

## N-TUPLE ORBITS AND N-TUPLE WEAK ORBITS TENDING TO INFINITY

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**Abstract.** In this paper we give a sufficient condition for  $n$  pairwise commuting and bounded linear operators on an infinite dimensional complex Banach space  $X$ , which will imply that the space contains a dense set of vectors each with a corresponding  $n$ -tuple orbit tending to infinity. The same condition is sufficient to imply that the product of  $X$  and its dual space contains a dense set of pairs, each with a corresponding  $n$ -tuple weak orbit tending to infinity.

### 1. INTRODUCTION

Throughout this paper, unless otherwise stated,  $X$  will denote a complex, infinite dimensional Banach space,  $B(X)$  the algebra of all bounded linear operators on  $X$  and  $X^*$  the dual space of  $X$  i.e., the space of all bounded linear functionals  $x^* : X \rightarrow \mathbb{C}$ . As usual, for  $x \in X$  and  $x^* \in X^*$  we will denote  $\langle x, x^* \rangle := x^*(x)$ . For the direct product  $X \times X^*$  we assume that is a Banach space, in a sense of the direct sum of  $X$  and  $X^*$ , with one of the following norms:  $\|(x, x^*)\|_\infty = \max\{\|x\|, \|x^*\|\}$  or  $\|(x, x^*)\|_p = (\|x\|^p + \|x^*\|^p)^{1/p}$  for  $1 \leq p < \infty$ .  $\mathbb{Z}_+$  will denote the set of all nonnegative integers and

$$\mathbb{Z}_+^n = \{(k_1, k_2, \dots, k_n) : k_i \in \mathbb{Z}_+, 1 \leq i \leq n\}.$$

If  $T_1, T_2, \dots, T_n \in B(X)$  are pairwise commuting operators, the  $n$ -tuple orbit of the vector  $x \in X$  (or the orbit of  $x$  under the  $n$ -tuple  $\mathbf{T} = (T_1, T_2, \dots, T_n)$ ) is the set

$$\text{Orb}(\{T_i\}_{i=1}^n, x) = \text{Orb}(\mathbf{T}, x) = \left\{ T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x : (k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n \right\}, \quad (1.1)$$

and the  $n$ -tuple weak orbit of the pair  $(x, x^*) \in X \times X^*$  is the set

$$\begin{aligned} \text{Orb}(\{T_i\}_{i=1}^n, x, x^*) &= \text{Orb}(\mathbf{T}, x, x^*) \\ &= \left\{ \left\langle T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x, x^* \right\rangle : (k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n \right\}. \end{aligned} \quad (1.2)$$

By the definition given in [15], the  $n$ -tuple orbit (1.1) tends to infinity if

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$$\lim_{k_i \rightarrow \infty} \left\| T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x \right\| = \infty, \text{ for every } k_j \in \mathbb{Z}_+, j \neq i, \text{ and every } 1 \leq i \leq n.$$

In [8] and [10] we gave a similar definition for  $n$ -tuple weak orbits: the  $n$ -tuple weak orbit (1.2) *tends to infinity* if

$$\lim_{k_i \rightarrow \infty} \left| \left\langle T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x, x^* \right\rangle \right| = \infty, \text{ for every } k_j \in \mathbb{Z}_+, j \neq i, \text{ and every } 1 \leq i \leq n.$$

For  $n = 1$ , the sets in (1.1) and (1.2) are sequences of form:

$$\text{Orb}(T, x) = \{T^n x : n = 0, 1, 2, \dots\} \subset X,$$

and

$$\text{Orb}(T, x, x^*) = \{\langle T^n x, x^* \rangle : n = 0, 1, 2, \dots\} \subset \mathbb{C}.$$

These sequences are usually referred as *single orbit* (or simply *orbit*) of the vector  $x \in X$  and *single weak orbit* (or simply *weak orbit*) of the pair  $(x, x^*) \in X \times X^*$  under the operator  $T$ , respectively. Clearly, if  $\text{Orb}(\{T_i\}_{i=1}^n, x)$  tends to infinity, then  $\text{Orb}(T_i, x)$  will also tend to infinity, for every  $i \in \{1, 2, \dots, n\}$ . The same holds for the weak orbits: if  $\text{Orb}(\{T_i\}_{i=1}^n, x, x^*)$  tends to infinity, then  $\text{Orb}(T_i, x, x^*)$  will also tend to infinity, for every  $i \in \{1, 2, \dots, n\}$ . As corollaries of the main results in [7]-[10], we've obtained that, if  $T_1, T_2, \dots, T_n \in B(X)$  are operators such that  $r(T_i) > 1$ , for all  $i \in \{1, 2, \dots, n\}$ , then:

