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COMPARATIVE STRONG FILTERS IN HYPER QUASI-ORDERED RESIDUATED SYSTEMS

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Abstract. The concept of quasi-ordered residuated systems as a generalization of both quasi-ordered commutative residuated lattices and hoop-algebras was developed in 2018 by Bonzio and Chajda. The ideas of the theory of hyper structures were applied to this system by this author and designed the concept of hyper quasi-ordered residuated system. In this paper, the concept of comparative strong filters in a hyper quasi-ordered residuated system is constructed and analyzed. This concept is compared to a (strong) filter in this algebraic structure, also.

1. Introduction

Quasi-ordered residuated system is a commutative residuated integral monoid ordered under a quasi-order, introduced by S. Bonzio and I. Chajda in [1]. In the last few years, the theory of quasi-ordered residuated systems was enriched with more results on ideals and filters in them (for example, see [7, 8, 9, 10, 11, 12, 13, 14]). This algebraic structure is a generalization of both commutative residuated lattices and hoop-algebras.

Hyperstructure theory is introduced in [6] (1934) when F. Marty gave definition of a hypergroup and illustrated some applications. Till now, the hyper structures have been studied from the theoretical point of view for their applications to many subjects of pure and applied mathematics. Some fields of applications of the mentioned structures are lattices, graphs, coding, ordered sets, median algebra, automata, and cryptography (see, for example [5]). Many researchers have worked on this area. For example, R. A. Borzooei et al. in [2, 3] and O. Zahiri er al. [17] introduced and studied hyper residuated lattices.

The application of the theory of hyper structures to quasi-ordered residuated systems is the subject of the articles [15, 16]. There, in [15], among other things, the concept of (strong) deductive systems and the concept of (strong) filters in a

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hyper quasi-ordered residuated system were developed.The paper [16] is dedicated to the concept of implicative strong filters in a hyper quasi-ordered residuated system.

In this article, as a direct continuation of the aforementioned research [15, 16], the concept of comparative (strong) filters is considered. (Instead of this term, the term 'positive implicative filter' is often used.) In addition to registering its essential characteristics, this newly established notion is connected with the concept of (strong) filters. Also, several examples are given that illustrate this newly designed notion.

2. Preliminaries

In this section, the necessary notions and notations and some of their interrelationships, mostly taken from papers [1, 7, 9], are listed in the order to enable a reader to comfortably follow the presentation in this report. It should be pointed out here that the notations for logical conjunction, logical implication and others have a literal meaning. Also, all formulas in this text are written in the standard usual way of writing formulas in mathematical logic. If one of the formulas is not closed by some quantifier, then the variables that appear in it should be seen as free variables. Thus, for example, the label $H \vDash Q$ has the meaning that the consequent Q can be demonstrated from the hypothesis H. The notation =: in the formula $A =: B$ serves to indicate that A in it is the abbreviation for the formula B.

2.1. **A few words about hyper structures.**

Now, we recall some basic notions of the hypergroup theory from [5]: Let H be a non-empty set. A hypergroupoid is a pair $(H, \circ)$, where $\circ : H \times H \longrightarrow \mathcal{P}(H) \setminus \{\emptyset\}$ is a binary hyperoperation on H. If $a \circ (b \circ c) = (a \circ b) \circ c$ holds, for all $a, b, c \in H$ then $(H, \circ)$ is called a semihypergroup, and it is said to be commutative if $\circ$ is commutative. An element $1 \in H$ is called a unit, if $a \in (1 \circ a) \cap (a \circ 1)$, for all $a \in H$ and it is called a scaler unit, if $\{a\} = 1 \circ a = a \circ 1$, for all $a \in H$. Note that if $A, B \subseteq H$, then $A \circ B = \bigcup_{a \in A, b \in B} (a \circ b)$.

In addition to the previous one, in what follows the following notations will also be used ([3]):

Let $(H, \preccurlyeq)$ be a quasi-ordered set and A, B be two subsets of H. The term 'quasi-order' means that the relation $\preccurlyeq$ is reflexive and transitive. Then we write

- $A \ll B$, if there exist $a \in A$ and $b \in B$ such that $a \preccurlyeq b$.
- $A \preccurlyeq B$ if for any $a \in A$, there exists $b \in B$ such that $a \preccurlyeq b$.
- We will write $A \preccurlyeq b$ instead of $A \preccurlyeq \{b\}$.

