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ON FREQUENTLY HYPERCYCLIC C-DISTRIBUTION COSINE FUNCTIONS IN FRÉCHET SPACES

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Abstract. In this paper, we analyze frequently hypercyclic C-distribution cosine functions in separable infinite-dimensional complex Fréchet spaces. The notion of frequent hypercyclicity seems to be new even for cosine operator functions in separable infinite-dimensional Banach spaces (real or complex). Our results, formulated also in terms of global fractionally integrated C-cosine functions, are illustrated with several instructive examples.

1. Introduction and preliminaries

The class of frequently hypercyclic linear continuous operators on separable Fréchet spaces was introduced by F. Bayart and S. Grivaux in 2006 ([2]). We can freely say that the frequent hypercyclicity and various generalizations of this concept are the central objects of investigations in the field of linear topological dynamics now (see [3]-[4], [6], [13] and references cited therein for more details on the subject).

The class of frequently hypercyclic strongly continuous semigroups was introduced by E. M. Mangino, A. Peris in [24] and further studied by E. M. Mangino, M. Murillo-Arcila in [25], while the class of frequently hypercyclic C-distribution semigroups has been recently introduced by the author in [22], where some generalizations of this concept have been also examined.

Hypercyclicity and topologically mixing property for cosine operator functions in Banach spaces have been analyzed by A. Bonilla, P. Miana [5], T. Kalmes [15] and the author [21]. As mentioned in the abstract, the notion of frequent hypercyclicity has not been yet considered for cosine operator functions in Banach spaces. The main aim of this paper is to go a step further by analyzing frequently hypercyclic C-distribution

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cosine functions in separable infinite-dimensional complex Fréchet spaces. We reformulate our results for fractionally integrated C-cosine functions, providing also several illustrative examples, applications and open problems. The paper is very simply organized, containing two separate subsections concerning C-distribution semigroups and fractionally integrated C-semigroups in Fréchet spaces as well as C-distribution cosine functions and fractionally integrated C-cosine functions in Fréchet spaces (Subsection 1.1 and Subsection 1.2); our main results are formulated and proved in Section 2.

We use the standard notation throughout the paper. By E we denote a separable infinite-dimensional complex Fréchet space. If Y is also a complex Fréchet space, then by $L(E, Y)$ we denote the space consisting of all continuous linear mappings from E into Y; $L(E) \equiv L(E, E)$. We will always assume henceforth that $C \in L(E)$ and C is injective. Let A be a closed linear operator with domain $D(A)$ and range $R(A)$ contained in E, and let $CA \subseteq AC$. By $\sigma_p(A)$ and $N(A)$ we denote the point spectrum and kernel space of A, respectively. Set $D_\infty(A) := \bigcap_{k\in\mathbb{N}} D(A^k)$. The part of A in a linear subspace $\tilde{E}$ of E, $A_{|\tilde{E}}$ shortly, is defined through $A_{|\tilde{E}} := \{(x, y) \in A : x,\ y \in \tilde{E}\}$ (we will identify an operator and its graph henceforth). Recall that the C-resolvent set of A, denoted by $\rho_C(A)$, is defined by

$$\rho_C(A) := \Big\{\lambda \in \mathbb{C} : \lambda - A \text{ is injective and } (\lambda - A)^{-1}C \in L(E)\Big\}.$$

In our framework, the C-resolvent set of A consists of those complex numbers λ for which the operator $\lambda - A$ is injective and $R(C) \subseteq R(\lambda - A)$. In the sequel, we assume that any regularizing operator $C,\ C_1, \cdots$ commutes with A. By $\chi_T(\cdot)$ we denote the characteristic function of set T. All operator families considered in this paper will be non-degenerate.