- (i)  $X$  will contain a dense set  $D$  such that  $\text{Orb}(T_i, x)$  tends to infinity for all  $x \in D$  and all  $i \in \{1, 2, \dots, n\}$  and if, in addition, the operators  $T_1, T_2, \dots, T_n$  are pairwise commuting and have at least one of the following properties:
  - (P.1)  $T_i$  is bounded bellow, for every  $i \in \{1, 2, \dots, n\}$ ,
  - (P.2)  $(T_i^k - T_j^k)_{k \geq 0}$  is a norm bounded sequence, for all  $i, j \in \{1, 2, \dots, n\}$ ,
then the  $m$ -tuple orbit  $\text{Orb}(\{T_i\}_{i=1}^m, x)$  will tend to infinity, for every  $2 \leq m \leq n, 1 \leq i_1 < i_2 < \dots < i_m \leq n$  and  $x \in D$ ,
- (ii)  $X \times X^*$  will contain a dense set  $D'$  such that  $\text{Orb}(T_i, x, x^*)$  tends to infinity, for all  $(x, x^*) \in D'$  and all  $i \in \{1, 2, \dots, n\}$  and if, in addition, the operators  $T_1, T_2, \dots, T_n$  are pairwise commuting and have the property (P.2), then the  $m$ -tuple weak orbit  $\text{Orb}(\{T_i\}_{i=1}^m, x, x^*)$  will tend to infinity for every  $2 \leq m \leq n, 1 \leq i_1 < i_2 < \dots < i_m \leq n$  and  $(x, x^*) \in D'$ .

The conditions (P.1) and (P.2) are quite rigorous. Moreover, for any operators  $T_1, T_2, \dots, T_n$  such that  $r(T_i) > 1, i \in \{1, 2, \dots, n\}$ , the condition (P.2) will imply that all these operators must have the same spectral radius. In this paper we are going to show that vectors in  $X$  with  $n$ -tuple orbits and pairs in  $X \times X^*$  with  $n$ -tuple weak orbits tending to infinity exist whenever  $T_1, T_2, \dots, T_n$  are pairwise commuting operators such that  $r(T_i) > 1$  for every  $i \in \{1, 2, \dots, n\}$ , without any additional conditions.

## 2. PRELIMINARIES

As usual, for a single operator  $T \in B(X)$ ,  $\sigma(T)$ ,  $\sigma_p(T)$  and  $\sigma_{\text{ap}}(T)$  will denote the spectrum, the point spectrum and the approximate point spectrum of  $T$ .

If  $\mathbf{T} = (T_1, T_2, \dots, T_n)$  is an  $n$ -tuple of pairwise commuting operators on  $X$ , the joint approximate point spectrum (or the left approximate spectrum) of  $\mathbf{T}$  is the set

$$\begin{aligned}\sigma_\pi(\mathbf{T}) &= \sigma_\pi(T_1, T_2, \dots, T_n) \\ &= \{(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{C}^n : (\forall \varepsilon > 0)(\exists x \in X) \text{ s.t. } \|x\| = 1 \wedge \\ &\quad \|(T_i - \lambda_i)x\| < \varepsilon, 1 \leq i \leq n\}.\end{aligned}$$

For alternative equivalent definitions of the joint approximate point spectrum, we refer to [1], [3] and [11]. For every  $n$ -tuple of pairwise commuting operators  $\mathbf{T} = (T_1, T_2, \dots, T_n)$ ,  $\sigma_\pi(\mathbf{T})$  is nonvoid and compact set ([3, Property 2]), which has the following property, usually referred as the spectral mapping theorem for the joint approximate point spectrum.

**Theorem 1.** [3, Theorem 1] *If  $\mathbf{T} = (T_1, T_2, \dots, T_n)$  is an  $n$ -tuple of pairwise commuting operators and  $f$  is an  $m$ -tuple of polynomials in  $n$  variables (so that  $f(\mathbf{T})$  is defined and is an  $m$ -tuple of commuting operators), then  $\sigma_\pi(f(\mathbf{T})) = f(\sigma_\pi(\mathbf{T}))$ .*

Clearly,  $\sigma_{\text{ap}}(T) = \sigma_\pi(T)$  for every operator  $T \in B(X)$  and, by [4, Theorem 1],

$$r(T) = \max \{|\lambda| : \lambda \in \sigma_{\text{ap}}(T)\}, \text{ for every } T \in B(X). \quad (2.1)$$

We also need the following two results.