In light of the foregoing determination, we have $x \preccurlyeq y$ if and only if $\{x\} \preccurlyeq \{y\}$. Also, we will write $a \ll B$ instead of $\{a\} \ll B$.

One can easily conclude that the relation $\preccurlyeq$ is a quasi-order on $\mathcal{P}(H)$. Indeed. Since reflexivity is obvious, let's show transitivity. Let $A, B, C \subseteq H$ be such that $A \preccurlyeq B$ and $B \preccurlyeq C$. Then for any $a \in A$ there exists an element $b = b(a) \in B$ such that $a \preccurlyeq b(a)$ and for any $b \in B$ there exists an element $c = c(b) \in C$ such that

$b \preccurlyeq c(b)$. So, for any $a \in A$ there exists an element $c = c(b(a)) \in C$ such that $a \preccurlyeq c$. This means that $A \preccurlyeq C$. In the general case, this relation is not antisymmetric.

Also, it is easy to see that $A \preccurlyeq B \implies A \ll B$. In addition to the previous one, the following applies

$$(\forall a \in H)(\forall b \subseteq H)(a \ll B \iff a \preccurlyeq B).$$

In the special case, for $B = \{b\}$, we have $(\forall a, b \in H)(a \ll b \iff a \preccurlyeq b)$. Finally, let's point out that the following holds

$$(\forall a \in H)(\forall B \subseteq H)(a \in B \implies (a \preccurlyeq B \wedge a \ll B)).$$

Also $\emptyset \neq A \subseteq B \implies B \ll A$ holds for $A, B \subseteq H$. Indeed. $A \subseteq B$ means that $(\forall a \in H)(a \in A \implies a \in B)$ holds. Therefore, one can find $b \in B$ such that $b \in A$. Since $b \preccurlyeq b$, we have $B \ll A$.

2.2. **Quasi-ordered residuated systems.** In article [1], S. Bonzio and I. Chajda introduced and analyzed the concept of residual relational systems.

Definition 2.1 ([1]). *A* quasi-ordered residuated system *is a structure* $\mathfrak{A} =: \langle A, \cdot, \rightarrow, 1, \preccurlyeq\rangle$, *where* $\langle A, \cdot, \rightarrow, 1\rangle$ *is an algebra of type* $\langle 2, 2, 0\rangle$ *and* $\preccurlyeq$ *is a quasi-order relation on A and satisfying the following properties:*
(1) $(A, \cdot, 1)$ *is a commutative monoid;*
(2) $(\forall x \in A)(x \preccurlyeq 1)$*;*
(3) $(\forall x, y, z \in A)(x \cdot y \preccurlyeq, z \iff x \preccurlyeq y \rightarrow z)$*.*
We will refer to the operation $\cdot$ *as (commutative) multiplication, to* $\rightarrow$ *as its residuum and to condition (3) as residuation.*

The concept of filters in a quasi-ordered residuated system was introduced in the article [7]. This concept is somewhat different from the filter concept in both hoop-algebras and residuated lattices.

Definition 2.2 ([7], Definition 3.1)**.** *For a subset F of a quasi-ordered residuated system* $\mathfrak{A}$ *we say that it is a* filter *of* $\mathfrak{A}$ *if it satisfies conditions*
(F2) $(\forall u, v \in A)((u \in F \wedge u \preccurlyeq v) \implies v \in F)$, *and*
(F3) $(\forall u, v \in A)((u \in F \wedge u \rightarrow v \in F) \implies v \in F)$*.*

Let it note that the empty subset of A satisfies the conditions (F2) and (F3). Therefore, $\emptyset$ is a filter in $\mathfrak{A}$. It is shown ([7], Proposition 3.4 and Proposition 3.2), that if a non-empty subset F of a quasi-ordered system $\mathfrak{A}$ satisfies the condition (F2), then it also satisfies the following conditions:
(F0) $1 \in F$ and
(F1) $(\forall u, v \in A)((u \cdot v \in F \implies (u \in F \wedge v \in F))$.

A reader can find several examples of filters in this algebraic system in the articles [7, 9, 11].