The Schwartz space of rapidly decreasing functions $\mathcal{S}$ is topologized by the following system of seminorms $p_{m,n}(\psi) := \sup_{x\in\mathbb{R}} |x^m \psi^{(n)}(x)|$, $\psi \in \mathcal{S}$, $m,\ n \in \mathbb{N}_0$ (see [26] for more details about vector-valued distribution spaces considered below). The Schwartz spaces $\mathcal{D} = C_0^\infty(\mathbb{R})$ and $\mathcal{E} = C^\infty(\mathbb{R})$ carry the usual topologies. For any $\emptyset \neq \Omega \subseteq \mathbb{R}$, the symbol $\mathcal{D}_\Omega$ denotes the subspace of $\mathcal{D}$ consisting of those functions $\varphi \in \mathcal{D}$ for which $\operatorname{supp}(\varphi) \subseteq \Omega$; $\mathcal{D}_0 \equiv \mathcal{D}_{[0,\infty)}$. The space $\mathcal{D}'(E) := L(\mathcal{D}, E)$ consisting of all continuous linear function from $\mathcal{D}$ into E carries the usual topology, whereas the symbol $\mathcal{D}'_\Omega(E)$ stands for its subspace containing E-valued distributions whose supports are contained in Ω; $\mathcal{D}'_0(E) \equiv \mathcal{D}'_{[0,\infty)}(E)$. By δ_t we denote the Dirac distribution centered at point t ($t \in \mathbb{R}$). If $\varphi,\ \psi : \mathbb{R} \to \mathbb{C}$ are measurable functions, then we define $\varphi *_0 \psi(t) := \int_0^t \varphi(t-s)\psi(s)\,ds,\ t \in \mathbb{R}$. The convolution of vector-valued distributions will be taken in the sense of [23, Proposition 1.1].

Let $\zeta \in \mathcal{D}_{[-2,-1]}$ be a fixed test function satisfying $\int_{-\infty}^{\infty} \zeta(t)\,dt = 1$. We define $I(\varphi)$ $(\varphi \in \mathcal{D})$ through

$$I(\varphi)(\cdot) := \int\limits_{-\infty}^{\cdot} \left[\varphi(t) - \zeta(t)\int_{-\infty}^{\infty} \varphi(u)\,du\right] dt.$$

Then $I(\varphi) \in \mathcal{D}$, $I(\varphi') = \varphi$, $\frac{d}{dt}I(\varphi)(t) = \varphi(t) - \zeta(t)\int_{-\infty}^{\infty}\varphi(u)\,du$, $t \in \mathbb{R}$ and, for every $G \in \mathcal{D}'(L(E))$, the primitive G^{-1} of G is defined by $G^{-1}(\varphi) := -G(I(\varphi))$, $\varphi \in \mathcal{D}$. We have $G^{-1} \in \mathcal{D}'(L(E))$, $(G^{-1})' = G$, i.e., $-G^{-1}(\varphi') = G(I(\varphi')) = G(\varphi)$, $\varphi \in \mathcal{D}$; moreover, $\operatorname{supp}(G) \subseteq [0,\infty)$ implies $\operatorname{supp}(G^{-1}) \subseteq [0,\infty)$.

For any $s \in \mathbb{R}$, we define $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$. Set $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$ for $t > 0$, where $\Gamma(\cdot)$ denotes the Gamma function $(\zeta > 0)$, and $g_0(t) \equiv$ the Dirac δ-distribution. Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $f \in \mathcal{S}$ and $n = \lceil \alpha \rceil$. Then the Weyl fractional derivative $W_+^\alpha f$ is defined by

$$W_+^\alpha f := (-1)^n \frac{d^n}{dt^n}\int_t^\infty g_{n-\alpha}(s-t)f(s)\,ds, \quad t \in \mathbb{R}.$$

If $\alpha \in \mathbb{N}$, then we set $W_+^\alpha f := (-1)^n f^{(n)},\ f \in \mathcal{S}$.

