Математички Билтен
47 (LXXIII) 2 (2023), 97–104

UDC: 515.124:515.126.4
DOI: 10.37560/matbil23472097d

COMMON FIXED POINT THEOREMS FOR PAIRS OF SELF-MAPPINGS ON $(3,2)$-W-SYMMETRIZABLE SPACES

TOMI DIMOVSKI [1] AND DONČO DIMOVSKI [2]

Abstract. In this paper, we prove the existence and the uniqueness of a fixed point for pairs of self-mappings on $(3,2)$-W-symmetrizable spaces using self-mappings similar to T_F contractions in ordinary metric spaces, and we obtain two corollaries.

1. Introduction

Fixed points theory plays a basic role in applications of many branches of mathematics. The Banach fixed point theorem [2] is a very simple and powerful theorem with a wide range of applications. This theorem has been generalized and extended by many authors in various ways and directions.

The geometric properties, their axiomatic classification and generalizations of metric spaces have been considered in a lot of papers: Menger [15], Aleksandrov, Nemytskii [1], Mamuzić [14], Gähler [12], Nedev, Choban [19, 21, 20], Kopperman [13], Dhage, Mustafa, Sims [5, 17].

The notion of an (n,m,ρ)-metric, $n>m$, generalizing the usual notion of a pseudometric (the case $n=2, m=1$), and the notion of an $(n+1)$-metric (as in [15] and [12] was introduced in [6]. Connections between some of the topologies induced by a $(3,1,\rho)$-metric and topologies induced by a pseudo-o-metric, o-metric and symmetric (as in [21]), are given in [7]. Some other characterizations of $(3,j,\rho)$-metrizable topological spaces, $j\in\{1,2\}$, are given in [3, 4, 8, 9].

The existence and uniqueness of a fixed point for self-mappings satisfying contractive conditions in $(3,2)$-W-symmetrizable spaces are proven in [11].

There are many fixed point theorems of various contraction type mappings. Moradi and Beiranvand in [16] gave some results concerning T_F contractive mappings on complete metric spaces.

Here we consider $(3,2)$-symmetric spaces and $(3,2)$-W-symmetrizable spaces. The purpose of this paper is to prove the existence and the uniqueness of a fixed point

---

2010 *Mathematics Subject Classification.* Primary: 45H10 Secondary: 54H25.

Key words and phrases. $(3,2)$-W-symmetrizable space, Metric space, Self-mapping, Fixed point.

for pairs of self-mappings on $(3,2)$-W-symmetrizable spaces by using mappings similar to T_F contractions in ordinary metric spaces.

2. Preliminaries

We give the basic definitions for $(3,2,\rho)$-metric spaces and $(3,2)$-metric spaces, as in [3].

Let $M \neq \varnothing$ and $M^{(3)} = M^3/\alpha$, where α is the equivalence relation on M^3 defined by:

$$(x,y,z)\alpha(u,v,w) \Leftrightarrow \pi(u,v,w) = (x,y,z),$$

where π stands for permutation. We will use the same notation (x,y,z) for the elements in $M^{(3)}$ keeping in mind that $(x,y,z) = (u,v,w)$ in $M^{(3)}$ iff (x,y,z) is a permutation of (u,v,w).

Let $d : M^{(3)} \to \mathbb{R}_0^+$. We state three conditions for such map:

$(M0)$ $d(x,x,x) = 0$, for all $x \in M$;
$(M1)$ $d(x,y,z) \leq d(x,a,b) + d(a,y,b) + d(a,b,z)$, for all $x,y,z,a,b \in M$;
(Ms) $d(x,x,y) = d(x,y,y)$, for all $x,y \in M$.

Let ρ be a subset of $M^{(3)}$. We consider the following two conditions for such a set:

$(E0)$ $(x,x,x) \in \rho$, for all $x \in M$;
$(E1)$ $(x,a,b),(a,y,b),(a,b,z) \in \rho \implies (x,y,z) \in \rho$, for any $x,y,z,a,b \in M$.

