

SOME CLASSIFICATIONS OF SUBRINGS OF THE RING OF CONTINUOUS FUNCTIONS

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Abstract. In this work, for a topological space (X, τ) and the ring of functions $C(X, \tau) = \{f \mid f : (X, \tau) \rightarrow (\mathbb{R}, \tau_{st}) \text{ and } f \text{ is continuous}\}$, where τ_{st} is the standard topology on the real line, for a closed, sublattice and subring \mathcal{A} of $C(X, \tau)$ we investigate sufficient conditions under which is not possible to find a topology $\tau' \subset \tau$ on X such that $\mathcal{A} = C(X, \tau')$.

1. INTRODUCTION

Let \mathbb{R}^X is the collection of all real valued functions on a set X . Let f and g be functions defined on a set X , and let $(f + g)(x) = f(x) + g(x)$ and $(f \cdot g)(x) = f(x) \cdot g(x)$ for all $x \in X$. Then \mathbb{R}^X is a ring under these operations of addition and multiplication. Let X is non empty set and τ be a topology on it. We denote the ring of real valued continuous functions from X to \mathbb{R} by $C(X, \tau)$ and the ring of bounded, real valued, continuous functions from X to \mathbb{R} by $C^*(X, \tau)$. We simply write $C(X)$, (resp. $C^*(X)$) instead of $C(X, \tau)$, (resp. $C^*(X, \tau)$) in case there is no change for confusion. The partial ordering on \mathbb{R}^X is defined by; $f \geq g$ if and only if $f(x) \geq g(x)$ for all $x \in X$. For any f, g in \mathbb{R}^X , $f \vee g$ and $f \wedge g$ exist in \mathbb{R}^X . In fact $(f \vee g)(x) = \max(f(x), g(x))$ for all $x \in X$ and $(f \wedge g)(x) = \min(f(x), g(x))$ for all $x \in X$. Thus \mathbb{R}^X is lattice-ordered ring. For every $f \in \mathbb{R}^X$, since $|f|(x) = \max(f(x), -f(x)) = |f(x)|$ for all $x \in X$, that $|f| = f \vee (-f) \in \mathbb{R}^X$. Since $(f \vee g)(x) = \max\{f(x), g(x)\} = \frac{1}{2}[f(x) + g(x) + |f(x) - g(x)|]$, $f, g \in C(X)$ that $f \vee g \in C(X)$. Similarly, $f \wedge g \in C(X)$ is obtained. Therefore $C(X)$ is a subalgebra and sublattice of \mathbb{R}^X . We know from [3] $C^*(X)$ is a subalgebra and sublattice of $C(X)$. If (\mathbb{R}, d) is a metric space (bounded metric) and X is a topological space, then $\bar{d}(f, g) = \sup\{d(f(x), g(x)), x \in X\}$ determines a metric on \mathbb{R}^X . The topology determined by this metric is the uniform convergence topology. It is clear that, with this topology, $C(X)$ is closed subring of \mathbb{R}^X .

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Let X be any set and \mathcal{A} be a family of functions from X to \mathbb{R} . We will investigate the necessary and sufficient conditions for a $\tau' \subset \tau$ topology to exist such that $\mathcal{A} = C(X, \tau') \subset C(X, \tau)$.

In this article for a closed, sublattice and subring \mathcal{A} of $C(X, \tau)$ we examine the existence of a topology τ' such that $\mathcal{A} = C(X, \tau')$. In the second section, we give some basic definitions and theorems. In the third section, by defining a special subring, we show some basic properties of this subring. In fourth section we found sufficient conditions under which it is not possible to find a topology $\tau' \subset \tau$ on X such that $\mathcal{A} = C(X, \tau')$.

2. PRELIMINARIES

Throughout this paper, we will define φ as follows, let $i : X \rightarrow Y$ be continuous and the map $\varphi : C(Y) \rightarrow C(X)$, $f \in C(Y)$ with $\varphi(f) = f \circ i$ is a ring homomorphism.