Before we move ourselves to Subsection 1.1, we need to recall a few definitions about lower and upper densities:

Definition 1. (i) *Suppose that $T \subseteq \mathbb{N}$. The lower density of T, denoted by $\underline{d}(T)$, is defined through:*

$$\underline{d}(T) := \liminf_{n\to\infty} \frac{|T \cap [1,n]|}{n}.$$

(ii) *A linear operator A on E is said to be frequently hypercyclic iff there exists an element $x \in D_\infty(A)$ (frequently hypercyclic vector of A) such that for each open non-empty subset V of E we have that the set $\{n \in \mathbb{N} : A^n x \in V\}$ has positive lower density.*

Denote by $m(\cdot)$ the Lebesgue measure on $[0,\infty)$. The following continuous counterpart of Definition 1 is well known in the existing literature (see e.g. [24]):

Definition 2. *Suppose that $T \subseteq [0,\infty)$. Then the lower density of T, denoted by $\underline{d}(T)$, is defined through:*

$$\underline{d}_c(T) := \liminf_{t\to\infty} \frac{m(T \cap [0,t])}{t}.$$

1.1. **C-Distribution semigroups and fractionally integrated C-semigroups.** Suppose that $C \in L(E)$ is an injective operator, $\mathcal{G} \in \mathcal{D}_0'(L(E))$ and $C\mathcal{G} = \mathcal{G}C$. Then $\mathcal{G}$ is called a C-distribution semigroup, shortly (C-DS), iff $\mathcal{G}$ satisfies the following two conditions:

(i) $\mathcal{G}(\varphi *_0 \psi)C = \mathcal{G}(\varphi)\mathcal{G}(\psi)$, φ, $\psi \in \mathcal{D}$;

(ii) $\mathcal{N}(\mathcal{G}) := \bigcap_{\varphi \in \mathcal{D}_0} N(\mathcal{G}(\varphi)) = \{0\}$.

If $\mathcal{G}$ satisfies only (i), then we say that $\mathcal{G}$ is a pre-C-distribution semigroup, shortly pre-(C-DS).

Assume now that $\mathcal{G}$ is a (C-DS), $T \in \mathcal{E}_0'$, i.e., T is a scalar-valued distribution with compact support contained in $[0, \infty)$. Define

$$G(T)x := \big\{(x, y) \in E \times E : \mathcal{G}(T * \varphi)x = \mathcal{G}(\varphi)y \text{ for all } \varphi \in \mathcal{D}_0\big\}.$$

Then $G(T)$ is a closed linear operator. We define the (infinitesimal) generator of a (C-DS) $\mathcal{G}$ by $A := G(-\delta')$. By $D(\mathcal{G})$ we denote the set consisting of those elements $x \in E$ for which $x \in D(G(\delta_t))$, $t \geq 0$ and the mapping $t \mapsto G(\delta_t)x$, $t \geq 0$ is continuous.

Definition 3. *Let $\alpha \geq 0$, and let A be a closed linear operator. If there exists a strongly continuous operator family $(S_\alpha(t))_{t \geq 0} \subseteq L(E)$ such that:*

(i) $S_\alpha(t)A \subseteq AS_\alpha(t)$, $t \geq 0$,

(ii) $S_\alpha(t)C = CS_\alpha(t)$, $t \geq 0$,

(iii) *for all $x \in E$ and $t \geq 0$: $\int_0^t S_\alpha(s)x\,ds \in D(A)$ and*

$$A\int_0^t S_\alpha(s)x\,ds = S_\alpha(t)x - g_{\alpha+1}(t)Cx,$$

then we say that A is a subgenerator of a (global) α-times integrated C-semigroup $(S_\alpha(t))_{t \geq 0}$.

If $\alpha = 0$, then $(S_0(t))_{t \geq 0}$ is also said to be a C-regularized semigroup with subgenerator A. The integral generator of $(S_\alpha(t))_{t \geq 0}$ is defined by setting

$$\hat{A} := \left\{(x, y) \in E \times E : S_\alpha(t)x - g_{\alpha+1}(t)Cx = \int_0^t S_\alpha(s)y\,ds,\ t \geq 0\right\}.$$

The integral generator $\hat{A}$ of $(S_\alpha(t))_{t \geq 0}$ is a closed linear operator which extends any subgenerator of $(S_\alpha(t))_{t \geq 0}$ and satisfies $\hat{A} = C^{-1}AC$. We refer the reader to [17] for the notion of an exponentially equicontinuous, analytic α-times integrated C-semigroup in a general locally convex space.

