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AN ALGORITHM FOR A CLASS OF (n, j, k) - GOOD MATRICES RELATED TO NUMERICAL SEMIGROUPS WITH EMBEDDING DIMENSION 4

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Abstract. In this paper, first we recall the definitions of (n, j)-good 2×2 and (n, j, k)-good 3×3 integer matrices, connected to numerical semigroups of embedding dimension 3 and 4, respectively. Then, for given natural numbers n, j and k where $1 < j, k < n$ and $k \neq j$, we present an algorithm for obtaining all the (n, j, k)-good matrices $M = \begin{bmatrix} a & -u & -p \\ -b & v & -q \\ -c & -w & r \end{bmatrix}$ corresponding to a given (n, j)-good 2×2 matrix $K_0 = \begin{bmatrix} a_0 & -u_0 \\ -b_0 & v_0 \end{bmatrix}$ such that $a \leq a_0$ and $v \leq v_0$.

1. Introduction and Preliminaries

This paper has been motivated by [7], where we have presented a class of integer 3×3 matrices related to numerical semigroups with embedding dimension 4, generated by four integers n, x, y and z such that $x \equiv 1 \pmod{n}, y \equiv j \pmod{n}, z \equiv k \pmod{n}$ and $n < x < y < z$. In fact, this work is a continuation of the works [1], [2], [3], [4], [6] and [9] where we have reformulated the examinations of the Apéry set of a numerical semigroup of embedding dimension 4 with respect to a generator n to the examinations of a presentation of the cyclic group of order n, by three generators and three relations - written as an integer 3×3 matrix. Using a system for computational discrete algebra GAP(Groups, Algorithms, Programming), we examined such matrices. For notions about codes used on GAP Package we refer to [8] and [5].

We recall some notions from [10] and [11]. We say that a submonoid S of the additive monoid $(\mathbb{N}_0, +)$ is an additive semigroup. A numerical semigroup G is an additive semigroup with finite complement in $\mathbb{N}_0$. Any numerical semigroup G is finitely generated, i.e. there is a nonempty set of generators $A =$

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$\{a_1, a_2, \ldots, a_t\} \subseteq G$, such that the elements of G are exactly the linear combinations of $a_1, a_2, \ldots, a_t$ with nonnegative integer coefficients. We denote it by $G = \langle A \rangle$ or $G = \langle a_1, a_2, \ldots, a_t \rangle$. A set A is the minimal set of generators for G if A generates G and no proper subset of A has this propery. The cardinality of the minimal set of generators of G is called embedding dimension of G and it is denoted by $ed(G)$. The Apéry set of an additive semigroup G with respect to a nonzero element $a \in G$ is defined by $Ap(G, a) = \{g \in G | g - a \notin G\}$.

Definition 1.1. For an integer 2×2 matrix $M = \begin{bmatrix} a & -u \\ -b & v \end{bmatrix}$ we say that it is (n, j)-*good*, if:

1) $a, b, u, v \in \mathbb{N}_0, a > 1, v > 1, a > b, a > u, v > u$ and $a + v \leq n + 1$;

2) $detM = av - bu = n$, and

3) The commutative group $H(M)$ with the presentation $\langle g, h | g^a = h^u, h^v = g^b \rangle$ is the cyclic group of order n, generated by g and $h = g^j$ in $H(M)$.

For an (n, j)-good matrix M, let

$$\begin{aligned} P(M) = \{(\alpha, \beta) | \alpha, \beta \in \mathbb{N}_0, 0 \leq \alpha < a - b, 0 \leq \beta < v\} \\ \cup \{(\alpha, \beta) | \alpha, \beta \in \mathbb{N}_0, 0 \leq \alpha < a, 0 \leq \beta < v - u\}. \end{aligned}$$

Definition 1.2. Let $n, j, k \in \mathbb{N}_0, n > j, k > 1$ and $j \neq k$. We say that a matrix

$$M = \begin{bmatrix} a & -u & -p \\ -b & v & -q \\ -c & -w & r \end{bmatrix}$$

is (n, j, k)-*nice* if:

(1.1) $a, b, c, v, u, w, r, p, q$ are nonnegative integers smaller than n;

(1.2) $a, v, r > 1$; $a + v + r \leq n + 2$;

(1.3) $a > b, c; a > u + p; v > u, w; v > q; a + v > u + p + b + q; r > p, q$;

(1.4) if $a - b - c \leq 0$ then $0 < v - u - w, 0 < r - p - q$ and $(0 < v - 2u - w$ or $0 < r - 2p - q)$;

(1.5) if $v - u - w \leq 0$ then $0 < a - b - c, 0 < r - p - q$ and $(0 < a - 2b - c$ or $0 < r - p - 2q)$;

(1.6) if $r - p - q \leq 0$, then $0 < a - b - c, 0 < v - u - w$ and $(0 < a - b - 2c$ or $0 < v - u - 2w)$;

(1.7) $ar - pc > ur + pw, av - ub > uq + vp$ and $av - ub > bp + aq$;

(1.8) if d is a divisor of a, u, p, then d is a divisor of n;

(1.9) if d is a divisor of b, v, q, then d is a divisor of n;

(1.10) if d is a divisor of c, w, r, then d is a divisor of n;

(1.11) n is a divisor of $detM$;

(1.12) If $r \geq c + w$ then for each $r' < r$ with $r'k \equiv c' + w'j (mod n), c' + w' > r'$ and

(1.13) The commutative group $H(M)$ with the presentation $\langle g, h, f | g^a = h^u f^p, h^v = g^b f^q, f^r = g^c h^w > is \rangle$ is the cyclic group of order n, generated by $g, h = g^j$ in $H(M)$ and $f = g^k$ in $H(M)$.

Definition 1.3. Let M be an (n, j, k)-nice matrix. Define $P(M) \subseteq \mathbb{N}^3$ to be the maximal set of triples of nonnegative integers (α, β, γ) satisfying the following conditions:

(1) if $(\alpha, \beta, \gamma) \in P(M), \alpha > 0$ then $(\alpha - 1, \beta, \gamma) \in P(M)$,
(2) if $(\alpha, \beta, \gamma) \in P(M), \beta > 0$ then $(\alpha, \beta - 1, \gamma) \in P(M)$,
(3) if $(\alpha, \beta, \gamma) \in P(M), \gamma > 0$ then $(\alpha, \beta, \gamma - 1) \in P(M)$,
(4) $(a-1, 0, 0), (0, v-1, 0), (0, 0, r-1) \in P(M)$,
(5) $(a, 0, 0), (0, v, 0), (0, 0, r) \notin P(M)$,
(6) if $g^\alpha = h^\beta f^\gamma$ in $H(M)$ and $0 \leq \alpha < a$, then $(0, \beta, \gamma) \notin P(M)$,
(7) if $h^\beta = g^\alpha f^\gamma$ in $H(M)$ and $0 \leq \beta < v$, then $(\alpha, 0, \gamma) \notin P(M)$,
(8) if $f^\gamma = g^\alpha h^\beta$ in $H(M)$ and $0 \leq \gamma < r$, then $(\beta, \gamma, 0) \notin P(M)$, and
(9) if $1 = g^\alpha h^\beta f^\gamma$ in $H(M)$ and $(\alpha, \beta, \gamma) \neq (0, 0, 0)$, then $(\alpha, \beta, \gamma) \notin P(M)$.

If $|P(M)| = n$, we say that M is (n, j, k)-*good*.

Here we restate some of the results from [7].