The concepts of two recognizable special filters in a quasi-ordered residuated system $\mathfrak{A}$ are given by the following definition:

Definition 2.3 ([8, 9]). *Let $\mathfrak{A}$ be a quasi-residuated system and F be a non-empty subset of A which satisfies the condition (F2).*

For F we say that it is an implicative filter *in $\mathfrak{A}$ if the following holds*
(IF) $(\forall u, v, z \in A)((u \rightarrow (v \rightarrow z) \in F \wedge u \rightarrow v \in F) \Longrightarrow u \rightarrow z \in F)$.

For F we say that it is a comparative filter *in $\mathfrak{A}$ if the following holds*
(FC) $(\forall u, v, z \in A)((u \rightarrow ((v \rightarrow z) \rightarrow v) \in F \wedge u \in F) \Longrightarrow v \in F)$.

A reader can find information about these filters and their relationships in [8, 9, 11, 14].

2.3. **Hyper quasi-ordered residuated systems.** A hyper quasi-ordered redissued system is introduced by the following definition:

Definition 2.4 ([15], Definition 3.1). *A hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A} =: (A, \circ, 1, \rightarrow, \preccurlyeq)$, (hyper QRS, by briefly), is a non-empty quasi-ordered set $(A, \preccurlyeq)$ endowed with two binary hyper operations $\circ$ and $\rightarrow$ and the element 1 such that satisfying the following conditions:*
(H1) $(A, \circ, 1)$ *is a commutative semihypergroup with* 1 *as the unit.*
(H0) $(\forall x \in A)(x \in 1 \circ x)$.
(H2) $(\forall x \in A)(x \preccurlyeq 1)$.
(H3) $(\forall x, y, z \in A)(x \circ y \ll z \Longleftrightarrow x \ll y \rightarrow z)$.

The following two propositions list some of the important fundamental features of this hyper system.

Proposition 2.1 ([15], Theorem 1). *In any hyper quasi-ordered residuated system $(A, \circ, 1, \rightarrow, \preccurlyeq)$, the following holds:*
(a) $(\forall B \subseteq A)(1 \ll B \Longrightarrow 1 \in B)$.
(b) $(\forall x, y \in A)(x \preccurlyeq y \Longrightarrow 1 \in x \rightarrow y)$.
(c) $(\forall x \in A)(1 \in x \rightarrow x)$.
(d) $(\forall x \in A)(1 \in x \rightarrow 1)$.
(e) $(\forall B.C, D \subseteq A)(B \ll C \rightarrow D \Longleftrightarrow B \circ C \ll D)$.
(f) $(\forall x, y \in A)(x \circ y \ll x \wedge x \circ y \ll y)$.
(g) $(\forall B, C \subseteq A)(B \circ C \ll B \wedge B \circ C \ll C)$.
(h) $(\forall x, y \in A)(x \ll y \rightarrow x)$.
(i) $(\forall x, y \in A)(1 \in x \rightarrow (y \rightarrow x))$.
(j) $(\forall B, C \subseteq A)(B \ll C \rightarrow B)$.
(k) $(\forall x, y \in A)(x \circ (x \rightarrow y) \ll x \wedge x \circ (x \rightarrow y) \ll y)$.
(l) $(\forall x, y, z \in A)(x \rightarrow (y \rightarrow z) \preccurlyeq (x \circ y) \rightarrow z \preccurlyeq x \rightarrow (y \rightarrow z) \preccurlyeq y \rightarrow (x \rightarrow z))$.
(m) $(\forall x, y \in A)(x \ll y \rightarrow (x \circ y))$.

Proposition 2.2 ([15], Theorem 2). *In any hyper quasi-ordered residuated system $(A, \circ, 1, \rightarrow, \preccurlyeq)$, the following holds:*
(n) $(\forall x, y, z \in A)(x \preccurlyeq y \Longrightarrow x \circ z \ll y \circ z)$.
(p) $(\forall x, y, z \in A)(x \preccurlyeq y \Longrightarrow (z \rightarrow x \preccurlyeq z \rightarrow y \wedge y \rightarrow z \preccurlyeq x \rightarrow z))$.
(q) $(\forall x, y, z \in A)(x \rightarrow y \preccurlyeq (y \rightarrow z) \rightarrow (x \rightarrow z))$.
(r) $(\forall x, y, z \in A)((x \rightarrow y) \circ (y \rightarrow z) \ll x \rightarrow z)$.
(s) $(\forall x, y, z \in A)(y \rightarrow z \ll (x \rightarrow y) \rightarrow (x \rightarrow z))$.