Let A be a closed linear operator on E. Denote by $Z_1(A)$ the space consisting of those elements $x \in E$ for which there exists a unique continuous mapping $u : [0,\infty) \to E$ satisfying $\int_0^t u(s,x)\,ds \in D(A)$ and $A\int_0^t u(s,x)\,ds = u(t,x) - x,\ t \geq 0$, i.e., the unique mild solution of the corresponding Cauchy problem (ACP_1) :

$$(ACP_1) : u'(t) = Au(t),\ t \geq 0,\ u(0) = x.$$

If A is a subgenerator of a global α-times integrated C-semigroup $(S_\alpha(t))_{t\geq 0}$ for some $\alpha \geq 0$, then there is only one (trivial) mild solution of (ACP_1) with $x = 0$, so that $Z_1(A)$ is a linear subspace of X. The space $Z_1(A)$ consists exactly of those elements $x \in E$ for which the mapping $t \mapsto C^{-1}S_{\lceil\alpha\rceil}(t)x,\ t \geq 0$ is well defined and $\lceil\alpha\rceil$-times continuously differentiable on $[0,\infty)$, where $S_{\lceil\alpha\rceil}(t)x := (g_{\lceil\alpha\rceil-\alpha} *_0 S_\alpha(\cdot)x)(t),\ t \geq 0$, $x \in E$; see e.g. [17]. Define

$$\mathcal{G}(\varphi)x := (-1)^{\lceil\alpha\rceil}\int\limits_0^\infty \varphi^{(\lceil\alpha\rceil)}(t)S_{\lceil\alpha\rceil}(t)x\,dt, \quad \varphi \in \mathcal{D},\ x \in E \tag{1}$$

and

$$G\big(\delta_t\big)x := \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}C^{-1}S_{\lceil\alpha\rceil}(t)x, \quad t \geq 0,\ x \in Z_1(A).$$

Then $\mathcal{G}$ is a C-distribution semigroup generated by $C^{-1}AC$ and $Z_1(A) = D(\mathcal{G})$ (see e.g. [16], [18] and [19, Proposition 1.2]).

The notion of an entire C-regularized group will be taken in the sense of [17, Definition 2.2.9]; cf. also the monograph [11] by R. deLaubenfels for more details about C-regularized semigroups and their applications.

1.2. **C-Distribution cosine functions and fractionally integrated C-cosine functions.** Let $C \in L(E)$ be an injective operator, and let $\mathbf{G} \in \mathcal{D}_0'(L(E))$ satisfy $C\mathbf{G} = \mathbf{G}C$. Then we say that $\mathbf{G}$ is a C-distribution cosine function, shortly (C-DCF), iff $\mathbf{G}$ satisfies the following two conditions:

$(C - DCF_1)$:

$$\mathbf{G}^{-1}(\varphi *_0 \psi)C = \mathbf{G}^{-1}(\varphi)\mathbf{G}(\psi) + \mathbf{G}(\varphi)\mathbf{G}^{-1}(\psi),\ \varphi,\ \psi \in \mathcal{D};$$

$(C - DCF_2)$:

$$x = y = 0 \text{ iff } \mathbf{G}(\varphi)x + \mathbf{G}^{-1}(\varphi)y = 0,\ \varphi \in \mathcal{D}_0.$$

If $\mathbf{G}$ satisfies only $(C - DCF_1)$, then we say that $\mathbf{G}$ is a pre-C-distribution cosine function, shortly pre-(C-DCF).