Theorem 1. *If $\varphi : C(Y) \rightarrow C(X)$ is a ring homomorphism, then φ is the lattice homomorphism.*

Proof. The proof is trivial. □

Theorem 2. *If $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ is continuous, $\varphi : C(Y) \rightarrow C(X)$ is ring homomorphism and $f(X)$ is dense subset of Y , then φ is one to one.*

Proof. The proof is trivial. □

Definition 2.1. [4, 5] Let A be a nonempty subset of topological space (X, τ) . The collection $\tau|_A = \{A \cap U : U \in \tau\}$ of subsets of A is a topology on A called the *subspace topology*. The topological space (A, τ_A) is said to be a *subspace* of (X, τ) .

Let (X, τ) is a topological space and $A \subseteq X$. Then the closure of A is denoted by \bar{A} .

Let (X, τ) be a topological space and $x \in X$. Then the open neighborhood system at x is denoted by $V(x) = \{V \in \tau : x \in V\}$.

3. CHARACTERISTICS OF THE SPECIAL SUBRING

In this section, we will investigate and identify the subring of the ring of continuous functions $C(X)$ that possess the properties of being a subalgebra, sublattice, and closed.

Lemma 1. *Let Y is bounded metric space, $X \subset Y$ and $i : X \rightarrow Y$ be continuous. If $\overline{i(X)} = Y$ and $\varphi(C(Y)) \subset C(X)$, then $\varphi(C(Y))$ is closed in $C(X)$*

Proof. Firstly we show that φ is isometric. We know that

$$\overline{d_X(foi, goi)} = \sup\{d((foi)(x), (goi)(x)), x \in X\}$$

since

$$\{d((foi)(x), (goi)(x)), x \in X\} \subseteq \{d(f(y), g(y)), y \in Y\}$$

then

$$\sup\{d((foi)(x), (goi)(x)), x \in X\} \leq \sup\{d(f(y), g(y)), y \in Y\}$$

Now let's take $h = |f - g| \wedge 1$, here $\overline{d}_Y(f, g) = \sup\{h(y), y \in Y\}$ and so

$$\overline{d}_X((foi)(x), (goi)(x), x \in X) = \sup\{(hoi)(x), x \in X\}$$

Suppose that

$$\sup\{(hoi)(x), x \in X\} < \sup\{h(y) : y \in Y\}$$

Let $\sup\{(hoi)(x), x \in X\} = a$. Then, since

$$\overline{d}_Y(f, g) = \sup\{h(y), y \in Y\} > a$$

there must exist at least one element, y_0 , in the set Y such that $h(y_0) > a$. Since the function $h : Y \rightarrow \mathbb{R}$ is continuous at the point $y_0 \in Y$, then for each open set V containing $h(y_0)$ there exist an open set U containing Y such that $h(U) \subset V$. Since $i : X \rightarrow Y$ is continuous and $\overline{i(X)} = Y$, then for every open set $U \subset Y$, $U \cap i(X) \neq \emptyset$. Let's take a point $y' \in U \cap i(X)$ and $V = (a, +\infty)$. Since $h(U) \subset V$, then $h(y') > a$ for $y' \in U$. Since $i : X \rightarrow Y$ is continuous, then there is a $x' \in X$ such that $i(x') = y'$. Since

$$\sup\{(hoi)(x'), x' \in X\} = a$$

then $(hoi)(x') \leq a$. However, this contradicts the fact that $h(y') > a$, as $(hoi)(x') = h(i(x')) = h(y')$. Thus, it is shown that

$$\sup\{h(y) : y \in Y\} = \sup\{(hoi)(x), x \in X\}$$

In other words,

$$\overline{d}_Y(f, g) = \overline{d}_X(foi, goi)$$

and so φ is an isometry.