Theorem 1. *Let $G = \langle n, x, y \rangle$ be a numerical semigroup with $ed(G) = 3$, $n < x < y$, and $x \equiv 1 \pmod{n}, y \equiv j \pmod{n}$. Then, there is a unique (n, j)-good matrix $M = \begin{bmatrix} a & -u \\ -b & v \end{bmatrix}$ with $vy > bx$, so that the Apéry set of G with respect to n is:*

$$Ap(G, n) = \{\alpha x + \beta y | (\alpha, \beta) \in P(M)\}.\square$$

Theorem 2. *Let $G = \langle n, x, y, z \rangle$ be a numerical semigroup with $ed(G) = 4$, $n < x < y < z$, and $x \equiv 1 \pmod{n}, y \equiv j \pmod{n}$, $z \equiv k \pmod{n}$. Then, there is a unique (n, j, k)-good matrix $M = \begin{bmatrix} a & -u & -p \\ -b & v & -q \\ -c & -w & r \end{bmatrix}$ with $rz > cx + wy$ and $vy > bx + qz$, so that the Apéry set of G with respect to n is:*

$$Ap(G, n) = \{\alpha x + \beta y + \gamma z | (\alpha, \beta, \gamma) \in P(M)\}.\square$$

2. Main result

Bellow, we present an algorithm that for a given (n, j)-good 2×2 matrix

$$K_0 = \begin{bmatrix} a_0 & -u_0 \\ -b_0 & v_0 \end{bmatrix}$$

and any $1 < k < n, k \neq j$, with $c' + w'j \equiv k \ (modn)$, $(c', w') \in P(K_0)$ and $c' \neq 0 \neq w'$, describes all the (n, j, k)-good 3×3 matrices

$$M = \begin{bmatrix} a & -u & -p \\ -b & v & -q \\ -c & -w & r \end{bmatrix}$$

with $a \leq a_0$ and $v \leq v_0$.

Step 1. Let $K_0 = \begin{bmatrix} a_0 & -u_0 \\ -b_0 & v_0 \end{bmatrix}$ and $(c', w') \in P(K_0)$ be as above.

Let

$$r_0 = min\{\lceil \frac{a_0}{c'} \rceil, \lceil \frac{v_0}{w'} \rceil, max\{\lceil \frac{a_0 - b_0}{c'} \rceil, \lceil \frac{v_0 - u_0}{w'} \rceil\},$$

where, for example, $\lceil \frac{a_0}{c'} \rceil$ denotes the upper integer part of $\frac{a_0}{c'}$.

Then $M_0 = \begin{bmatrix} a_0 & -u_0 & 0 \\ -b_0 & v_0 & 0 \\ -c_0 & -w_0 & r_0 \end{bmatrix}$ is (n, j, k)-good matrix, where $(c_0, w_0) \in P(K_0)$, such that

$$c_0 + w_0 j \equiv \; r_0 k (mod n).$$

Step 2. Let

$$M = \begin{bmatrix} a & -u & -p \\ -b & v & -q \\ -c & -w & r \end{bmatrix}$$

be (n, j, k)-good matrix with $a \leq a_0$ and $v \leq v_0$.

(1) Let $D_{yz}, E_{yz}, D_{xz}, E_{xz}, D_{xy}, E_{xy}$, be the following sets:

$D_{yz} = \{(0, \beta, \gamma) | 1 \leq \beta < v, 1 \leq \gamma < r, (0, \beta, \gamma) \notin P(M), (0, \beta - 1, \gamma), (0, \beta, \gamma - 1) \in P(M)\}$;

$E_{yz} = \{(-a_i, u_i, p_i) | (0, u_i, p_i) \in D_{yz}, a_i \leq a, a_i \equiv u_i j + p_i k (mod n), a_i > u_i + p_i, a_i + v > u_i + p_i + b + q\}$;

$D_{xz} = \{(\alpha, 0, \gamma), 1 \leq \alpha < a, 1 \leq \gamma < r, (\alpha, 0, \gamma) \notin P(M)$; $(\alpha - 1, 0, \gamma)$,$(\alpha, 0, \gamma - 1) \in P(M)\}$;

$E_{xz} = \{(b_i, -v_i, q_i) | (b_i, 0, q_i) \in D_{xz}, v_i \leq v, v_i j \equiv b_i + q_i k (mod n), v_i > q_i, a + v_i > u + p + b_i + q_i\}$;

$D_{xy} = \{(\alpha, \beta, 0), 1 \leq \alpha < a, 1 \leq \beta < v, (\alpha, \beta, 0) \notin P(M); (\alpha - 1, \beta, 0), (\alpha, \beta - 1, 0) \in P(M)\}$;

$E_{xy} = \{(c_i, w_i, -r_i) | (c_i, w_i, 0) \in D_{xy}, r_i < r, r_i k \equiv c_i + w_i j (mod n)\}$.