In order to determine the filter concept in this algebraic structure, the following formulas were observed:

(HF0) $1 \in F$.
(HF1) $(\forall x, y \in A)(x \circ y \subseteq F \Longrightarrow (x \in F \wedge y \in F))$.
(HF2) $(\forall x, y \in A)((x \preccurlyeq y \wedge x \in F) \Longrightarrow y \in F)$.
(HF3) $(\forall x, y \in A)((x \in F \wedge x \rightarrow y \subseteq F) \Longrightarrow y \in F)$.
(sHF3) $(\forall x, y \in A)((x \in F \wedge F \ll x \rightarrow y) \Longrightarrow y \in F)$.
(dHF3) $(\forall x, y \in A)(((x \rightarrow y) \cap F \neq \emptyset \wedge x \in F) \Longrightarrow y \in F)$.
(SH) $(\forall x, y \in A)((x \in F \wedge y \in F) \Longrightarrow x \circ y \subseteq F)$.
(wSH) $(\forall x, y \in A)((x \in F \wedge y \in F) \Longrightarrow F \ll x \circ y)$.

The concept of filters in a hyper residuated lattice was introduced in [17] and discussed in more detail in [3] as follows: A nonempty subset F of a hyper resuduated lattice L satisfying (HF2) and (SH) is a filter of L. A nonempty subset F of a hyper resuduated lattice L satisfying (HF2) and (wSH) is a weak filter of L. In [3] it was shown (Theorem 3.4) that: A non-empty subset F of a hyper residuated lattice L is a weak filter if and only if it satisfies the condition (HF2) and

(wHF4) $(\forall x, y \in A)((x \in F \wedge y \in F) \Longrightarrow (x \circ y) \cap F \neq \emptyset)$.

The concept of filters in a hyper hoop-algebra can be found in [4] and is determined in the following way: A nonempty subset F of a hyper hoop-algebra H satisfying (HF2) and (wHF4) is a weak filter of H. A nonempty subset F of a hyper hoop-algebra H satisfying (HF2) and (SH) is a filter of H. In addition, it was shown there: A non-empty subset F of a hyper hoop-algebra H is a weak filter of H if and only if it satisfies conditions (HF2) and (wSH).

In accordance with our earlier orientations - the omission that requires that a filter in a semigroup A be a subsemigroup of the semigroup A (see, for example [7]), we determined the concept of (strong) filters in a hyper QRS as follows:

Definition 2.5 ([15], Definition 3.5). *A subset F of a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$ is a* filter *of $\mathfrak{h}\mathfrak{A}$ if it satisfies the conditions (HF2) and (HF3).*

It is not difficult to conclude that $\emptyset$ and A are filters in a quasi-ordered reduced system $\mathfrak{h}\mathfrak{A}$, since the empty set $\emptyset$ and set A satisfy the conditions (HF2) and (HF3).

Definition 2.6 ([15], Definition 3.6). *A subset F of a hyper quasi-ordered residuated system $\mathfrak{h}\mathfrak{A}$ is a* strong filter *of $\mathfrak{h}\mathfrak{A}$ if it satisfies the conditions (HF2) and (sHF3).*

In [15] it is shown (Theorem 3.3)

$$[\mathfrak{H}], F \neq \emptyset, (HF2) \models (dHF3) \Longleftrightarrow (sHF3).$$

A filter in hyper QRS $\mathfrak{h}\mathfrak{A}$ does not have to be a strong filter in $\mathfrak{h}\mathfrak{A}$, as the following example shows:

Example 2.1: Let $A = \langle -\infty, 1] (\subseteq \mathbb{R})$. Then $(A, \leqslant)$ with the natural ordering is a partially ordered set. Define the hyperoperations $\circ$ and $\rightarrow$ on A as follows: $a \circ b =: min\{a, b\}$ and $a \rightarrow b =: \{1\}$ if $a \leqslant b$ and $a \rightarrow b =: [b, 1]$ if $b < a$. It is not difficult to check that $(A, \circ, 1, \rightarrow, \leqslant)$ is a hyper (quasi-)ordered residuated system. For each $a \leqslant 1$, the set $[a, 1]$ is a filter in $\mathfrak{h}\mathfrak{A}$ but it is not a strong filter in $\mathfrak{h}\mathfrak{A}$. Take

$x, y \in A$ such that $x \in [a, 1]$ and $y < x$. Then $x \to y = [y, 1]$. If we assume that $x \to y = [y, 1] \subseteq [a, 1]$ holds, then it must be $y \in [a, 1]$. However, for elements $x, y \in A$ from $x \in [a, 1]$ and $(x \to y) \cap [a, 1] = [y, 1] \cap [a, 1] \neq \emptyset$ not must follow $y \in [a, 1]$.