We will use the following well-known result (see e.g. [27]):

Lemma 1. *Let* $\mathbf{G} \in \mathcal{D}_0'(L(E))$ *and* $\mathbf{G}(\varphi)C = C\mathbf{G}(\varphi)$, $\varphi \in \mathcal{D}$. *Then* $\mathbf{G}$ *is a pre-(C-DCF) in* E *iff* $\mathcal{G} \equiv \begin{pmatrix} \mathbf{G} & \mathbf{G}^{-1} \\ \mathbf{G}' - \delta \otimes C & \mathbf{G} \end{pmatrix}$ *is a pre-($\mathcal{C}$-DS) in* $E \times E$, *where* $\mathcal{C} \equiv \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$. *Moreover,* $\mathcal{G}$ *is a ($\mathcal{C}$-DS) iff* $\mathbf{G}$ *is a pre-(C-DCF) which satisfies* $(C - DCF_2)$.

Assume $\mathbf{G}$ is a $(C-DCF)$ and $T \in \mathcal{E}_0'$. Then the (infinitesimal) generator A of $\mathbf{G}$ is defined by

$$A := G(\delta'') := \Big\{(x,y) \in E \times E : \mathbf{G}^{-1}\big(\varphi''\big)x = \mathbf{G}^{-1}(\varphi)y \text{ for all } \varphi \in \mathcal{D}_0\Big\}.$$

Then A is a closed linear operator on E, $C^{-1}AC = A$ and $\mathcal{A} \subseteq \mathcal{B}$, where $\mathcal{A} \equiv \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ and $\mathcal{B}$ is the generator of $\mathcal{G}$, as well as $(x,y) \in A \Leftrightarrow \Big(\binom{x}{0}, \binom{0}{y}\Big) \in \mathcal{B}$.

Definition 4. *Let* $\alpha \geq 0$, *and let* A *be a closed linear operator. If there exists a strongly continuous operator family* $(C_\alpha(t))_{t\geq 0} \subseteq L(E)$ *such that:*

(i) $C_\alpha(t)A \subseteq AC_\alpha(t)$, $t \geq 0$,
(ii) $C_\alpha(t)C = CC_\alpha(t)$, $t \geq 0$,
(iii) *for all* $x \in E$ *and* $t \geq 0$: $\int_0^t (t-s)C_\alpha(s)x\,ds \in D(A)$ *and*

$$A\int_0^t (t-s)C_\alpha(s)x\,ds = C_\alpha(t)x - g_{\alpha+1}(t)Cx,$$

then it is said that A *is a subgenerator of a (global)* α*-times integrated* C*-cosine function* $(C_\alpha(t))_{t\geq 0}$.

If $\alpha = 0$, then $(C_0(t))_{t\geq 0}$ is also said to be a C-regularized cosine function with subgenerator A. The integral generator of $(C_\alpha(t))_{t\geq 0}$ is defined by

$$\hat{A} := \left\{(x,y) \in E \times E : C_\alpha(t)x - g_{\alpha+1}(t)Cx = \int_0^t (t-s)C_\alpha(s)y\,ds,\ t \geq 0\right\}.$$

The integral generator of $(C_\alpha(t))_{t\geq 0}$ is a closed linear operator which is an extension of any subgenerator of $(C_\alpha(t))_{t\geq 0}$. Furthermore, the integral generator of $(C_\alpha(t))_{t\geq 0}$ is its maximal subgenerator with respect to the set inclusion and we have $\hat{A} = C^{-1}AC$. It is worth noting that if A is a subgenerator of both a global α-times integrated C-cosine function $(C_\alpha(t))_{t\geq 0}$ and a global β-times integrated C_1-semigroup $(\mathrm{C}_\beta(t))_{t\geq 0}$ ($\alpha \geq 0$, $\beta \geq 0$), then $(C_\alpha(t))_{t\geq 0}$ and $(\mathrm{C}_\beta(t))_{t\geq 0}$ share any frequent hypercyclic property considered below; a similar assertion holds for integrated C-semigroups.