(2) We consider two cases:

- If $E_{yz} \neq \emptyset$, let $A = max\{a_i | (-a_i, u_i, p_i) \in E_{yz}\}$. Then, there is some t, so that $a_t = A$. Let $U = u_t, P = p_t$ and let $M_x = \begin{bmatrix} A & -U & -P \\ -b & v & -q \\ -c & -w & r \end{bmatrix}$.
- If $E_{yz} = \emptyset$, let $M_x = M$.

(3) We consider two cases:

- If $E_{xz} \neq \emptyset$, let $V = max\{v_i | (b_i, -v_i, q_i) \in E_{xz}\}$. Then, there is some t, so that $v_t = V$. Let $B = b_t$, $Q = q_t$ and let $M_y = \begin{bmatrix} a & -u & -p \\ -B & V & -Q \\ -c & -w & r \end{bmatrix}$.
- If $E_{xz} = \emptyset$, let $M_y = M$.

(4) Let $r < c + w$ and $F_{xy} = \{(\alpha, \beta, 0) | (\alpha, \beta, 0) \in P(M)\}$. Since M is (n, j, k) - good matrix, it follows that $(c, w, 0) \in F_{xy}$. Let m be the largest integer such that

$(mc, mw, 0) \in F_{xy}$ and $((m+1)c, (m+1)w, 0) \notin F_{xy}$.

Let s_m be the unique integer such that $(mc, mw, s_m) \notin P(M)$ and $(mc, mw, s_m - 1) \in P(M)$. Let $R = mr + s_m$, and let $(C, W) \in P(K_0)$ be the pair such that $Rk \equiv C + Wj(modn)$. We consider the following three cases:

- If $c \neq 0$ and $w \neq 0$, let

$$M_z = \begin{bmatrix} a & -u & -p \\ -b & v & -q \\ -C & -W & R \end{bmatrix}$$

- If $c \neq 0$ and $w = 0$, let

$$M_z = \begin{bmatrix} c & 0 & -r \\ -B & v & -Q \\ -C & -W & R \end{bmatrix},$$

where $B = b - [\frac{b}{c}]c$, $Q = [\frac{b}{c}]r + q$ and $[\frac{b}{c}]$ is the integer part of $\frac{b}{c}$.

- If $c = 0$ and $w \neq 0$, let

$$M_z = \begin{bmatrix} a & -U & -P \\ 0 & w & -r \\ -C & -W & R \end{bmatrix},$$

where $U = u - [\frac{u}{w}]w$, $P = [\frac{u}{w}]r + p$ and $[\frac{u}{w}]$ is the integer part of $\frac{u}{w}$.

(5) For $r \geq c + w$, let $M_z = M$.

Step 3. For an (n, j, k)-good matrix M we define the set

$$H_1(M) = \{M^{'} | M^{'} \in \{M_x, M_y, M_z\}, M^{'} \ is \ (n, j, k) - good\} \setminus \{M\},$$

where it is possible $H_1(M)$ to be the empty set.

Next, we define a set $H_m(M)$ by induction as follows:

If $H_i(M) = \{M_1, M_2, \ldots, M_s\}$ then

$$H_{i+1}(M) = (H_1(M_1) \cup H_1(M_2) \cup \ldots \cup H_1(M_s)) \backslash K_i(M),$$

where $K_i(M) = \{M\} \cup H_1(M) \cup , ..., \cup H_i(M)$.