Now let's show that $\varphi(C(Y))$ is closed. Let's take a sequence $(g_n) \in \varphi(C(Y))$. Since $g_n \rightarrow g$ is a convergent sequence, then it is a Cauchy sequence. Then there exists an $n \in \mathbb{N}$ such that $\overline{d}_X(g_m, g_n) < \varepsilon$ for every $\varepsilon > 0$ and every $m, n > n_0$. Also, since $(f_n oi) = (g_n)$ and $(f_m oi) = (g_m)$, then $\overline{d}_X(f_m oi, f_n oi) < \varepsilon$. Due to the isometry, $\overline{d}_Y(f_m, f_n) < \varepsilon$ is provided. Then, (f_n) is a Cauchy sequence. Since $C(Y)$ is complete, then there is a $f \in C(Y)$ such that $f_n \rightarrow f$. Because of the isometry,

$$f_n \rightarrow f \Leftrightarrow \overline{d}_Y(f_n, f) \rightarrow 0$$

and as a result $\overline{d}_X(f_n oi, foi) \rightarrow 0$. Since $f_n oi \rightarrow foi$ and $(f_n oi) = (g_n)$, then $g_n \rightarrow foi$. Simultaneously, since $g_n \rightarrow g$, then $g = foi$. Thus, $\varphi(f) = foi = g$, $g \in \varphi(C(Y))$. This show that $\varphi(C(Y))$ is closed. \square

Theorem 3. *Let X and Y be two topological spaces, and let $i : X \rightarrow Y$ be a continuous function such that $i(X)$ is a dense subset of Y and $Y \neq i(X)$. Additionally, let*

$$i : X \rightarrow i(X)$$

be a homeomorphism and let

$$I = \{f \circ i : f(y) = 0, \forall y \in (Y - i(X))\} \subseteq \varphi(C(Y)) \subseteq C(X)$$

then,

- (1) $I \cap C^*(X)$ is ideal of $C^*(X)$.
- (2) If $g \in I, h \in C^*(X)$ then $g.h \in I$.

Proof.

(1) Let take $f \circ i \in I \cap C^*(X)$ and $g \in C^*(X)$. Assume that $(f \circ i).g = h$. We defined as

$$\bar{h}(y) = \begin{cases} 0, & y \in Y - i(X) \\ ((f \circ i).g)(x), & i(x) = y, y \in i(X) \end{cases}$$

Let us show the continuity of the \bar{h} for three different cases.

Case 1 Let $y_0 \in Y - i(X)$. Then, $\bar{h}(y_0) = 0$. Let's take $(-\varepsilon, \varepsilon)$ as a neighborhood of 0 in \mathbb{R} for every $\varepsilon > 0$. Since $g \in C^*(X)$, there exists a $r \in \mathbb{R}$ such that $r > 0$ and $|g(x)| \leq r$ for all $x \in X$. Simultaneously, since the function f is continuous, there exists a neighborhood U of y_0 such that $f(U) \subseteq (\frac{-\varepsilon}{r}, \frac{\varepsilon}{r})$. For $y \in U$, if $y \in i(X)$, then $\bar{h}(y) = ((f \circ i).g)(x)$, where $i(x) = y$, and thus $f(y) \cdot g(x) \in (-\varepsilon, \varepsilon)$. Therefore, $\bar{h}(U) \subseteq (-\varepsilon, \varepsilon)$ becomes true. Thus, \bar{h} is continuous.

Case 2 Let $y_0 \in i(X)$ and $f(y_0) = 0$. Since $g \in C^*(X)$, there exists a $r \in \mathbb{R}$ such that $r > 0$ and $|g(x)| \leq r$ for all $x \in X$. Now let's show that $\bar{h}(U) \subseteq (-\varepsilon, \varepsilon)$. For $y_0 \in U$, if $y_0 \in Y - i(X)$, then $\bar{h}(y_0) = 0 \in (-\varepsilon, \varepsilon)$. For $y_0 \in U$, if $y_0 \in i(X)$, then $\bar{h}(y_0) = ((f \circ i).g)(x)$ and $i(x) = y_0$, and thus $f(y_0) \cdot g(x) \in (-\varepsilon, \varepsilon)$. Therefore, $\bar{h}(U) \subseteq (-\varepsilon, \varepsilon)$ holds true. Thus, the function \bar{h} is continuous at the point y_0 .