**Theorem 2.** [13, Theorem V.37.14] *Let  $X$  and  $Y$  be Banach spaces and  $(T_n)_{n \geq 1}$  be a sequence of operators in  $B(X, Y)$ . Let  $(a_n)_{n \geq 1}$  be sequence of positive numbers such that  $\sum_{n=1}^{\infty} a_n < \infty$ . Then there exists  $x \in X$  such that  $\|T_n x\| \geq a_n \|T_n\|$ , for all  $n \geq 1$ . Moreover, it is possible to choose such an  $x$  in each ball in  $X$  of radius greater than  $\sum_{n=1}^{\infty} a_n$ .*

**Theorem 3.** [13, Theorem V.39.5] *Let  $X$  and  $Y$  be Banach spaces and  $(T_n)_{n \geq 1}$  be a sequence of operators in  $B(X, Y)$ . Let  $(a_n)_{n \geq 1}$  be sequence of positive numbers with  $\sum_{n=1}^{\infty} a_n^{1/2} < \infty$ . Then there are  $x \in X$  and  $y^* \in Y^*$  such that  $|\langle T_n x, y^* \rangle| \geq a_n \|T_n\|$ , for all  $n \geq 1$ . Moreover, given balls  $B \subset X$  and  $B^* \subset Y^*$  of radii greater than  $\sum_{n \geq 1} a_n^{1/2} < \infty$ , then it is possible to find  $x \in B$  and  $y^* \in B^*$  with this property.*

### 3. N-TUPLE ORBITS TENDING TO INFINITY

**Theorem 4.** *If  $\mathbf{T} = (T_1, T_2, \dots, T_n)$  is an  $n$ -tuple of pairwise commuting operators on an infinite dimensional complex Banach space  $X$  such that  $r(T_i) > 1$ , for every  $1 \leq i \leq n$ , then there is a dense set  $D_1 \subset X$  such that the  $n$ -tuple orbit  $\text{Orb}(\{T_i\}_{i=1}^n, x)$  tends to infinity for every  $x \in D_1$ .*

*Proof.* Let  $x_0 \in X$  and  $\varepsilon > 0$ . Since  $r(T_i) > 1$ , for all  $1 \leq i \leq n$ , by (2.1) there are  $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$  such that  $\lambda_i \in \sigma_{\text{ap}}(T_i)$  and  $|\lambda_i| = r(T_i) > 1, 1 \leq i \leq n$ . Let  $q \in \mathbb{R}$  and  $C > 0$  are such that

$$1 < q < \min \{|\lambda_1|, |\lambda_2|, \dots, |\lambda_n|\}, \quad (3.1)$$

$$C \left( \frac{q}{q-1} \right)^n < \varepsilon. \quad (3.2)$$

If  $p_1 < p_2 < \dots < p_n$  are the first  $n$  prime numbers, let  $g : \mathbb{Z}_+^n \rightarrow \mathbb{Z}_+$  be the injective mapping defined with  $g(k_1, k_2, \dots, k_n) = p_1^{k_1} p_2^{k_2} \dots p_n^{k_n}$  and let

$$a_{g(k_1, k_2, \dots, k_n)} = \frac{C}{q^{k_1 + k_2 + \dots + k_n}} > 0, \text{ for } (k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n,$$

$$S_{g(k_1, k_2, \dots, k_n)} = T_1^{k_1} T_2^{k_2} \dots T_n^{k_n}, \text{ for } (k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n.$$

By the first inequality in (3.1) and by (3.2) we have

$$\begin{aligned} \sum_{(k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n} a_{g(k_1, k_2, \dots, k_n)} &= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \dots \sum_{k_n=0}^{\infty} \frac{C}{q^{k_1 + k_2 + \dots + k_n}} \\ &= C \prod_{i=1}^n \left( \sum_{k_i=0}^{\infty} \frac{1}{q^{k_i}} \right) = C \left( \frac{q}{q-1} \right)^n < \varepsilon. \end{aligned}$$

Hence, applying Theorem 2 on the sequence  $\{a_{g(k_1, k_2, \dots, k_n)} : (k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n\}$  and the sequence  $\{S_{g(k_1, k_2, \dots, k_n)} : (k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n\}$ , we can find a vector  $x \in X$  such that  $\|x - x_0\| < \varepsilon$  and

$$\begin{aligned} \left\| T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x \right\| &\geq \frac{C}{q^{k_1 + k_2 + \dots + k_n}} \left\| T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} \right\| \\ &\geq \frac{C}{q^{k_1 + k_2 + \dots + k_n}} r(T_1^{k_1} T_2^{k_2} \dots T_n^{k_n}), \forall (k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n. \end{aligned} \quad (3.3)$$