This procedure has to stop after some time, i.e. for some $T, H_T(M)$ is going to be the empty set. Hence, for a given (n, j)-good 2×2 matrix $K_0 = \begin{bmatrix} a_0 & -u_0 \\ -b_0 & v_0 \end{bmatrix}$

and any $1 < k < n, k \neq j$, the set of all the (n, j, k)-good 3×3 matrices

$$M = \begin{bmatrix} a & -u & -p \\ -b & v & -q \\ -c & -w & r \end{bmatrix}$$

with $a \leq a_0$ and $v \leq v_0$ is the set

$$\{M_0\} \cup H_1(M_0) \cup H_2(M_0) \cup ... \cup H_T(M_0) = K_T(M_0).$$

Next, we present the code in GAP for obtaining all (n, j, k)-good matrices M_x and M_y as follows:

- First, we need to create all the possible coordinates, to do so we will use a cartesian product: coordinates := Cartesian $([0..0], [0..v-1], [0..r-1])$ and coordinates := Cartesian$([0..a-1], [0..0], [0..r-1]$ and from them we select elements that meet the conditions defined for sets D_{yz} and D_{xz} respectively. Then for the obtained elements of D_{yz} and D_{xz} filter the elements which meet the conditions defined for sets E_{yz} and E_{xz}, respectively.
- From the obtained elements $(-a_i, u_i, p_i)$ in E_{yz} we find the triple with maximum a_i and denote it by $(A, -U, -P)$. So we get the matrix M_x=[[A,-U,-P],[-b,v,-q],[-c,-w,r]]. Next test if the obtained matrix M_x fulfills the conditions of (n, j, k)-good matrix by using the codes presented on [3] and [4]. If it does add it to the set H_1.
- From the obtained elements $(-b_i, v_i, -q_i)$ in E_{xz} we select the triple with maximum v_i and denote it by $(-B, V, -Q)$. So we get the matrix M_y=[[a,-u,-p],[-B,V,-Q],[-c,-w,r]]. Next test if the obtained matrix fulfills the conditions of (n, j, k)-good matrix. If it does add it at the set of H_1.
- The same procedure continues for all (n, j, k)-good matrices M_x and M_y until we obtain that there is no matrix that satisfies the necessary conditions.

We now execute the algorithm we described above:

gap> D_{yz}:= Filtered(Cartesian$([0..0], [1..v-1], [1..r-1]), f \to$ not $([0, f[2], f[3]]$ in $PM)$ and $= [0, f[2]-1, f[3]]$ in PM and $[0, f[2], f[3]-1]$ in $PM)$;;

gap> total2:=Filtered($Cartesian([1..a-1], [1..v-1], [1..r-1]), f \to f[1]mod(n) = (f[2]*j + f[3]*k)mod(n)))$ and $f[1]mod(n) \leqslant amod(n)$ and $f[1]mod(n) > (f[2] + f[3])mod(n)$ and $(f[1] + v)mod(n) > (f[2] + f[3] + b + q)mod(n))$;;

gap> E_{yz} := Filtered(total2, $f \to [0, f[2], f[3]]$ in D_{yz});

gap> D_{xz} := $Filtered$(Cartesian $([1..a-1], [0..0], [1..r-1]), f \to$ not $([f[1], 0, f[3]]$ in PM) and $[f[1]-1, 0, f[3]]$ in PM and $[f[1], 0, f[3]-1]$ in $PM)$;;

gap> total2:=Filtered(Cartesian$([1..a-1], [1..v-1], [1..r-1]), f \to (f[2]*j)mod(n) = (f[1] + f[3]*k)mod(n));;$

gap> E_{xz}:=Filtered(total2, $f \to [f[1], 0, f[3]]$ in D_{xz});

gap> $D_{xy} := Filtered(Cartesian([1..a-1], [1..v-1], [0..0]), f \to not([f[1], f[2], 0]$ in $PM)$ and $[f[1]-1, f[2], 0]$ in PM and $[f[1], f[2]-1, 0]$ in $PM);;$

gap> total2:= Filtered$(Cartesian([1..a-1],[1..v-1],[1..r-1]), f \rightarrow (f[3]*k)mod(n) = (f[1]+f[2]*j)mod(n))$ and $f[2]mod(n) \leqslant vmod(n)$ and $f[2]mod(n) > f[3]mod(n)$ and $(a+f[2])mod(n) > (u+p+f[1]+f[3])mod(n))$;;
gap> $E_{xy} := Filtered(total2, f \rightarrow [f[1], f[2], 0]$ in $D_{xy})$;

Next, we present the code for obtaining the possible (n, j, k)-good matrices M_z on GAP as follows:

First define F_{xy} and D_0 then find m and s_m by using for loop and if statement, according to the conditions defined on step 2, (4). This gives us the element R and then using for loop and if statement again we find C and W.

