 6: Element x is chain related to y in X, and we denote it by $x \underset{X}{\sim} y$ or $x \sim y$ in X if for every covering $\mathcal{U}$ of X there exists a chain in $\mathcal{U}$ that connects x and y.

The chain relation is an equivalence relation.

2. CHAIN SEPARATED SETS IN A FAMILY COVERINGS SPACE

Let X be a family coverings space, and let $A, B \subseteq X$.

Definition 7: The nonempty sets A and B are *c*hain separated in X, if there exists a covering $\mathcal{U}$ of X such that for every point $x \in A$ and every $y \in B$, there is no chain in $\mathcal{U}$ that connects x and y.

From the definition, it follows that if A and B are chain separated in a family coverings space X, then any pair of nonempty sets C and D, where $C \subseteq A$ and $D \subseteq B$, are chain separated in X.

The following proposition will show us that a pair of sets, which are chain separated in a family coverings space, are also chain separated in every subspace that contain them.

Let X be a family coverings space, let $Y \subseteq X$, and let A and B be nonempty subsets of Y.

Proposition 2.1. *If A and B are chain separated in X, then A and B are chain separated in Y.*

Remark 2.1: The most important case of the previous proposition is when $Y = A \cup B$.

The next example shows that converse claim does not hold in general.

Example 1: For the space $X = \{1, 2, 3\}$ with the family with one covering:

$$\mathcal{U} = \{\{1, 2\}, \{2, 3\}\},$$

and the subspace $Y = \{1, 3\}$; the sets $A = \{1\}$ and $B = \{3\}$ are chain separated in $Y = A \cup B$, but they are not chain separated in X.

Now let's consider statement that give criterion for chain connected set by using the notion of chain separatedness.

Let X be a family coverings space and $C \subseteq X$.

Theorem 2. *The set C is chain connected in X, if and only if C cannot be represented as a union of two chain separated sets A and B in X.*

Proof. ($\Rightarrow$) If C can be represented as a union of chain separated sets A and B in X, then C is not chain connected in X.

($\Leftarrow$) Let C not be chain connected in X. It follows that C has at least two elements, and that there exists a covering $\mathcal{U}$ of X, for which there is no chain in $\mathcal{U}$ that connects some elements x and y of C. We consider the set $V = V_{CX}(x, \mathcal{U})$. Since x is not chain related to y in X, it follows that $y \in C \backslash V$. So, C is represented as a union of two chain separated sets V and $C \backslash V$. ☐

Remark 2.2: The most important case of the previous theorem is when $C = X$.

Theorem 3. *Let $X = A \cup B$, where A and B are chain separated sets in X, and C is a chain connected set in X. Then $C \subseteq A$ or $C \subseteq B$.*

Proof. If there exists $x, y \in C$ such that $x \in A$ and $y \in B$, since C is a chain connected set, it follows that for every covering $\mathcal{U}$ of $X = A \cup B$ there exists a chain in $\mathcal{U}$ that connects x and y. The last claim contradicts the claim that sets A and B are chain separated in X. So, $C \subseteq A$ or $C \subseteq B$. ☐

3. WEAKLY CHAIN SEPARATED SETS IN A FAMILY COVERINGS SPACE

Let $X = (X, \underline{\mathcal{U}})$ be a family coverings space, and let $A, B \subseteq X$.

Definition 8: The nonempty sets A and B are *w*eakly chain separated in X, if for every point $x \in A$ and every $y \in B$, there exists a covering $\mathcal{U} = \mathcal{U}(x, y)$ of X such that there is no chain in $\mathcal{U}$ that connects x and y.

The notion is similar to the notion of pair of chain separated sets in a family coverings space. Therefore, the analogue theorems to chain separated sets, are valid for weakly chain separated sets.

From the definition, it follows that:

Proposition 3.1. *If A and B are weakly chain separated in X, then any pair of nonempty sets C and D, where $C \subseteq A$ and $D \subseteq B$, are weakly chain separated in X.*

The following theorem will show us that two sets, which are weakly chain separated in a family coverings space, are also weakly chain separated in every subspace that contain them.

Let X be a family coverings space, let $Y \subseteq X$, and let A and B be nonempty subsets of Y.

Theorem 4. *If A and B are weakly chain separated in X, then A and B are weakly chain separated in Y.*

Proof. Let the sets A and B be weakly chain separated in X and let $x \in A$ and $y \in B$. It follows that there exists a covering $\mathcal{U}$ of X such that there is no chain in $\mathcal{U}$ that connects x and y. Then:

$$\mathcal{U}_Y = \mathcal{U} \cap Y = \{U \cap Y \,|U \in \mathcal{U}\}$$

is covering of Y in Y such that there is no chain in $\mathcal{U}_Y$ that connects x and y. □

Remark 3.1: The most important case of the previous theorem is when $Y = A \cup B$.