If  $(k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n$  and  $p_{k_1, k_2, \dots, k_n} : \mathbb{C}^n \rightarrow \mathbb{C}$  is the polynomial defined with,

$$p_{k_1, k_2, \dots, k_n}(z_1, z_2, \dots, z_n) = z_1^{k_1} z_2^{k_2} \dots z_n^{k_n},$$

then, by Theorem 1,

$$\begin{aligned} \sigma_{\text{ap}}(T_1^{k_1} T_2^{k_2} \dots T_n^{k_n}) &= \sigma_{\text{ap}}(p_{k_1, k_2, \dots, k_n}(T_1, T_2, \dots, T_n)) \\ &= p_{k_1, k_2, \dots, k_n}(\sigma_{\pi}(T_1, T_2, \dots, T_n)) \\ &= \left\{ z_1^{k_1} z_2^{k_2} \dots z_n^{k_n} : (z_1, z_2, \dots, z_n) \in \sigma_{\pi}(T_1, T_2, \dots, T_n) \right\}. \end{aligned} \quad (3.4)$$

On the other hand, if  $p_i : \mathbb{C}^n \rightarrow \mathbb{C}$  are the polynomials defined with,

$$p_i(z_1, z_2, \dots, z_n) = z_i, \quad 1 \leq i \leq n,$$

then (again by Theorem 1),

$$p_i(\sigma_{\pi}(T_1, T_2, \dots, T_n)) = \sigma_{\pi}(p_i(T_1, T_2, \dots, T_n)) = \sigma_{\text{ap}}(T_i), \text{ for all } 1 \leq i \leq n. \quad (3.5)$$

Since  $\lambda_i \in \sigma_{\text{ap}}(T_i)$ , (3.5) implies that there are  $\mu_1^{(i)}, \dots, \mu_{i-1}^{(i)}, \mu_{i+1}^{(i)}, \dots, \mu_n^{(i)} \in \mathbb{C}$  such that,

$$(\mu_1^{(i)}, \dots, \mu_{i-1}^{(i)}, \lambda_i, \mu_{i+1}^{(i)}, \dots, \mu_n^{(i)}) \in \sigma_{\pi}(T_1, T_2, \dots, T_n).$$

Then, by (2.1), (3.3) and (3.4),

$$\begin{aligned}
& \left\| T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x \right\| \\
& \geq \frac{C}{q^{k_1+k_2+\dots+k_n}} r(T_1^{k_1} T_2^{k_2} \dots T_n^{k_n}) \\
& = \frac{C}{q^{k_1+k_2+\dots+k_n}} \max \left\{ |\lambda| : \lambda \in \sigma_{\text{ap}}(T_1^{k_1} T_2^{k_2} \dots T_n^{k_n}) \right\} \\
& = \frac{C}{q^{k_1+k_2+\dots+k_n}} \max \left\{ \left| z_1^{k_1} z_2^{k_2} \dots z_n^{k_n} \right| : (z_1, z_2, \dots, z_n) \in \sigma_{\pi}(T_1, T_2, \dots, T_n) \right\} \quad (3.6) \\
& \geq C \frac{|\lambda_i|^{k_i}}{q^{k_i}} \left( \prod_{\substack{j=1 \\ j \neq i}}^n \frac{|\mu_j^{(i)}|^{k_j}}{q^{k_j}} \right), \text{ for all } (k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n.
\end{aligned}$$

Since  $|\lambda_i| > q$ , from (3.6) we obtain that,  $\lim_{k_j \rightarrow \infty} \left\| T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x \right\| = \infty$ , for all  $k_j \in \mathbb{Z}_+$ ,  $j \neq i$ . And this holds for every  $1 \leq i \leq n$ .  $\square$

Before we state some corollaries of Theorem 4, we'll give one simple example.