Definition 2.1. *If ρ satisfies $(E0)$ and $(E1)$ we say that ρ is a $(3,2)$-equivalence.*

Example 2.1. The set $\rho_d = \{(x,y,z) | (x,y,z) \in M^{(3)}, d(x,y,z) = 0\}$ where d satisfies $(M0)$ and $(M1)$ is a $(3,2)$-equivalence on M.

Definition 2.2. *i) If d satisfies $(M0)$ and $(M1)$ we say that d is a $(3,2,\rho)$-metric on M, and the pair (M,d) is a $(3,2,\rho)$-metric space.*
ii) If d satisfies $(M0)$,$(M1)$ and (Ms) we say that d is a $(3,2,\rho)$-symmetric on M, and the pair (M,d) is a $(3,2,\rho)$-symmetric space.
If $\rho = \Delta = \{(x,x,x) | x \in M\}$, we write $(3,2)$ instead of $(3,2,\Delta)$.

Example 2.2. Let M be a nonempty set. The map $d : M^{(3)} \to \mathbb{R}_0^+$ defined by:

$$d(x,y,z) = \begin{cases} 0 & , x = y = z \\ 1 & , \text{otherwise} \end{cases},$$

is a $(3,2)$-metric on M (the discrete 3-metric).

Proposition 2.1. *If d is a $(3,2,\rho)$-metric on M, then*

(i) $d(x,x,y) \leq 2d(x,a,b) + d(y,a,b)$;
(ii) $d(x,x,y) \leq 2d(x,y,y)$;
(iii) $d(x,x,y) \leq 2d(x,z,z) + d(y,z,z)$,

for any $x,y,z,a,b \in M$.

Proof. Follows directly from Definition 2.2. □

Definition 2.3. *Let d be a $(3, 2, \rho)$-metric on M, $x, y \in M$ and $\epsilon > 0$. We define an ϵ-ball with center at x and radius ϵ by:*
$B(x, \epsilon) = \{z | z \in M$*, there is a* $v \in M$ *such that* $d(x, z, v) < \epsilon\}$.

Definition 2.4. *Let d be a $(3, 2, \rho)$-metric on M and $U \subseteq M$. We define the topology $\tau(W, d)$ by: $U \in \tau(W, d)$ iff for any $x \in U$, there is an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.*

Definition 2.5. *We say that a topological space (M, τ) is $(3, 2)$-W-metrizable if there is a $(3, 2)$-metric d such that $\tau = \tau(W, d)$.*

Definition 2.6. *We say that a topological space (M, τ) is $(3, 2)$-W-symmetrizable if there is a $(3, 2)$-symmetric d such that $\tau = \tau(W, d)$.*

Proposition 2.2. *For any $(3, 2, \rho)$-metric d on M and any sequence (x_n) in M, the following conditions are equivalent:*

$(C1)$ $d(x_n, x_m, x_p) \to 0$*, as* $n, m, p \to \infty$*; and*
$(C2)$ $d(x_n, x_m, x_m) \to 0$*, as* $n, m \to \infty$.

Proof. Let d satisfy the condition (C1). For any $n, m \in \mathbb{N}$ we choose $p, q > \max \{m, n\}$. By the previous proposition we obtain

$$d(x_n, x_m, x_m) \leq d(x_n, x_p, x_q) + 2d(x_m, x_p, x_q).$$

Thus, $d(x_n, x_m, x_m) \to 0$, as $n, m \to \infty$.
Let d satisfy the condition (C2). For any $n, m, p \in \mathbb{N}$ we choose $q > \max \{m, n, p\}$ and we obtain

$$d(x_n, x_m, x_p) \leq d(x_n, x_q, x_q) + d(x_m, x_q, x_q) + d(x_p, x_q, x_q).$$

Thus, $d(x_n, x_m, x_p) \to 0$ as $n, m, p \to \infty$. □

Definition 2.7. *A sequence (x_n) in a $(3, 2, \rho)$-metric space (M, d) is called $(3, 2)$-Cauchy if it satisfies $(C1)$ or $(C2)$.*

In the following we use notations and results from [10].