For each of the obtained matrices M_z test if they satisfy conditions defined for (n, j, k)-good matrix, until we obtain that there is no matrix that satisfies the necessary conditions and stop the procedure.

We now execute the algorithm we described:
gap> $F_{xy} := []$;
gap> for α in $[1..a-1]$ do
gap> for β in $[1..v-1]$ do
gap> $A := [\alpha, \beta, 0]$;
gap> if $[\alpha, \beta, 0]$ in PM then
gap> Add(F_{xy}, A);
gap> fi;
gap> od;od;
gap> $M := []$; ;
gap> t :=Maximum(a, v); ;
gap> for m in $[1..t]$ do
gap> for s_m in $[1..r]$ do
gap> $A := [m, s_m]$; ;
gap> if $[m*c, m*w, 0]$ in F_{xy} and not$([(m+1)*c, (m+1)*w, 0]$ in $F_{xy})$ and $[m*c, m*w, s_m-1]$ in PM and not$([m*c, m*w, s_m]$ in $PM)$ then
gap> Add(M, A);
gap> fi;
gap> od;od;
gap> M;
gap> total1:= []; ;
gap> for α in $[0..a_0-1]$ do
gap> for beta in $[0..b_0-1]$ do
gap> D_1:= $[\alpha, \beta]$;
gap> Add(total1, D1);
gap> od;od;
gap>total2:=[];;
gap> for α in $[a_0-b_0..a_0-1]$ do
gap> for β in $[u_0-v_0..b_0-1]$ do
gap>$D_2 := [\alpha, \beta]$;

```
gap>Add(total2, D_2);
gap>od;od;
gap>SubtractSet(total1,total2);
gap> D_0 :=total1;;
gap> Size(D_0);
gap> R := m * r + s_m;
gap> total2:=[];;
gap> for C in [0..a - 1] do
gap> for W in [0..v - 1] do
gap> A := [C, W];
gap> if (R * k)mod(n) =(C + W * j) mod(n) and ([C, W] in D_0) then
gap>Add(total2,A); gap>fi;
gap>od;od;
gap>total2;
```

We have examined example how to describe the set of all (n, j, k)-good matrices for a given (n, j, k)-good matrix using GAP.

For simplicity instead of the notation of the matrix $\begin{bmatrix} a & -u & -p \\ -b & v & -q \\ -c & -w & r \end{bmatrix}$ we will use the notation $[a; v; r]$.

Example 1. Let $K_0 = \begin{bmatrix} 327 & -180 \\ -238 & 197 \end{bmatrix}$. It is easy to show that K_0 is (n, j)-good matrix where n=21579 and j=15227. Let k=3665.

Step 1. Since

$$K_0 = \begin{bmatrix} 327 & -180 \\ -238 & 197 \end{bmatrix}$$

is (n, j)-good, there is a unique $(c', w') = (14, 30) \in P(K_0)$ such that

$$c' + w'j \equiv k(modn).$$

We calculate

$$r_0 = \min\{\lceil \frac{327}{14} \rceil, \lceil \frac{197}{30} \rceil\ , \max\{\lceil \frac{327-238}{14} \rceil, \lceil \frac{197-180}{30} \rceil\}\} = 7.$$

Then $M_0 = \begin{bmatrix} 327 & -180 & 0 \\ -238 & 197 & 0 \\ -9 & -193 & 7 \end{bmatrix}$ is (n, j, k)-good matrix, where $(c_0, w_0) = (9, 193) \in P(K_0)$ such that

$$9 + 193j \equiv 7k(modn).$$

Step 2. $M = M_0 = \begin{bmatrix} 327 & -180 & 0 \\ -238 & 197 & 0 \\ -9 & -193 & 7 \end{bmatrix}$ is (n, j, k)-good matrix,