**Example 3.1.** Let  $\{e_n : n \in \mathbb{N}\}$  be the canonical base of  $\ell^1 \equiv \ell^1(\mathbb{N})$  and  $B : \ell^1 \rightarrow \ell^1$  be the backward shift,

$$B e_n = \begin{cases} 0, & \text{if } n = 1 \\ e_{n-1}, & \text{if } n \geq 2 \end{cases}, \quad n \in \mathbb{N}.$$

For this operator (see, for example [5, Corollary 6.6]),

$$\sigma_p(B) = \{\lambda \in \mathbb{C} : |\lambda| < 1\},$$

$$\text{Ker}(B - \lambda) = \{\alpha(1, \lambda, \lambda^2, \dots) : \alpha \in \mathbb{C}\}, \text{ for every } \lambda \in \sigma_p(B),$$

$$\sigma(B) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\} = \sigma_{\text{ap}}(B).$$

Let  $a_1, a_2, \dots, a_n \in \mathbb{R}$  and  $\lambda_0 \in \mathbb{C}$  are such that

$$1 < |\lambda_0|^{-1} < a_1 < a_2 < \dots < a_n, \quad (3.7)$$

and let

$$T_i = a_i B, \quad 1 \leq i \leq n.$$

It can be easily verified, directly or by applying the spectral mapping theorems for the spectrum and the approximate point spectrum (the later one can be regarded as a special case of Theorem 1 for one operator and the polynomials  $p_i : \mathbb{C} \rightarrow \mathbb{C}$  defined with  $p_i(z) = a_i z$ ,  $1 \leq i \leq n$ ) that

$$\sigma(T_i) = \sigma(a_i B) = \{\lambda \in \mathbb{C} : |\lambda| \leq a_i\} = \sigma_{\text{ap}}(a_i B),$$

and

$$\sigma_p(T_i) = \sigma_p(a_i B) = \{\lambda \in \mathbb{C} : |\lambda| < a_i\},$$

for all  $1 \leq i \leq n$ .

Clearly,  $T_1, T_2, \dots, T_n$  are pairwise commuting operators. But, none of these operators is bounded below (for example,  $\|T_i e_1\| = 0 < C \|e_1\|$ , for all  $C > 0$  and  $1 \leq i \leq n$ ) and they do not satisfy the condition (P.2) (since  $r(T_i) = a_i > 1$ , if the operators satisfy (P.2), they will have the same spectral radius, which contradicts (3.7)).

Independently of Theorem 4 we will show that in every open ball in  $\ell^1$  there is a vector  $x$  such that  $\text{Orb}(\{T_i\}_{i=1}^n, x)$  tends to infinity.

Let  $y = (y_n)_{n \geq 1} \in \ell^1$  and  $\varepsilon > 0$ . By the choice of  $\lambda_0$ , there is  $n_0 = n_0(\varepsilon) \in \mathbb{N}$  such that  $\sum_{j=n_0+1}^{\infty} |y_j| < \varepsilon/3$  and  $\sum_{j=n_0+1}^{\infty} |\lambda_0|^{j-1} < \varepsilon/3$ . Let

$$x_{\lambda_0} = (y_1, \dots, y_{n_0}, \lambda_0^{n_0}, \lambda_0^{n_0+1}, \dots) = \sum_{j=1}^{n_0} y_j e_j + \sum_{j=n_0+1}^{\infty} \lambda_0^{j-1} e_j.$$

Then,

$$\|y - x_{\lambda_0}\| = \sum_{j=n_0+1}^{\infty} |y_j - \lambda_0^{j-1}| \leq \sum_{j=n_0+1}^{\infty} |y_j| + \sum_{j=n_0+1}^{\infty} |\lambda_0|^{j-1} < \varepsilon,$$

and, if  $(k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n$  is such that  $k_1 + k_2 + \dots + k_n \geq n_0$ ,

$$\|T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x_{\lambda_0}\| = \|a_1^{k_1} a_2^{k_2} \dots a_n^{k_n} B^{k_1+k_2+\dots+k_n} x_{\lambda_0}\| = a_1^{k_1} a_2^{k_2} \dots a_n^{k_n} \frac{|\lambda_0|^{k_1+k_2+\dots+k_n}}{1 - |\lambda_0|}.$$

Since (3.7) implies that  $a_i |\lambda_0| > 1$ , for all  $1 \leq i \leq n$ , we have

$$\begin{aligned} \lim_{k_i \rightarrow \infty} \|T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x_{\lambda_0}\| &= \left[ \frac{1}{1-|\lambda_0|} \prod_{\substack{j=1 \\ j \neq i}}^n (a_j |\lambda_0|)^{k_j} \right] \lim_{k_i \rightarrow \infty} (a_i |\lambda_0|)^{k_i} \\ &= \infty, \text{ for all } k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_n \in \mathbb{Z}_+. \end{aligned}$$