The definition of pair of chain separated sets, the definition of pair weakly chain separated sets, and the properties of quantifiers, leads to the following statement.

Let X be a family coverings space and let A and B be nonempty subsets of X.

Theorem 5. *If the sets A and B are chain separated in X, then A and B are weakly chain separated in X.*

Remark 3.2: The most important case of the previous theorem is when $X = A \cup B$.

The next example shows that the converse statement does not hold in general.

Example 2: Let $A = \{0\}$, $B = \left\{ \frac{1}{n} \middle| \, n \in \mathbb{N} \right\}$, and $X = A \cup B$ with the family of coverings $\underline{\mathcal{U}} = \{\mathcal{U}_n | \, n \in \mathbb{N}\}$, where:

$$\mathcal{U}_n = \left\{ \{0\} \cup \left\{ \frac{1}{n+k} \middle| k \in \mathbb{N} \right\}, \left\{ \frac{1}{n} \right\}, \left\{ \frac{1}{n-1} \right\}, ..., \left\{ \frac{1}{2} \right\}, \{1\} \right\}.$$

The sets A and B are weakly chain separated in X, but A and B are not chain separated in $X = A \cup B$.

Indeed, let $b \in B$. Then $b = \frac{1}{n_0}$ for some $n_0 \in \mathbb{N}$, and for the covering:

$$\mathcal{U}_{n_0} = \left\{ \{0\} \cup \left\{ \frac{1}{n_0+k} \middle| k \in \mathbb{N} \right\}, \left\{ \frac{1}{n_0} \right\}, \left\{ \frac{1}{n_0-1} \right\}, ..., \left\{ \frac{1}{2} \right\}, \{1\} \right\},$$

there is no chain in $\mathcal{U}_{n_0}$ that connects 0 and b. It follows that A and B are weakly chain separated in X.

On the other hand, every element of arbitrary covering of X that contains the point 0, also contains a point from the set B. It follows that A and B are not chain separated in X.

Theorem 6. *Singletons A and B are weakly chain separated in X if and only if they are chain separated in X.*

Proposition 3.2. *Two sets A and B are weakly chain separated in X if and only if for every $x \in A$ and $y \in B$, $x \underset{X}{\not\sim} y$.*

The last proposition in case of chain separateness is valid in one direction. From the last proposition it follows the next statement:

Corollary 6.1. *Let $x, y \in X$. Then $x \underset{X}{\sim} y$, if and only if X cannot be represented as a union of two weakly chain separated sets A and B in X that contain x and y, respectively.*

4. STRONGLY CHAIN CONNECTED SETS IN A FAMILY COVERINGS SPACE

Let X be a family coverings space, and $C \subseteq X$.

Definition 9: The set C is strongly chain connected in X if C cannot be represented as a union of two weakly chain separated sets A and B in X.

From the definition it follows that a family coverings space X is strongly chain connected in X if it cannot be represented as a union of two weakly chain separated sets A and B in X.

Since the property of weak chain separatedness is weaker than the property of chain separatedness, we expect the property of strong chain connectedness to be stronger then the property of chain connectedness. Actually, the next theorem shows that they are equivalent.

Theorem 7. *The set C is strongly chain connected in X if and only if C is chain connected in X.*

Proof. ($\Rightarrow$) If the set C is not chain connected in X, then there exists a pair of chain separated sets A and B in X, such that $C = A \cup B$. But then A and B are weakly chain separated in X i.e. C is not strongly chain connected in X.

($\Leftarrow$) If C is not strongly chain connected in X i.e. C can be represented as a union of two weakly chain separated sets A and B in X it follows that there exist $x, y \in C$ and a covering $\mathcal{U}$ of X such that there is no chain in $\mathcal{U}$ that connects x and y. Therefore C is not chain connected in X. $\square$

Remark 4.1: The most important case of the theorem is when $C = X$.

From the last definition it follows that if the set C is strongly chain connected in X then for every covering $\mathcal{U}$ of X and every $x, y \in C$, there is a chain in $\mathcal{U}$ that connects x and y.

The last theorem allows us to prove the properties of the chain connected set by using the notion of pair of weakly chain separated sets.

5. ISOLATED POINTS IN A FAMILY COVERINGS SPACE

In a T_1 topological space, the point $x \in X$ is an isolated point if and only if there exists a covering $\mathcal{U}$ of X such that there is no chain in $\mathcal{U}$ that connects x and y for every $y \in X \backslash \{x\}$. We generalise the notion to a family coverings space.

Definition 10: The point $x \in X$ is an isolated point of a family coverings space $X = (X,\underline{\mathcal{U}})$ if there exists a covering $\mathcal{U}$ of X such that there is no chain in $\mathcal{U}$ that connects x and y for every $y \in X\backslash\{x\}$.