(1) D_{yz}, E_{yz}, D_{xz}, E_{xz}, D_{xy}, E_{xy} are the following sets:

$D_{yz} = \{(0, 17, 6), (0, 47, 5), (0, 77, 4), (0, 107, 3), (0, 137, 2), (0, 167, 1)\}$,

$E_{yz} = \{(-252, 167, 1), (-266, 137, 2), (-280, 107, 3), (-294, 77, 4), (-308, 47, 5), (-322, 17, 6)\}$,

$D_{xz} = \{(5, 0, 6), (19, 0, 5), (33, 0, 4), (47, 0, 3), (61, 0, 2), (75, 0, 1)\}$,

$E_{xz} = \{(5, -163, 6), (19, -133, 5), (33, -103, 4), (47, -73, 3), (61, -43, 2), (75, -13, 1)\}$,
$D_{xy} = \{(14, 30, 0), (89, 17, 0)\}$,
$E_{xy} = \{(14, 30, -1)\}$.

(2) $A = \max\{a_i | (-a_i, u_i, p_i) \in E_{yz}\} = 322$. Then, $U = 17$, $P = 6$ and

$$M_x = \begin{bmatrix} 322 & -17 & -6 \\ -238 & 197 & 0 \\ -9 & -193 & 7 \end{bmatrix} = [322; 197; 7].$$

(3) $V = \max\{v_i | (b_i, -v_i, q_i) \in E_{xz}\} = 163$. Then, $B = 5$, $Q = 6$ and

$$M_y = \begin{bmatrix} 327 & -180 & 0 \\ -5 & 163 & -6 \\ -9 & -193 & 7 \end{bmatrix} = [327; 163; 7].$$

(4) Since M is (n, j, k)-good matrix, it follows that $(c, w, 0) \in F_{xy}$. $m = 1$ is the largest integer such that $(mc, mw, 0) \in F_{xy}$ and $((m+1)c, (m+1)w, 0) \notin F_{xy}$. $s_m = 1$ is the unique integer such that $(mc, mw, s_m) \notin P(M)$ and $(mc, mw, s_m - 1) \in P(M)$. $R = mr + s_m = 8$, and $(172, 9) \in P(K_0)$ is the pair such that $8k \equiv 172 + 9j (mod n)$. In this case, $c \neq 0$ and $w \neq 0$, so

$$M_z = \begin{bmatrix} 327 & -180 & 0 \\ -238 & 197 & 0 \\ -172 & -9 & 8 \end{bmatrix}.$$

(5) After testing we obtain that $M_x = [322; 197; 7]$ and $M_z = [327; 197; 8]$ are (n, j, k)-good matrices, but $M_y = [327; 163; 7]$ is not since it's corresponding $P(M)$ size is 21285, which is not equal to n. So $H_1(M_0) = \{[322; 197; 7], [327; 197; 8]\}$.

Next by applying the above five steps for each matrix in $H_1(M_0)$ we obtain the set $H_2(M_0)$. Continuing in the same way we obtain the following sets of (n, j, k)-good matrices with $a \leq a_0$ and $v \leq v_0$:

$H_2(M_0) = \{[308; 197; 7], [322; 197; 8]\}$,
$H_3(M_0) = \{[294; 197; 7], [308; 197; 8], [322; 163; 8]\}$,
$H_4(M_0) = \{[280; 197; 7], [294; 197; 8], [308; 163; 8], [322; 133; 8]\}$,
$H_5(M_0) = \{[266; 197; 7], [280; 197; 8], [294; 163; 8], [308; 133; 8], [322; 103; 8]\}$,
$H_6(M_0) = \{[252; 197; 7], [266; 197; 8], [280; 163; 8], [294; 133; 8], [308; 103; 8], [322; 73; 8]\}$,
$H_7(M_0) = \{[252; 197; 8], [266; 163; 8], [280; 133; 8], [294; 103; 8], [308; 73; 8], [322; 43; 8]\}$,
$H_8(M_0) = \{[247; 197; 8], [247; 163; 8], [247; 133; 8], [247; 103; 8], [347; 73; 8], [247; 43; 8]\}$,
$H_9(M_0) = \{[233; 163; 8], [233; 133; 8], [233; 103; 8], [233; 73; 8], [247; 43; 9]\}$,
$H_{10}(M_0) = \{[219; 163; 8], [219; 133; 8], [219; 103; 8], [233; 73; 9], [233; 43; 9], [247; 13; 9]\}$,
$H_{11}(M_0) = \{[205; 163; 8], [205; 133; 8], [219; 103; 9], [219; 73; 9], [158; 43; 9], [247; 13; 26]\}$,
$H_{12}(M_0) = \{[205; 133; 9], [205; 103; 9], [158; 73; 9], [83; 43; 9], [247; 13; 53]\}$,
$H_{13}(M_0) = \{[191; 133; 9], [158; 103; 9], [144; 73; 9], [83; 43; 53]\}$,
$H_{14}(M_0) = \{[158; 133; 9], [144; 103; 9], [83; 73; 9], [77; 43; 53]\}$,

$H_{15}(M_0) = \{[144;133;9],[130;103;9],[69;73;9],[69;43;53]\}$,
$H_{16}(M_0) = \{[130;133;9],[144;103;9],[83;103;9],[69;73;53]\}$,
$H_{17}(M_0) = \{[69;103;9],[69;68;53]\}$,
$H_{18}(M_0) = \{[69;63;53]\}$,
$H_{19}(M_0) = \{[69;58;53]\}$,
$H_{20}(M_0) = \{[69;53;53]\}$,
$H_{21}(M_0) = \{[69;48;53]\}$.

We observe that $H_1([69;48;53]) = \{[69;43;53]\}$ and since $[69;43;53] \in K_{21}(M_0)$ it follows that $H_{22}(M_0) = \emptyset$. Thus $K_{21}(M_0) = \{M_0\} \cup H_1(M_0) \cup \ldots \cup H_{21}(M_0)$ is the set of all the (n,j,k)-good matrices $\begin{bmatrix} a & -u & -p \\ -b & v & -q \\ -c & -w & r \end{bmatrix}$ with $a \leqslant 327$, $v \leqslant 197$.

We give the complete matrices for some members of $K_{21}(M_0)$:

$$\begin{bmatrix} 308 & -47 & -5 \\ -238 & 197 & 0 \\ -9 & -193 & 7 \end{bmatrix}, \begin{bmatrix} 322 & -17 & -6 \\ -5 & 163 & -6 \\ -172 & -9 & 8 \end{bmatrix}, \begin{bmatrix} 266 & -137 & -2 \\ -238 & 197 & 0 \\ -172 & -9 & 8 \end{bmatrix}, \begin{bmatrix} 247 & -4 & -7 \\ -33 & 103 & -4 \\ -172 & -9 & 8 \end{bmatrix},$$

$$\begin{bmatrix} 219 & -64 & -5 \\ -19 & 133 & -5 \\ -172 & -9 & 8 \end{bmatrix}, \begin{bmatrix} 83 & -8 & -8 \\ -61 & 43 & -2 \\ -8 & -5 & 9 \end{bmatrix}, \begin{bmatrix} 144 & -51 & -6 \\ -33 & 103 & -4 \\ -8 & -5 & 9 \end{bmatrix}, \begin{bmatrix} 144 & -51 & -6 \\ -19 & 133 & -5 \\ -8 & -5 & 9 \end{bmatrix},$$

$$\begin{bmatrix} 69 & -38 & -7 \\ -61 & 43 & -2 \\ -34 & 0 & 53 \end{bmatrix}, \begin{bmatrix} 69 & -38 & -7 \\ -53 & 48 & -11 \\ -34 & 0 & 53 \end{bmatrix}.$$

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