Definition 2.8. *We say that a sequence (x_n) in a $(3, 2, \rho)$-metric space (M, d):*

(i) *1-converges to* $x \in M$ *if* $d(x, x, x_n) \to 0$ *as* $n \to \infty$;
(ii) *2-converges to* $x \in M$ *if* $d(x, x_n, x_n) \to 0$ *as* $n \to \infty$;
(iii) *3-converges to* $x \in M$ *if* $d(x, x_n, x_m) \to 0$ *as* $n, m \to \infty$.

Theorem 1. [10] *For any sequence (x_n) in a $(3, 2, \rho)$-metric space (M, d) the following conditions are equivalent:*

(i) (x_n) *1-converges to* $x \in M$;
(ii) (x_n) *2-converges to* $x \in M$;
(iii) (x_n) *3-converges to* $x \in M$.

Definition 2.9. *We say that a sequence (x_n) in a $(3, 2, \rho)$-metric space (M, d) is $(3, 2)$-convergent if it satisfies any of the conditions in the previous theorem.*

Let (M, d) be a $(3, 2, \rho)$-symmetric space, $x, y \in M$ and (x_n), (y_n) be sequences in M.

Lemma 1. *If* $d(x_n, x, x) \to 0, d(y_n, y, y) \to 0$ *as* $n \to \infty$*, then* $d(x_n, y, y) \to d(x, y, y)$ *and* $d(x_n, y_n, y_n) \to d(x, y, y)$ *as* $n \to \infty$.

Proof. From $d(x_n, y, y) = d(x_n, x_n, y) \leq 2d(x_n, x, x) + d(y, x, x) = 2d(x_n, x, x) + d(x, y, y)$ we obtain

$$d(x_n, y, y) - d(x, y, y) \leq 2d(x_n, x, x). \tag{2.1}$$

From $d(x, y, y) = d(x, x, y) \leq 2d(x, x_n, x_n) + d(y, x_n, x_n) = 2d(x_n, x, x) + d(x_n, y, y)$ we obtain

$$d(x_n, y, y) - d(x, y, y) \geq -2d(x_n, x, x). \tag{2.2}$$

From (2.1) and (2.2) it follows that

$$|d(x_n, y, y) - d(x, y, y)| \leq 2d(x_n, x, x),$$

from where $d(x_n, y, y) \to d(x, y, y)$ as $n \to \infty$.
From $d(x_n, y_n, y_n) = d(x_n, x_n, y_n) \leq 2d(x_n, x, x) + d(y_n, x, x) = 2d(x_n, x, x) + d(y_n, y_n, x) \leq 2d(x_n, x, x) + 2d(y_n, y, y) + d(x, y, y)$ we obtain

$$d(x_n, y_n, y_n) - d(x, y, y) \leq 2(d(x_n, x, x) + d(y_n, y, y)). \tag{2.3}$$

Similarly we obtain

$$d(x_n, y_n, y_n) - d(x, y, y) \geq -2(d(x_n, x, x) + d(y_n, y, y)). \tag{2.4}$$

From (2.3) and (2.4) it follows that

$$|d(x_n, y, y) - d(x, y, y)| \leq 2(d(x_n, x, x) + d(y_n, y, y)),$$

from where $d(x_n, y_n, y_n) \to d(x, y, y)$ as $n \to \infty$. ☐

Definition 2.10. *Let* (M, d) *be a* $(3, 2)$*-metric space. We say that* (M, d) *is* $(3, 2)$*-complete if any* $(3, 2)$*-Cauchy sequence in* M *is* $(3, 2)$*-convergent* $($*with respect to* $d)$.

3. Main results

Let Θ be the set of all continuous non-decreasing functions $f : [0, \infty) \to [0, \infty)$ such that:
i) $f^{-1}(0) = \{0\}$, and
ii) $f(a + b) \leq f(a) + f(b)$ for all $a, b \in [0, \infty)$.
Let (M, d) be a complete $(3, 2)$-symmetric space. In this paper for simplicity, for any mapping $g : M \to M$, we will write gx for $g(x)$.