Case 3 Let $f(y_0) \neq 0$. Assume that $f(y_0) = m$. If $i : X \rightarrow i(X)$ is a bijective homeomorphism and $i(x_0) = y_0$, and $g(x_0) = n$, then $\bar{h}(y_0) = ((f \circ i).g)(x) = f(y).g(x) = m.n$ holds true. Now we can show that function \bar{h} is continuous at point y_0 . Since $g \in C^*(X)$, there exists a $r \in \mathbb{R}$ such that $r > 0$ and $|g(x)| \leq r$ for all $x \in X$. Additionally, since the function g is continuous at point x_0 , there exists a neighborhood of x_0 called V such that $g(V) \subseteq \left(n - \frac{\varepsilon}{2(|m|+1)}, n + \frac{\varepsilon}{2(|m|+1)}\right)$ for each $\varepsilon > 0$. Simultaneously, since the function f is also continuous at the point y_0 , then $y_0 \in i(X)$ and there is an open set $U \subseteq Y$ such that $|m| < \frac{\varepsilon}{r}$, $f(U) \subseteq \left(m - \frac{\varepsilon}{r}, m + \frac{\varepsilon}{r}\right)$ and $0 \notin \left(m - \frac{\varepsilon}{r}, m + \frac{\varepsilon}{r}\right)$. Therefore, $y_0 \in U$. Now we show that for any $\varepsilon > 0$, $\bar{h}(U) \subseteq (m.n - \varepsilon, m.n + \varepsilon)$. Since i is continuous, then for $x_0 \in X$ there is a open set $U \subseteq i(X)$ such that $i(x_0) = y_0 \in U$. We know that the function $i^{-1} : i(X) \rightarrow X$ is also continuous. Let's take a W neighborhood of the point y_0 such that $i^{-1}(W) = M$. If $W = U$ is chosen, then $i(M \cap V) = i(M) \cap i(V) = U \cap i(V) = N$ and $N \subseteq U$ is an open neighborhood of the point y_0 . Let's take a $y \in N$. Since $N \subseteq i(M \cap V)$, there is a $x \in M \cap V$ such that $i(x) = y$. Then, since $f(U) \subseteq \left(m - \frac{\varepsilon}{2r}, m + \frac{\varepsilon}{2r}\right)$, then $f(y) \in \left(m - \frac{\varepsilon}{2r}, m + \frac{\varepsilon}{2r}\right)$. Since $g(x) \in \left(n - \frac{\varepsilon}{2(|m|+1)}, n + \frac{\varepsilon}{2(|m|+1)}\right)$ then,

$$\begin{aligned} |g(x).f(y) - m.n| &= |g(x).(f(y) - m) + (g(x) - n).m| \\ &\leq |g(x)| \cdot |f(y) - m| + |g(x) - n| |m| \end{aligned}$$

$$\begin{aligned}
&< r \cdot |f(y) - m| + |g(x) - n| \cdot (|m| + 1) \\
&\leq r \cdot \frac{\varepsilon}{2r} + \frac{\varepsilon}{2 \cdot (|m| + 1)} \cdot (|m| + 1) \\
&= \varepsilon
\end{aligned}$$

Thus, $\overline{h}(N) \subseteq (m.n - \varepsilon, m.n + \varepsilon)$ and so \overline{h} is continuous.

Now we show that N is a open subset of Y . We know $N = U \cap i(V)$ and i is a homomorphism. If $|m| < \frac{\varepsilon}{r}$, then there exists an open set $U \subseteq Y$, with $U \subseteq i(X)$, such that the image of f over U is contained in the interval $(m - \frac{\varepsilon}{r}, m + \frac{\varepsilon}{r})$ for any point $y_0 \in U$. For $i(x_0) = y_0 \in U$ since both $i^{-1}(U)$ and $V \subseteq X$ are an open neighborhood of the point x_0 , then $V \cap i^{-1}(U)$ is also an open neighborhood of point x_0 . Now, let's take an image of $V \cap i^{-1}(U)$.

$$\begin{aligned}
&i(V \cap i^{-1}(U)) \\
&= i(V) \cap i(i^{-1}(U)) \\
&= i(V) \cap U \\
&= (i(V) \cap U) \\
&\subseteq U^{open} \\
&\subseteq Y
\end{aligned}$$

Thus, $i(V) \cap U$ is open subset of Y .