3. Comparative strong filters

This section is the main part of this paper. Here we introduce and analyze the conditions obtained by generalizing in a hyper quasi-ordered residuated system the condition (CF) for mentioned filters in a quasi-ordered residuated system. In order to determine the concept of comparative strong filters in a hyper quasi-ordered residuated system $\mathfrak{hA}$, we will analyze the following conditions:

(dHCF3) $((x \to ((y \to z) \to y)) \cap F \neq \emptyset \wedge x \in F) \Longrightarrow y \in F$

(HCF3) $(x \to ((y \to z) \to y) \subseteq F \wedge x \in F) \Longrightarrow y \in F$

(sHCF3) $(F \ll x \to ((y \to z) \to y) \wedge F \ll x) \Longrightarrow F \ll y$.

The first conclusion is given by the following theorem:

Theorem 1. *Let F be a non-empty subset of a hyper QRS* $\mathfrak{hA}$ *satisfying the condition (HF2). Then* $(dHCF3) \Longleftrightarrow (sHCF3)$.

Proof. $(\Longrightarrow)$ Let $\mathfrak{hA}$ be a hyper quasi-ordered residuated system and let F be a non-empty subset in A satisfying the conditions (HF2) and (dHCF3). Let us take elements $x, y, z \in A$ such that $F \ll x \to ((y \to z) \to y)$ and $F \ll x$. This means that there exist elements $s, t \in F$ and $u \in x \to ((y \to z) \to y)$ such that $s \preccurlyeq u$ and $t \preccurlyeq x$. Thus $u, x \in F$ by (HF2). Hence $(x \to ((y \to z) \to y) \cap F \neq \emptyset$ and $x \in F$. From here we get $y \in F$ according to (dHCF3). This proves the validity of the condition (sHCF3).

$(\Longleftarrow)$ Let $\mathfrak{hA}$ be a hyper quasi-ordered residuated system and let F be a non-empty subset in A satisfying the condition (sHCF3). Let us take elements $x, y, z \in A$ such that $(x \to (y \to z)) \cap F \neq \emptyset$ and $x \in F$. This means that there exists $u \in (x \to (y \to z)) \cap F$. From here we conclude that $F \ll (x \to (y \to z)$ and $F \ll x$ are valid since $u \preccurlyeq u$ and $x \preccurlyeq x$ due to the transitivity of the relation $\preccurlyeq$. Thus $F \ll y$ by (sHCF3). Thus, there exist $t \in F$ such that $t \preccurlyeq y$. Hence, $y \in F$ by (HF2), which proves the validity of the condition (dHCF3). □

The following definitions introduce the concept of comparative strong filters in a hyper quasi-ordered residuated system.

Definition 3.1. *Let* $\mathfrak{hA} =: (A, \circ, 1, \to, \preccurlyeq)$ *be a hyper quasi-ordered residuated system and let F be a non-empty subset of A that satisfies the condition (HF2). Then F is called* comparative strong filter *in* $\mathfrak{hA}$ *if (sHCF3) holds.*

According to what was shown in Theorem 1, the condition (sHCF3) can be replaced by the condition (dHCF3) in the determination of the concept of comparative strong filters in a hyper quasi-ordered residuated system.

To show that a strong comparative filter in hyper quasi-ordered residuated system $\mathfrak{hA}$ is a strong filter in $\mathfrak{hA}$, we need the following lemma:

Lemma 1 ([16], Lemma 3.5). *Let F be a non-empty subset of a hyper quasi-ordered residuated system $\mathfrak{hA}$ which satisfies the condition (HF2). Then:*
(i) $(\forall x \in A)(x \in F \iff (1 \to x) \cap F \neq \emptyset)$;
(ii) $(\forall x, y \in A)((x \to y) \cap F \neq \emptyset \implies (1 \to (x \to y)) \cap F \neq \emptyset)$.

The following theorem connects the concept of strong filters and the concept of comparative strong filters in this algebraic structure.