Theorem 2. *Let* $S_1, S_2, T : M \to M, f \in \Theta$ *and* T *be bijective. If there are* $\alpha > 0$,$\beta \geq 0$ *such that* $2\alpha + \beta \in (0, 1)$ *and*

$$\begin{aligned} f(d(TS_1x, TS_1x, TS_2y)) \leq \alpha(f(d(Tx, Tx, TS_1x) + f(d(Ty, Ty, TS_2y))) \\ + \beta f(d(Tx, Tx, Ty)), \end{aligned}$$

for all $x, y \in M$*, then* S_1 *and* S_2 *have a unique fixed point.*

Proof. Let $x_0 \in M$ be an arbitrary point. We define a sequence (x_k) by $x_{2k+1} = S_1x_{2k}, x_{2k+2} = S_2x_{2k+1}$, for $k = 0, 1, 2, \ldots$. If there is $k \in \mathbb{N}_0$ such that $x_k = x_{k+1} = x_{k+2}$, then $u = x_k$ is the common fixed point of S_1 and S_2. So, we may assume that all three consecutive terms of the sequence are different. For simplicity, we take $y_n = Tx_n$. Then

$$\begin{aligned}
f(d(y_{2k+1}, y_{2k+1}, y_{2k})) &= f(d(TS_1x_{2k}, TS_1x_{2k}, TS_2x_{2k-1})) \\
&\leq \alpha(f(d(y_{2k}, y_{2k}, y_{2k+1})) + f(d(y_{2k-1}, y_{2k-1}, y_{2k}))) \\
&\quad + \beta f(d(y_{2k}, y_{2k}, y_{2k-1})),
\end{aligned}$$

from where and from the fact that d is a $(3,2)$-symmetric we obtain

$$(1-\alpha)f(d(y_{2k+1}, y_{2k+1}, y_{2k})) \leq (\alpha+\beta)f(d(y_{2k}, y_{2k}, y_{2k-1})),$$

i.e.

$$f(d(y_{2k+1}, y_{2k+1}, y_{2k})) \leq \frac{\alpha+\beta}{1-\alpha}\, f(d(y_{2k}, y_{2k}, y_{2k-1})).$$

Thus, $f(d(y_{2k+1}, y_{2k+1}, y_{2k})) \leq \lambda f(d(y_{2k}, y_{2k}, y_{2k-1}))$, where $\lambda = \frac{\alpha+\beta}{1-\alpha} \in (0,1)$. Similarly it can be shown that $f(d(y_{2k}, y_{2k}, y_{2k-1})) \leq \lambda f(d(y_{2k-1}, y_{2k-1}, y_{2k-2}))$, where $\lambda = \frac{\alpha+\beta}{1-\alpha} \in (0,1)$.
So, $f(d(y_{n+1}, y_{n+1}, y_n)) \leq \lambda f(d(y_n, y_n, y_{n-1}))$, for any $n \in \mathbb{N}$.
Let $m, n \in \mathbb{N}, m > n$. Then

$$\begin{aligned}
f(d(y_n, y_m, y_m)) &\leq f(d(y_n, y_{n+1}, y_{n+1})) + 2f(d(y_m, y_{n+1}, y_{n+1})) \\
&\leq f(d(y_n, y_{n+1}, y_{n+1})) + 4f(d(y_{n+1}, y_{n+2}, y_{n+2})) \\
&\quad + 2f(d(y_m, y_{n+2}, y_{n+2})) \\
&\leq f(d(y_n, y_{n+1}, y_{n+1})) + 4f(d(y_{n+1}, y_{n+2}, y_{n+2})) \\
&\quad + 4f(d(y_{n+2}, y_{n+3}, y_{n+3})) + 2f(d(y_m, y_{n+3}, y_{n+3})) \\
&\leq f(d(y_n, y_{n+1}, y_{n+1})) + 4f(d(y_{n+1}, y_{n+2}, y_{n+2})) \\
&\quad + 4f(d(y_{n+2}, y_{n+3}, y_{n+3})) + \ldots + 4f(d(y_{m-1}, y_m, y_m)) \\
&\leq 4(\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1})f(d(y_0, y_1, y_1)) \\
&\leq 4\frac{\lambda^n}{1-\lambda} f(d(y_0, y_1, y_1)).
\end{aligned}$$