(2) Since the boundedness of the function $f \circ i$ was not used in (1), the proof is the same as before. \square

Example 1. Let X be a locally compact but non-compact space, and let X^t be a one-point compactification of X . Let $\varphi(C(X^t))$ be defined as follows:

$$\varphi(C(X^t)) = \{g : g \in C(X), \exists L \in \mathbb{R} \text{ such that } \forall \varepsilon > 0, g(X - K) \subseteq (L - \varepsilon, L + \varepsilon),$$

for some compact set $K \subset X$.

Then:

- (1) $I = \{\varphi(f) : f(\infty) = 0\}$ is ideal of $\varphi(C(X^t))$.
- (2) I is an ideal of $C^*(X)$.

Theorem 4. Let X and Y be two topological space, $i : X \rightarrow Y$ is continuous and $i(X)$ is a dense subset of Y . If $Y \neq i(X)$, $i : X \rightarrow i(X)$ is a homeomorphism and $I = \{f \circ i : f(y) = 0, \forall y \in (Y - i(X))\} \subseteq \varphi(C(Y)) \subseteq C(X)$, then I is closed in $C(X)$.

Proof. Let's take a sequence $(g_n) \in I$. Since $g_n \rightarrow g$ is a convergent sequence, then it is a Cauchy sequence. Then, there is a $n \in \mathbb{N}$ such that $\overline{d}_X(g_m, g_n) < \varepsilon$ for every $\varepsilon > 0$ and every $m, n > n_0$. Since $(f_n \circ i) = (g_n)$ and $(f_m \circ i) = (g_m)$, then $\overline{d}_X(f_m \circ i, f_n \circ i) < \varepsilon$. Because of the isometry in Lemma 1, the inequality $\overline{d}_Y(f_m, f_n) < \varepsilon$ holds. Thus, (f_n) is a Cauchy sequence. Since $C(Y)$ is complete, then there is a $f \in C(Y)$ such that $f_n \rightarrow f$. Since $(f_n \circ i) \in I$, it follows that $f_n(y) = 0$ for all $y \in Y - i(X)$. Thus $f_n \rightarrow f \Leftrightarrow \overline{d}_Y(f_n, f) \rightarrow 0$. Conversely, suppose that $f(y) = m$, where $m \neq 0$ and there exists $y \in Y$ such that $y \notin i(X)$.

Then, we have $|f_n(y) - f(y)| = |0 - m| = |-m|$. If we choose $\varepsilon = \frac{|m|}{2}$, the inequality $|f_n(y) - f(y)| < \varepsilon$ is not satisfied. As a result, $f(y) = 0$ for all $y \in Y$ such that $y \notin i(X)$. Because of the isometry, $\bar{d}_x(f_n \circ i, f \circ i) \rightarrow 0$. Since $f_n \circ i \rightarrow f \circ i$ and $(f_n \circ i) = (g_n)$, we have that $g_n \rightarrow f \circ i$. Simultaneously, since $g_n \rightarrow g$, we have that $g = f \circ i$. As $f \circ i \in I$, it follows that g also belongs to I . Hence, I is a closed subspace of $C(X)$. \square

Theorem 5. *Let X and Y be two topological spaces, $i : X \rightarrow Y$ is continuous and $i(X)$ is a dense subset of Y . If $Y \neq i(X)$, $i : X \rightarrow i(X)$ is a homeomorphism and $I = \{f \circ i : f(y) = 0, \forall y \in (Y - i(X))\} \subseteq \varphi(C(Y)) \subseteq C(X)$, then I is sublattice of $C(X)$.*

Proof. Let take $f \circ i, g \circ i \in I$.

$$(f \circ i) \vee (g \circ i) = \frac{1}{2} ((f \circ i) + (g \circ i) + |(f \circ i) - (g \circ i)|)$$