**Remark 3.1:** For the vector  $x_{\lambda_0}$  in the previous example  $\text{Orb}(a_i B, x_{\lambda_0})$  tends to infinity for every  $1 \leq i \leq n$ . But the operators  $a_1 B, a_2 B, \dots, a_n B$  do not share the same set of vectors such that each one of them has an orbit tending to infinity under each of the operators. For example, if  $\mu \in \mathbb{C}$  is such that  $a_1 \leq |\mu|^{-1} < a_2$ , and  $x_{\mu}$  is the vector constructed in a similar way as  $x_{\lambda_0}$ , i.e.

$$x_{\mu} = (y_1, \dots, y_{n_1}, \mu^{n_1}, \mu^{n_1+1}, \dots) = \sum_{j=1}^{n_1} y_j e_j + \sum_{j=n_1+1}^{\infty} \mu^{j-1} e_j,$$

for some sufficiently large  $n_1$ , then  $\text{Orb}(\{a_i B\}_{i=2}^n, x_{\mu})$  and, consequently  $\text{Orb}(a_i B, x_{\mu})$ , will tend to infinity, for each  $2 \leq i \leq n$ . But  $\text{Orb}(a_1 B, x)$  does not tend to infinity:

$$\|T_1^{k_1} x_{\mu}\| = a_1^{k_1} \|B^{k_1} x_{\mu}\| = \frac{a_1^{k_1} |\mu|^{k_1}}{1 - |\mu|}, \text{ for all } k_1 > n_1,$$

and consequently, since  $a_1 \leq |\mu|^{-1}$ ,

$$\lim_{k_1 \rightarrow \infty} \|T_1^{k_1} x_{\mu}\| = \lim_{k_1 \rightarrow \infty} \frac{a_1^{k_1} |\mu|^{k_1}}{1 - |\mu|} = \begin{cases} \frac{1}{1-|\mu|}, & \text{if } a_1 = |\mu|^{-1} \\ 0, & \text{if } a_1 < |\mu|^{-1}. \end{cases}$$

**Remark 3.2:**  $T \in B(X)$  is *hypercyclic operator* if there is a vector  $x \in X$  such that  $\text{Orb}(T, x)$  is dense in  $X$ . The vector  $x$  with this property is said to be *hypercyclic vector* for  $T$ . If  $T$  is hypercyclic operator, then the set of all hypercyclic vectors for  $T$  is dense  $G_{\delta}$  set in  $X$  ([2, Lemma III.5.1], [13, Theorem V.38.2]). By definition, the  $n$ -tuple of pairwise commuting operators  $\mathbf{T} = (T_1, T_2, \dots, T_n)$  is a *hypercyclic*

$n$ -tuple if there is a vector  $x \in X$  such that  $\text{Orb}(\{T_i\}_{i=1}^n, x)$  is dense in  $X$  ([6]). If at least one of the operators  $T_1, T_2, \dots, T_n$  is hypercyclic, or the semigroup generated by  $T_1, T_2, \dots, T_n$  i.e.,  $\mathcal{T} = \left\{ T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} : (k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n \right\}$ , contains a hypercyclic operator  $S$  (which may occur even if none of the operators  $T_1, T_2, \dots, T_n$  is hypercyclic, a simple example will be  $\mathbf{T} = (2I, 2^{-1}B)$ , where  $I$  is the identity operator,  $B$  the backward shift on  $\ell^1$  and  $S = (2I)^2 \cdot (2^{-1}B) = 2B$ ), then the  $n$ -tuple  $\mathbf{T} = (T_1, T_2, \dots, T_n)$  is hypercyclic ([6, Proposition 2.1]). By [14, Theorem 1], each of the operators  $T_i = a_i B$ ,  $1 \leq i \leq n$ , in Example 3.1 hypercyclic. Hence  $\mathbf{T} = (T_1, T_2, \dots, T_n)$  is a hypercyclic  $n$ -tuple and  $\ell^1$  will contain at least one dense  $G_\delta$  set of vectors  $x$  such that,

$$\text{Orb}(\{a_i B\}_{i=1}^n, x) = \left\{ a_1^{k_1} a_2^{k_2} \dots a_n^{k_n} B^{k_1+k_2+\dots+k_n} x : (k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n \right\},$$

is dense in  $\ell^1$ .