Thus, $f(d(y_n, y_m, y_m)) \to 0$ as $n, m \to \infty$. But then, since $f \in \Theta$ we obtain that $d(y_n, y_m, y_m) \to 0$ as $n, m \to \infty$, i.e. (y_n) is a $(3,2)$-Cauchy sequence. Since (M, d) is $(3,2)$-complete, it follows that there is a $v \in M$, such that $d(y_n, v, v) \to 0$ as $n \to \infty$, i.e. $d(Tx_n, v, v) \to 0$ as $n \to \infty$.
We will prove that $T^{-1}(v) = u$ is a common fixed point of S_1 and S_2. From

$$\begin{aligned}
f(d(Tu, Tu, TS_1u) &= f(d(Tu, TS_1u, TS_1u)) \\
&\leq f(d(Tu, y_{2n+2}, y_{2n+2})) + 2f(d(TS_1u, y_{2n+2}, y_{2n+2})) \\
&= f(d(Tu, Tu, y_{2n+2})) + 2f(d(TS_2x_{2n+1}, TS_2x_{2n+1}, TS_1u)) \\
&= f(d(Tu, Tu, y_{2n+2})) + 2f(d(TS_1u, TS_1u, TS_2x_{2n+1})) \\
&\leq f(d(Tu, Tu, y_{2n+2}))
\end{aligned}$$

$$\begin{aligned}&+2\alpha(f(d(Tu,Tu,TS_1u)+f(d(y_{2n+1},y_{2n+1},TS_2x_{2n+1})))\\&+2\beta f(d(Tu,Tu,y_{2n+1})),\end{aligned}$$

it follows that

$$\begin{aligned}(1-2\alpha)f(d(Tu,Tu,TS_1u)\leq f(d(Tu,Tu,y_{2n+2}))+2\alpha f(d(y_{2n+1},y_{2n+1},y_{2n+2}))\\+2\beta f(d(y_{2n+1},Tu,Tu))\end{aligned}$$

Since f is continuous and by lemma 1, if we let $n\to\infty$, we obtain

$$(1-2\alpha)f(d(Tu,Tu,TS_1u)\leq f(d(Tu,Tu,v))+2\alpha f(0)+2\beta f(d(v,Tu,Tu))),$$

i.e.

$$(1-2\alpha)f(d(Tu,Tu,TS_1u)\leq(1+2(\alpha+\beta))f(0)=0,$$

i.e. $f(d(Tu,Tu,TS_1u)=0$. It follows that $d(Tu,Tu,TS_1u)=0$, which means that $Tu=TS_1u$, from where $u=S_1u$.
Since

$$\begin{aligned}f(d(Tu,Tu,TS_2u))&\leq 2f(d(Tu,TS_1u,TS_1u))+f(d(TS_2u,TS_1u,TS_1u))\\&=f(d(TS_1u,TS_1u,TS_2u))\\&\leq\alpha(f(d(Tu,Tu,TS_1u))+f(d(Tu,Tu,TS_2u))),\end{aligned}$$

it follows that

$$(1-\alpha)f(d(Tu,Tu,TS_2u))\leq 0.$$

Thus, $f(d(Tu,Tu,TS_2u))=0$, and $d(Tu,Tu,TS_2u)=0$, which means that $Tu=TS_2u$, from where $u=S_2u$.
Next we will show the uniqueness of the fixed point u. Let w be another common fixed point for S_1 and S_2. Then

$$\begin{aligned}f(d(Tu,Tu,Tw))&=f(d(TS_1u,TS_1u,TS_2w))\\&\leq\alpha(f(d(Tu,Tu,TS_1u))+f(d(Tw,Tw,TS_2w)))\\&+\beta f(d(Tu,Tu,Tw))\\&=\beta f(d(Tu,Tu,Tw)).\end{aligned}$$

It follows that

$$(1-\beta)f(d(Tu,Tu,Tw))\leq 0,$$

i.e. $f(d(Tu,Tu,Tw))=0$, which means $d(Tu,Tu,Tw)=0$. So, $Tu=Tw$, from where $u=w$. ☐

Corollary 2.1. *Let $S_1,S_2,T:M\to M,f\in\Theta$ and T be bijective. If there is an $\alpha\in(0,\frac{1}{2})$ such that*

$$f(d(TS_1x,TS_1x,TS_2y))\leq\alpha(f(d(Tx,Tx,TS_1x)+f(d(Ty,Ty,TS_2y)))$$

for all $x,y\in M$, then S_1 and S_2 have a unique fixed point.