We know that for every $y \in Y - i(X)$, $f(y) = g(y) = 0$. Since

$$(f \circ i)(x) + (g \circ i)(x) = ((f + g) \circ i)(x)$$

and

$$(f \circ i)(x) - (g \circ i)(x) = ((f - g) \circ i)(x)$$

then $(f + g)(y) = 0$ and $(f - g)(y) = 0$ for every $y \in Y - i(X)$. Thus,

$$(f \circ i) \vee (g \circ i) \in I$$

Similarly, we can show that $(f \circ i) \wedge (g \circ i) \in I$. Hence, I is sub-lattice of $C(X)$. \square

Lemma 2. *If $C(X, \tau) - C(X, \tau') \neq \emptyset$ then $C^*(X, \tau) - C^*(X, \tau') \neq \emptyset$*

Proof. Let $f \in C(X, \tau) - C(X, \tau')$. There is at least one point $x_0 \in X$ such that f is not $\tau' - \tau_{st}$ -continuous at this point. Let $f(x_0) = a$, $a \in \mathbb{R}$. For every neighborhood U of x_0 , there exists $\varepsilon_0 > 0$ such that $f(U) \not\subset (a - \varepsilon_0, a + \varepsilon_0)$. By defining

$$g = (f \vee (a - \varepsilon_0)) \wedge (a + \varepsilon_0)$$

the function g becomes $\tau' - \tau_{st}$ discontinuous and bounded at the point x_0 . Now, let's show this. Suppose that the function g is $\tau' - \tau_{st}$ -continuous at the point x_0 . Let us set $g(x_0) = a$ and consider $(a - \varepsilon_0, a + \varepsilon_0)$ as a neighborhood of a . Then, there exists a neighborhood U_1 of x_0 such that $g(U_1) \subset (a - \varepsilon_0, a + \varepsilon_0)$. Thus,

$$\begin{aligned} g(U_1) &= (f(U_1) \vee (a - \varepsilon_0)) \wedge (a + \varepsilon_0) \\ &\subset (a - \varepsilon_0, a + \varepsilon_0). \end{aligned}$$

For every $x \in U_1$,

$$(f(x) \vee (a - \varepsilon_0)) \wedge (a + \varepsilon_0) \in (a - \varepsilon_0, a + \varepsilon_0)$$

and so

$$\min \{ \max \{ (f(x), (a - \varepsilon_0)) \}, (a + \varepsilon_0) \} \in (a - \varepsilon_0, a + \varepsilon_0)$$

This can only be achieved by ensuring;

$$\max \{ (f(x), (a - \varepsilon_0)) \} < (a + \varepsilon_0)$$

and

$$\max \{(f(x), (a - \varepsilon_0))\} > (a - \varepsilon_0)$$

From this, we conclude that $f(x) < a + \varepsilon_0$ and $f(x) > a - \varepsilon_0$, which implies $f(x) \in (a - \varepsilon_0, a + \varepsilon_0)$ for all x . Hence, we reach a contradiction because $f(U_1) \subset (a - \varepsilon_0, a + \varepsilon_0)$ for every $x \in U_1$. \square

4. *-IDEAL

In this section, we will define a specific ideal and establish the conditions under which it is not possible to find a topology $\tau' \subset \tau$ such that $\mathcal{A} = C(X, \tau')$, by using this ideal.

Definition 4.1. Let $I \subset C(X)$ be a subalgebra, sub-lattice and closed. If for every $f \in I$ and $g \in C^*(X)$, $f, g \in I$, then I is called an **-ideal*.

Definition 4.2. For a function $f : A \rightarrow B$ to be an into function there will be one or more elements in set B that do not have a pre-image in set A.

Lemma 3. If X is a proper open subset of Y , $i : X \rightarrow Y$ is an into function, and Y is a T_4 space, then there exists a bounded function $f \in C(Y)$ such that $f(x) \neq 0$ for all $x \in X$ and $I = \{f \circ i : f \in C(Y), f(Y - X) = 0\}$.