**Corollary 4.1.** *If  $\mathbf{T} = (T_1, T_2, \dots, T_n)$  is an  $n$ -tuple of pairwise commuting operators on an infinite dimensional complex Banach space  $X$  such that  $r(T_i) > 1$  for all  $1 \leq i \leq n$ , then there is a dense set  $D_1^* \subset X^*$  such that the  $n$ -tuple orbit  $\text{Orb}(\{T_i^*\}_{i=1}^n, x^*)$  tends to infinity for every  $x^* \in D_1^*$ .*

*Proof.* If  $T_1, T_2, \dots, T_n$  are pairwise commuting operators on  $X$ , so will be their Banach space adjoints  $T_1^*, T_2^*, \dots, T_n^* \in B(X^*)$ . Having in mind that  $T^*$  has the same spectrum as  $T$  and hence,  $r(T^*) = r(T)$ , the conclusion follows by Theorem 4.  $\square$

**Corollary 4.2.** *If  $\mathbf{T} = (T_1, T_2, \dots, T_n)$  is an  $n$ -tuple of pairwise commuting invertible operators on an infinite dimensional complex Banach space  $X$  such that,*

$$\{\lambda \in \mathbb{C} : |\lambda| > 1\} \cap \sigma(T_i) \neq \emptyset \neq \{\lambda \in \mathbb{C} : |\lambda| < 1\} \cap \sigma(T_i), \quad (3.8)$$

*for all  $1 \leq i \leq n$ , then there is a dense set  $D_1^{(1)} \subset X$  such that the  $2n$ -tuple orbit  $\text{Orb}(\{T_i\}_{i=1}^n \cup \{T_i^{-1}\}_{i=1}^n, x)$  tends to infinity, for every  $x \in D_1^{(1)}$ .*

*Proof.* If  $T_1, T_2, \dots, T_n$  are pairwise commuting invertible operators on  $X$ , then  $T_1^{-1}, T_2^{-1}, \dots, T_n^{-1}$  will also pairwise commute and

$$T_i T_j^{-1} = T_j^{-1} T_i T_j T_j^{-1} = T_j^{-1} T_i T_j T_j^{-1} = T_j^{-1} T_i,$$

for all  $i, j \in \{1, 2, \dots, n\}$ . Since  $\sigma(T^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(T)\}$  for every invertible operator  $T \in B(X)$ , if  $T_1, T_2, \dots, T_n$  satisfy the conditions in (3.8), then  $r(T_i) > 1$  and  $r(T_i^{-1}) > 1$ , for all  $1 \leq i \leq n$ , and the conclusion follows from Theorem 4.  $\square$

**Remark 3.3:** Every invertible operator  $T \in B(X)$  is bounded below:

$$\|Tx\| \geq \left\| T^{-1} \right\|^{-1} \|x\|, \text{ for every } x \in X,$$

Hence, if  $T_1, T_2, \dots, T_n$  are pairwise commuting invertible operators, then the operators  $T_1, T_2, \dots, T_n, T_1^{-1}, T_2^{-1}, \dots, T_n^{-1}$  will satisfy the condition (P.1). If, in addition,

the operators satisfy the conditions in (3.8), then the conclusion in Corollary 4.2 can be derived from [7, Theorem 2.2].

In the next two corollaries we assume that  $T^*$  denotes the Hilbert space adjoint of the operator  $T$ .

**Corollary 4.3.** *If  $\mathbf{T} = (T_1, T_2, \dots, T_n)$  is an  $n$ -tuple of pairwise commuting operators on an infinite dimensional complex Hilbert space  $H$  such that  $r(T_i) > 1$  for all  $1 \leq i \leq n$ , then there is a dense set  $D_1^{(2)} \subset H$  such that the  $n$ -tuple orbit  $\text{Orb}(\{T_i^*\}_{i=1}^n, x)$  tends to infinity for every  $x \in D_1^{(2)}$ .*

*Proof.* If  $T_1, T_2, \dots, T_n$  are pairwise commuting operators on  $H$ , then the corresponding Hilbert space adjoints  $T_1^*, T_2^*, \dots, T_n^* \in B(H)$  will also commute pairwise. Since the spectrum of a Hilbert space adjoint  $T^*$  of an operator  $T \in B(H)$  satisfies  $\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\}$  and hence,  $r(T^*) = r(T)$ , the conclusion follows by Theorem 4.  $\square$

**Corollary 4.4.** *If  $\mathbf{T} = (T_1, T_2, \dots, T_n)$  is an  $n$ -tuple of pairwise commuting normal operators on an infinite dimensional complex Hilbert space  $H$  such that  $r(T_i) > 1$  for all  $1 \leq i \leq n$ , then there is a dense set  $D_1^{(3)} \subset H$  such that the  $2n$ -tuple orbit  $\text{Orb}(\{T_i\}_{i=1}^n \cup \{T_i^*\}_{i=1}^n, x)$  tends to infinity for every  $x \in D_1^{(3)}$ .*