Proof. It follows directly from the previous theorem for $\beta=0$.

☐

Corollary 2.2. *Let* $S_1, S_2, T : M \to M, f \in \Theta$ *and* T *be bijective. If there are* $\alpha > 0, \beta \geq 0, n, m \in \mathbb{N}$ *such that* $2\alpha + \beta \in (0,1)$ *and*

$$\begin{aligned} f(d(TS_1^n x, TS_1^n x, TS_2^m y)) &\leq \alpha(f(d(Tx, Tx, TS_1^n x) + f(d(Ty, Ty, TS_2^m y))) \\ &\quad + \beta f(d(Tx, Tx, Ty)), \end{aligned}$$

for all $x, y \in M$*, then* S_1 *and* S_2 *have a unique fixed point.*

Proof. Since S_1^n, S_2^m are mappings which satisfy the conditions of the previous theorem, it follows that they have a unique common fixed point u. Thus,

$$S_1 u = S_1(S_1^n u) = S_1^n(S_1 u), S_2 u = S_2(S_2^m u) = S_2^m(S_2 u),$$

i.e. $S_1 u$ is a fixed point of S_1^n and $S_2 u$ is a fixed point of S_2^m. But then $S_1 u = u = S_2 u$, i.e. the mappings S_1, S_2 have a common fixed point. Let w be another common fixed point of S_1 and S_2. Then w is a fixed common point of S_1^n and S_2^m. But, S_1^n and S_2^m have a unique fixed point u, so $w = u$. ☐

Theorem 3. *For a complete* $(3,2)$*-W-symmetrizable space* (M, τ) *via* $(3,2)$*-symmetric* d*,* $S_1, S_2, T : M \to M, f \in \Theta$ *such that* T *is bijective, if there are* $\alpha > 0$*,*$\beta \geq 0$ *such that* $2\alpha + \beta \in (0,1)$ *and*

$$\begin{aligned} f(d(TS_1 x, TS_1 x, TS_2 y)) &\leq \alpha(f(d(Tx, Tx, TS_1 x) + f(d(Ty, Ty, TS_2 y))) \\ &\quad + \beta f(d(Tx, Tx, Ty)), \end{aligned}$$

for all $x, y \in M$*, then* S_1 *and* S_2 *have a unique fixed point.*

Proof. Follows from the fact that each $(3,2)$-W-symmetrizable space is a $(3,2)$-symmetric space. ☐

Corollary 3.1. *For a complete* $(3,2)$*-W-symmetrizable space* (M, τ) *via* $(3,2)$*-symmetric* d*,* $S_1, S_2, T : M \to M, f \in \Theta$ *such that* T *is bijective, if there is an* $\alpha \in (0, \frac{1}{2})$ *such that*

$$f(d(TS_1 x, TS_1 x, TS_2 y)) \leq \alpha(f(d(Tx, Tx, TS_1 x) + f(d(Ty, Ty, TS_2 y)))$$

for all $x, y \in M$*, then* S_1 *and* S_2 *have a unique fixed point.*

Corollary 3.2. *For a complete* $(3,2)$*-W-symmetrizable space* (M, τ) *via* $(3,2)$*-symmetric* d*,* $S_1, S_2, T : M \to M, f \in \Theta$ *such that* T *is bijective, if there are* $\alpha > 0$*,*$\beta \geq 0, n, m \in \mathbb{N}$ *such that* $2\alpha + \beta \in (0,1)$ *and*

$$\begin{aligned} f(d(TS_1^n x, TS_1^n x, TS_2^m y)) &\leq \alpha(f(d(Tx, Tx, TS_1^n x) + f(d(Ty, Ty, TS_2^m y))) \\ &\quad + \beta f(d(Tx, Tx, Ty)), \end{aligned}$$

for all $x, y \in M$*, then* S_1 *and* S_2 *have a unique fixed point.*

References

[1] P. Alexandroff and V. Niemytzki, *Der allgemeine Metrisationssatz und das Symmetrieaxiom*, Rec. Math. [Mat. Sbornik] N.S. **45**(3) (1938), 663–672.