Corollary 7.1. *If $x \in X$ is an isolated point of X then the chain component of x in X is a singleton i.e. $V_X(x) = \{x\}$.*

Corollary 7.2. *If $x \in X$ is an isolated point of X then there exists a covering $\mathcal{U}_x$ of X such that $\{x\} \in \mathcal{U}_x$.*

Theorem 8. *The point $x \in X$ is an isolated point of the family coverings space X if and only if the sets $\{x\}$ and $X\backslash\{x\}$ are chain separated in X.*

Proof. Let x be an isolated point of the space X. It follows that there exists a covering $\mathcal{U}$ of X such that there is no chain in $\mathcal{U}$ that connects x and y for every $y \in X\backslash\{x\}$. So $\{x\}$ and $X\backslash\{x\}$ are chain separated in X.

Conversely, if $\{x\}$ and $X\backslash\{x\}$ are chain separated in X then there exists a covering $\mathcal{U}$ of X such that there is no chain in $\mathcal{U}$ that connects x and y for every $y \in X\backslash\{x\}$ i.e. x is an isolated point of the space X. □

Corollary 8.1. *If the point $x \in X$ is an isolated point of the family coverings space X then $\{x\}$ and $X\backslash\{x\}$ are weakly chain separated sets in X.*

The converse statement of the corollary does not hold in general.

Example 3: Let's consider the family coverings space X from Example 2. The sets $A = \{0\}$ and $B = X\backslash\{0\}$ are weakly chain separated in X, but they are not chain separated in X, and 0 is not an isolated point in X.

6. THE DISCRETE FAMILY COVERINGS SPACE

The topological space is the discrete if every point is an isolated point of the space.

Definition 11: The family coverings space X is the discrete if every $x \in X$ is an isolated point of X.

Corollary 8.2. *The family coverings space X is the discrete if for every $x \in X$ there exists a covering $\mathcal{U}_x$ of X such that there is no chain in $\mathcal{U}_x$ that connects x and y for every $y \in X\backslash\{x\}$.*

Theorem 9. *The family coverings space X is the discrete if and only if any for every $x \in X$ the subsets $\{x\}$ and $X\backslash\{x\}$ are chain separated in X.*

Proof. ($\Rightarrow$) Let X be discrete space and let $x \in X$. Then there exists a covering $\mathcal{U}_x$ of X such that $\{x\} \in \mathcal{U}_x$, and for any other member $B \in \mathcal{U}_x$, $B \neq \{x\}$, holds $B \subseteq X\backslash\{x\}$. Then for the covering $\mathcal{U}_x$ it follows that there is no chain in $\mathcal{U}_x$ that connects x and y, for every $y \in \mathcal{U}_x$ i.e. $\{x\}$ and $X\backslash\{x\}$ are chain separated in X.

($\Leftarrow$) If $\{x\}$ and $X\backslash\{x\}$ are chain separated in X then there exists a covering $\mathcal{U}_x$ of X, such that there is no chain in $\mathcal{U}_x$ that connects x and y, for every $y \in \mathcal{U}_x$ i.e. x is an isolated point. □

Theorem 10. *If any two disjoint nonempty subsets of X are chain separated in X, then the family coverings space X is the discrete.*

Proof. If any two disjoint nonempty subsets of X are chain separated in X, then firstly for every $x \in X$ the sets $\{x\}$ and $X \backslash \{x\}$ are chain separated in X and secondly from the last theorem it follows that X is the discrete. $\square$

The next example shows that converse claim does not hold in general.

Example 4: Let: $X = \{1,2,3,4\}$ and $\underline{\mathcal{U}} = \{\mathcal{U}_i | \, i \in \{1,2,3,4\}\}$, where:
$\mathcal{U}_1 = \{\{1\},\{2,3,4\}\}$, $\mathcal{U}_2 = \{\{1,3,4\},\{2\}\}$, $\mathcal{U}_3 = \{\{1,2,4\},\{3\}\}$ and $\mathcal{U}_4 = \{\{1,2,3\},\{4\}\}$. Then $A = \{1,2\}$ and $B = \{3,4\}$ are disjoint, but they are not chain separated in X.

7. CONCLUSIONS

The paper gives a generalization of notions of separatedness and chain separatedness of a family coverings space that is more general than a topological space and it consists of a set and a family of coverings of the set. In these spaces, the notion of pair of chain separated sets is defined and their properties are presented. Other notions of topological spaces to spaces are also generalized, such as types of spaces (discrete, totally separated, totally chain separated, totally weakly chain separated etc.) and other notions such as an isolated point, which for topological spaces are given in [1–6]. Most of the claims and proofs in this paper are analogous to [1–6]. Theorem 10 on a level of space holds in one direction, while the analogous Theorem 2.5 in [3] for topological space, holds in both directions. A number of statements from [1–6] cannot be generalized to the space level. Three examples for spaces that are not topological are given, to be shown that three statements do not hold in the converse direction.

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Received 29.11.2023
Revised 19.2.2024
Accepted 19.2.2024