Theorem 2. *Every comparative strong filter of a hyper quasi-ordered residuated system $\mathfrak{hA}$ is a strong filter of $\mathfrak{hA}$.*

Proof. Let F be a comparative strong filter in a hyper quasi-ordered residuated system $\mathfrak{hA}$. This means that the nonempty subset F of A satisfies conditions (HF2) and (dHCF3). Let us prove that F satisfies the condition (dHF3).

Let $x, y \in A$ be such that $(x \to y) \cap F \neq \emptyset$ and $x \in F$. Then $(1 \to (x \to y)) \cap F \neq \emptyset$ by Lemma 1. This means that there is an element $u \in 1 \to (x \to y)$ and $u \in F$. Hence we have $u \in (y \to y) \to (x \to y)$ according to (c) in Proposition 2.1. Thus $u \in x \to ((y \to y) - y)$ in accordance with (l) in Proposition 2.1. Therefore $(x \to ((y \to y) - y)) \cap F \neq \emptyset$. From this and from $x \in F$ we get $y \in F$ according to (dHFC3). This shows that (dHF3) is valid and, therefore, F is a strong filter in $\mathfrak{hA}$. □

The following theorem gives a criterion for recognizing a comparative strong filter in a hyper quasi-residuated system.

Theorem 3. *Let F be a non-empty subset of a hyper quasi-ordered residuated system $\mathfrak{hA}$ that satisfies condition (HF2). Then F is a comparative strong filter in $\mathfrak{hA}$ if and only if it is a filter in $\mathfrak{hA}$ and the following holds*

(dHCF4) $(\forall x, y \in A)(((x \to y) \to x) \cap F \neq \emptyset \implies x \in F)$

Proof. ($\Longrightarrow$) Let F be a comparative strong filter in $\mathfrak{hA}$. Let us take $x, y \in A$ such that $((x \to y) \to x) \cap F \neq \emptyset$. Then $(1 \to ((x \to y) \to x) \cap F \neq \emptyset$ by Lemma 1. From this and $1 \in F$ follows $x \in F$ according to (dHCF3). This proves the validity of the formula (dHCF4). On the other hand, F is a strong filter in $\mathfrak{hA}$ according to Theorem 2.

($\Longleftarrow$) Conversely, let F be a strong filter in $\mathfrak{hA}$ such that it satisfies the condition (dHCF4). Let $x, y, z \in A$ be such that $(x \to ((y \to z) \to y)) \cap F \neq \emptyset$ and $x \in F$. Since F is a strong filter in $\mathfrak{hA}$, then$((y \to z) \to y) \cap F \neq \emptyset$ by (dHF3) and so $y \in F$ by (dHCF4). Therefore, F is a comparative strong filter in $\mathfrak{hA}$. □

The notion of implicative strong filter in a hyper quasi-ordered residuated system $\mathfrak{hA}$ was introduced in the article [16] (Definition 3.1) as follows: A non-empty subset F of A is an implicative strong filter in $\mathfrak{hA}$ if F satisfies the conditions (HF2) and

(dHIF3) $((x \to (y \to z)) \cap F \neq \emptyset \wedge (x \to y) \cap F \neq \emptyset) \implies (x \to z) \cap F \neq \emptyset$.

Example 3.1: Let $A =: \{a, b, c, 1\}$ be a chain such that

$$\preccurlyeq =: \{(a, a), (a, b), (a, 1), (b, b), (b, 1), (c, c), (c, 1)\}.$$

Let us defile the hyper operations as follows

∘	a	b	c	1
a	{a}	{a}	{a,b,c}	{a}
b	{a}	{a,b}	{b,c}	{a,b}
c	{a,b,c}	{b,c}	{c}	{c}
1	{a}	{a,b}	{c}	{1}

and

→	a	b	c	1
a	{1}	{1}	{c}	{1}
b	{a.b,c}	{1}	{c}	{1}
c	{a,b}	{a,b}	{1}	{1}
1	{a}	{a,b}	{c}	{1}

Routine calculations show that $(A, \circ, 1, \rightarrow, \preccurlyeq)$ is a hyper (quasi-)ordered residuated system. The subsets $F_1 = \{1\}$, $F_2 =: \{c, 1\}$, $F_3 =: \{b, 1\}$ and $F_4 =: \{a, b, 1\}$ are comparative strong filters in $\mathfrak{hA}$. As shown in Example 3.2 in [16], these filters are also implicative strong filters in $\mathfrak{hA}$.