*Proof.* If  $T_1, T_2, \dots, T_n$  are pairwise commuting normal operators on  $H$  then, by the Fuglede-Putnam theorem  $T_1, T_2, \dots, T_n, T_1^*, T_2^*, \dots, T_n^*$  will be pairwise commuting normal operators on  $X$ . Since  $r(T^*) = \|T^*\| = \|T\| = r(T)$  for every normal operator  $T \in B(H)$ , the conclusion follows from Theorem 4.  $\square$

#### 4. N-TUPLE WEAK ORBITS TENDING TO INFINITY

In the section we are going to give only the corresponding result of Theorem 4 for  $n$ -tuple weak orbits.

**Theorem 5.** *If  $\mathbf{T} = (T_1, T_2, \dots, T_n)$  is an  $n$ -tuple of pairwise commuting operators on an infinite dimensional complex Banach space  $X$  such that  $r(T_i) > 1$  for all  $1 \leq i \leq n$ , then there is a dense set  $D_2 \subset X \times X^*$  such that the  $n$ -tuple weak orbit  $\text{Orb}(\{T_i\}_{i=1}^n, x, x^*)$  tends to infinity for every  $(x, x^*) \in D_2$ .*

*Proof.* Let  $(x_0, x_0^*) \in X \times X^*$  and  $\varepsilon > 0$ . If  $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$  are as in the proof of Theorem 4, let  $q \in \mathbb{R}$  and  $C > 0$  are such that,

$$1 < q < q^2 < \min\{|\lambda_1|, |\lambda_2|, \dots, |\lambda_n|\},$$

and

$$C \left( \frac{q}{q-1} \right)^n < \frac{\varepsilon}{2^{1/p}},$$

assuming that  $p = \infty$  if the norm on  $X \times X^*$  is the max-norm. Now, let

$$a_{g(k_1, k_2, \dots, k_n)} = \frac{C^2}{q^{2(k_1 + k_2 + \dots + k_n)}} > 0, \text{ for } (k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n,$$

where  $g : \mathbb{Z}_+^n \rightarrow \mathbb{Z}_+$  is as in the proof of Theorem 4. Then

$$\sum_{(k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n} a_{g(k_1, k_2, \dots, k_n)}^{1/2} = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \dots \sum_{k_n=0}^{\infty} \frac{C}{q^{k_1+k_2+\dots+k_n}} = C \left( \frac{q}{q-1} \right)^n < \frac{\varepsilon}{2^{1/p}},$$

and, by Theorem 3, there are  $x \in X$  and  $x^* \in X^*$  such that,

$$\|x - x_0\| < \frac{\varepsilon}{2^{1/p}}, \quad \|x^* - x_0^*\| < \frac{\varepsilon}{2^{1/p}}, \quad (4.1)$$

and

$$\left| \left\langle T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x, x^* \right\rangle \right| \geq \frac{C^2}{q^{2(k_1+k_2+\dots+k_n)}} \left\| T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} \right\|, \quad (4.2)$$

for all  $(k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n$ . By (4.1), in both cases,  $1 \leq p < \infty$  and  $p = \infty$ , we have

$$\|(x, x^*) - (x_0, x_0^*)\|_p = \|(x - x_0, x^* - x_0^*)\|_p < \varepsilon,$$

and, if  $\mu_1^{(i)}, \dots, \mu_{i-1}^{(i)}, \mu_{i+1}^{(i)}, \dots, \mu_n^{(i)} \in \mathbb{C}$  are as in the proof of Theorem 4, by (4.2) we have

$$\left| \left\langle T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x, x^* \right\rangle \right| \geq C \frac{|\lambda_i|^{k_i}}{q^{2k_i}} \left( \prod_{\substack{j=1 \\ j \neq i}}^n \frac{|\mu_j^{(i)}|^{k_j}}{q^{2k_j}} \right) \quad (4.3)$$

for all  $(k_1, k_2, \dots, k_n) \in \mathbb{Z}_+^n$ . Since  $|\lambda_i| > q^2$ , from (4.3) we obtain that, for every  $1 \leq i \leq n$ ,  $\lim_{k_i \rightarrow \infty} \left| \left\langle T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x, x^* \right\rangle \right| = \infty$ , for all  $k_j \in \mathbb{Z}_+$ ,  $j \neq i$ .  $\square$

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