[2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fundamenta Mathematicae **3**(1) (1922), 133–181.

[3] S. Čalamani and D. Dimovski, *Topologies induced by* $(3,j,\rho)$*-metrics,* $j\in\{1,2\}$, International Mathematical Forum **9**(22) (2014), 1075–1088.
[4] S. Čalamani, T. Dimovski and D. Dimovski, *Separation properties for some topologies induced by* $(3,j,\rho)$*-metrics,* $j\in\{1,2\}$, Math. Nat. Sci. Proceedings of the Sixth International Scientific Conference – FMNS2015 **1** (2015), 24–30.
[5] B. C. Dhage, *Generalized metric spaces mappings with fixed point*, Bull. Calcutta Math. Soc. **84** (1992), 329–336.
[6] D. Dimovski, *Generalized metrics –* (n,m,r)*-metrics*, Mat. Bilten **16**(42) (1992), 73–76.
[7] D. Dimovski $(3,1,\rho)$*-metrizable topological spaces*, Math. Maced. **3** (2005), 59–64.
[8] T. Dimovski and D. Dimovski, *On* $(3,2,\rho)$*-K-metrizable spaces*, Math. Nat. Sci. Proceedings of the Fifth International Scientific Conference – FMNS2013 **1** (2013), 73–79.
[9] T. Dimovski and D. Dimovski, *Some properties concerning* $(3,2,\rho)$*-K-metrizable spaces*, Math. Nat. Sci. Proceedings of the Fifth International Scientific Conference – FMNS2013 **1** (2015), 18–23.
[10] T. Dimovski and D. Dimovski, *Convergence of sequences in* $(3,j,\rho)$*-N-metrizable spaces,* $j\in\{1,2\}$, Mat. Bilten **42**(1) (2018), 21–27.
[11] T. Dimovski and D. Dimovski, *A fixed point theorem for mappings satisfying cyclical contractive conditions in* $(3,2)$*-W-symmetrizable spaces*, Kragujevac J. Math. **47**(2) (2023), 1021–1028.
[12] S. Gähler, *2-metrische Räume und ihre topologische Struktur*, Math. Nachr. **26** (1963), 115–148.
[13] R. Kopperman, *All topologies come from generalized metrics*, Amer. Math. Monthly **95**(2) (1988), 89–97.
[14] Z. Mamuzič, *Uvod u opštu topologiju I*, Volume 17 of Matematička biblioteka, Zavod za izdavanje udžbenika, Beograd, 1960.
[15] K. Menger, *Untersuchungen über allgemeine Metrik*, Math. Ann. **100** (1928), 75–163.
[16] S. Moradi and A. Beiranvand, *Fixed point of* T_F*-contractive single-valued mappings*, Iranian Journal of Mathematical Sciences and Informatics, **5** (2010), 25–32.
[17] Z. Mustafa and B. Sims, *Some Remarks Concerning D-Metric Spaces*, Proceedings of the International Conferences on Fixed Point Theory and Applications, Valencia (Spain) July (2003), 189–198.
[18] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal. **7**(2) (2006), 289–297.
[19] S. Nedev, *Generalized-metrizable spaces*, C. R. Acad. Bulgare Sci. **20**(6) (1967), 513–516.
[20] S. Nedev and M. Choban, *On metrization of topological groups*, Vestnik Moskovskogo Universiteta Seriya 1. Matematika Mekhanika. **6** (1968), 18–20.
[21] S. Nedev, *O-metrizable spaces*, Trudy Moskovskogo Matematicheskogo Obshchestva **24** (1971), 201–236.

[1] Ss. Cyril and Methodius University,
Faculty of Mechanical Engineering,
Rugjer Boshkovikj, Skopje 1000, Macedonia
Email address: `tomi.dimovski@gmail.com`

[2] Macedonian Academy of Sciences and Arts,
Av. Krste Misirkov 2, Skopje 1000, Macedonia
Email address: `ddimovskid@gmail.com`

Received 5.7.2023
Revised 25.7.2023
Accepted 28.7.2023