However, a strong filter in a hyper quasi-ordered residuated system $\mathfrak{hA}$ does not have to be a comparative strong filter in $\mathfrak{hA}$ as the following example shows:

Example 3.2: Let $A = \{0, a, b, c, 1\}$ and let the order on A be determined as follows $0 < a < b < c < 1$. In this example, we write a instead of $\{a\}$ for simplicity. Let's define hyper operations on A in the following way:

∘	0	a	b	c	1
0	0	0	0	0	0
a	0	{0,a}	{0,a}	{0,a}	a
b	0	{0,a}	{0,a,b}	{a,b}	b
c	0	{0,a}	{a,b}	c	c
1	0	a	b	c	1

and

→	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	b	1	1	1
c	a	b	b	1	1
1	0	a	b	c	1

Routine calculations show that $(A, \circ, 1, \rightarrow, \preccurlyeq)$ is a hyper (quasi-)ordered residuated system. The subset $F =: \{1\}$ is a strong filter in $\mathfrak{hA}$, but it is not a comparative strong filter in $\mathfrak{hA}$ because, for example, it holds

$$((c \rightarrow b) \rightarrow c) \cap \{1\} = (b \rightarrow c) \cap \{1\} = \{1\} \cap \{1\} = \{1\} \neq \emptyset$$

but $c \notin \{1\}$. Therefore, $\{1\}$ is not a comparative strong filter in $\mathfrak{hA}$ according to Theorem 3.

The following example also shows the relationship between filters, strong filters and comparative strong filters in a hyper quasi-ordered residuated system $\mathfrak{hA}$.

Example 3.3: Let $A =: \{1, a, b, c, d\}$ be a set with a partially ordered $\leqslant$ defined as follows $d \leqslant c \leqslant 1$, $d \leqslant a \leqslant b \leqslant 1$. Let the operations on A be determined as follows

∘	1	a	b	c	d
1	{1}	{a}	{a,b}	{c}	{d}
a	{a}	{a}	{a}	{d}	{d}
b	{a,b}	{a}	{a,b}	{d}	{d}
c	{c}	{d}	{d}	{c}	{d}
d	{d}	{d}	{d}	{d}	{d}

and

→	1	a	b	c	d
1	{1}	{a}	{a,b}	{c}	{d}
a	{1}	{1}	{1}	{c}	{c}
b	{1}	{a,b,c}	{1}	{c}	{c}
c	{1}	{a,b}	{a,b}	{1}	{a,b}
d	{1}	{1}	{1}	{1}	{1}

With a little effort it can be shown that $(A, \circ, \rightarrow, \leqslant, \{1\})$ is a hyper (quasi)ordered residuated system. Furthermore, with a standard calculation, it can be determined that the sets $F_0 = \{1\}$, $F_1 =: \{1, c\}$ and $F_2 =: \{1, b, a\}$ are strong filters in $\mathfrak{hA}$. However, the set $\{1, b\}$ is not a strong filter in $\mathfrak{hA}$ because, for example, we have $(b \rightarrow a) \cap \{1, b\} = \{a, b, c\} \cap \{1, b\} \neq \emptyset$ but $a \notin \{1, b\}$. Sets F_1 and F_2 are comparative strong filters in $\mathfrak{h}\mathfrak{A}$.

4. Conclusions and future works

This article is a continuation of research started with paper [15] on the application of hyper structure theory to quasi-ordered residuated systems, and continued with paper [16] in which the concept of implicative strong filters in hyper quasi- ordered residuated system was introduced, discussed about the comparative strong filters in such a system. In addition to defining this concept (Definition 3.1) some equivalent conditions (Theorem 1 and Theorem 3) for its determination are found. Apart from the previous one, the connection between strong filter and comparative strong filter is registered (Theorem 2) in this class of hyper algebraic structures. Also, several examples are given that illustrate this newly designed notion.

Previously mentioned papers and this report may be a substrate for further research on filters in this class of hyper structures. Apart from the above, this form of research could be continued not only by choosing the determinations of fantastic and normal strong filters in hyper quasi-ordered residuated systems, generalizing the results in the articles [12, 13], but also by registering their important properties.

Conflict of interests. The author declares that there is no conflict of interest related to the material presented in this paper.

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