Proof. Let $A = \{x\}$ be a singleton set for a $x \in X$. A and $Y - X$ are closed. Furthermore, let $g : (Y - X) \cup A \rightarrow \mathbb{R}$ be defined by

$$g(t) = \begin{cases} 1, & t = x \\ 0, & t \in Y - X \end{cases}$$

Now let us show that g is continuous. Let F be a closed subset of \mathbb{R} . Since

$$g^{-1}(F) = \begin{cases} A \cup (Y - X), & 0, 1 \in F \\ Y - X, & 0 \in F, 1 \notin F \\ A, & 0 \notin F, 1 \in F \\ \emptyset, & 0, 1 \notin F \end{cases}$$

then $g^{-1}(F)$ is closed. Thus g is continuous. Since the function g is bounded and $A \cup (Y - X)$ is closed, then according to Tietze's Expansion Theorem g has a continuous and bounded expansion of $f : Y \rightarrow \mathbb{R}$. Thus, for every $x \in X$ there is a bounded function f such that $f(x) \neq 0$. \square

Theorem 6. Let \mathcal{A} be a subalgebra, sublattice, and closed subset of $C(X, \tau)$ with $\mathcal{A} \neq C(X, \tau)$. If $I \subseteq \mathcal{A}$ is a **-ideal* and there exists $f \in I$ such that $f(x) \neq 0$ for all $x \in X$, then it is not possible to find a topology $\tau' \subset \tau$ such that $\mathcal{A} = C(X, \tau')$.

Proof. Suppose that, there exists a topology $\tau' \subset \tau$ such that $\mathcal{A} = C(X, \tau')$. Let's take a $g \in C^*(X, \tau) - C^*(X, \tau')$. Then there is at least one point $x_0 \in X$ such that g is not $\tau' - \tau_{st}$ -continuous at this point. Let $g(x_0) = n$, $n \in \mathbb{R}$. Let $f \in I$ be a function such that f is $\tau' - \tau$ -continuous and $f(x_0) \neq 0$. We can choose $f(x_0) = m > 0$, since if $f(x_0) < 0$, we can select $-f$ instead of f . There exists an $\varepsilon_0 > 0$ such that $m - \varepsilon_0 > 0$. Since f is $\tau' - \tau_{st}$ -continuous, there is a

neighborhood $U \in \tau'$ of x_0 such that $f(U) \subset (m - \varepsilon_0, m + \varepsilon_0)$. Then, $f(x) > 0$ for all x . The function $\frac{1}{f}$ is $\tau' - \tau_{std}$ -continuous on the set U . The product of the functions $f.g$ and $\frac{1}{f}$ is $\tau' - \tau_{std}$ -continuous on the set U . However, since g is not continuous at x_0 , there exists a neighborhood V of $g(x_0)$ such that $g(U) \not\subset V$ for any neighborhood U of x_0 . Since the restriction $\left(f.g.\frac{1}{f}\right)|_U = g|_U$ is continuous at point x_0 and $U \in \tau'$, then the function $\left(f.g.\frac{1}{f}\right)|_U = g|_U$ is $\tau' - \tau_{std}$ -continuous at point x_0 . Then $\left(f.g.\frac{1}{f}\right)(x_0) = g(x_0) = n$. There is a neighborhood W of x_0 such that $g(W) \subset V$ for the V neighborhood of n . Since $x_0 \in W \in \tau'$ and $x_0 \in U \in \tau'$, it follows that $x_0 \in U \cap W = U_2$. So, x_0 has a neighborhood $U_2 \in \tau'$ such that $g(U_2) \subset V$, which contradicts the fact that $g(U) \not\subset V$. \square

The following example shows that it is not possible to find a $\tau' \subset \tau$ such that $\mathcal{A} = C(X, \tau')$.

Example 2. Let $\mathcal{A} = C^*(X) \subset C(X)$. In this case, there is no $\tau' \subset \tau$ such that $C^*(X) = C(X, \tau')$.

5. QUESTION

If a topology τ' does not exist such that $\mathcal{A} = C(X, \tau')$, is there a larger space Y containing X such that $\mathcal{A} = C(Y)$?

This problem is open.

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