Relations between (degenerate) C-distribution cosine functions and (degenerate) integrated C-cosine functions in general locally convex spaces have been recently investigated in [27, Section 3]. For our further purposes, the following lemma will be sufficiently enough (see e.g. [27, Theorem 3.6] and [17]):

Lemma 2. *Let A be the integral generator of a global α-times integrated C-cosine function $(C_\alpha(t))_{t\geq 0}$. Set*

$$\mathbf{G}(\varphi)x := \int\limits_0^\infty W_+^\alpha \varphi(t) C_\alpha(t)x\,dt,\ \varphi \in \mathcal{D},\ x \in E.$$

Then $\mathbf{G}$ is a $(C-DCF)$ with the integral generator A.

The fundamental relation between fractionally integrated C-semigroups and fractionally integrated C-cosine functions in locally convex spaces is described as follows:

Lemma 3. *(see e.g.* [27, Lemma 5.1]*) Suppose A is a closed linear operator on E and $\alpha \geq 0$. Then the following assertions are equivalent:*

(i) *A is a subgenerator of an α-times integrated C-cosine function $(C_\alpha(t))_{t\geq 0}$ in E.*

(ii) *The operator $\mathcal{A} := \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ is a subgenerator of an $(\alpha+1)$-times integrated $\mathcal{C}$-semigroup $(S_{\alpha+1}(t))_{t\geq 0}$ in $E \times E$, where $\mathcal{C} := \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$.*

In this case:

$$S_\alpha(t) = \begin{pmatrix} \int_0^t C_\alpha(s)\,ds & \int_0^t (t-s)C_\alpha(s)\,ds \\ C_\alpha(t) - g_{\alpha+1}(t)C & \int_0^t C_\alpha(s)\,ds \end{pmatrix},\ \ t \geq 0,$$

and the integral generators of $(C_\alpha(t))_{t\geq 0}$ and $(S_{\alpha+1}(t))_{t\geq 0}$, denoted respectively by B and $\mathcal{B}$, satisfy $\mathcal{B} = \begin{pmatrix} 0 & I \\ B & 0 \end{pmatrix}$. Furthermore, the integral generator of $(C_\alpha(t))_{t\geq 0}$, resp. $(S_{\alpha+1}(t))_{t\geq 0}$, is $C^{-1}AC$, resp. $\mathcal{C}^{-1}\mathcal{A}\mathcal{C} \equiv \begin{pmatrix} 0 & I \\ C^{-1}AC & 0 \end{pmatrix}$.

For the sequel, we need the following notion (see [17] for the Banach space case). A function $u(t)$ is said to be a mild solution of the abstract Cauchy problem

$$(ACP_2) : u''(t) = Au(t),\ t \geq 0,\ u(0) = x,\ u'(0) = y,$$

if the mapping $t \mapsto u(t),\ t \geq 0$ is continuous, $\int_0^t (t-s)u(s)\,ds \in D(A)$ and $A\int_0^t (t-s)u(s)\,ds = u(t) - x - ty$, $t \geq 0$. It is well known that there exists at most one mild solution of (ACP_2) provided that there exists $\alpha \geq 0$ such that A is a subgenerator of a global α-times integrated C-cosine function. Denote by $Z_2(A)$ the vector subspace of E consisting of all $x \in E$ for which there exists a unique mild solution of (ACP_2) with $y=0$. Let $\pi_1 : E\times E \rightarrow E$ and $\pi_2 : E\times E\rightarrow E$ be the first and second projection, respectively, and let G be a $(C-DCF)$ generated by A. Then Lemma 1(i) implies that $\mathcal{G}$ is a $(\mathcal{C}-DS)$ generated by $\mathcal{A}$ and the solution space $Z_1(\mathcal{A})$ can be characterized as in the former subsection. In order not to make any abbuse of notation, the operator $G(\delta_t)\binom{x}{y}$ will be denoted henceforth by $\mathcal{G}_1(\delta_t)\binom{x}{y}$, for any $\binom{x}{y}\in Z_1(\mathcal{A})$. We have $Z_1(\mathcal{A})=D(\mathcal{G})$ and the mild solution $u(\cdot;\binom{x}{y})$ of (ACP_1) with initial value $\binom{x}{y}\in Z_1(\mathcal{A})$ is given by $u(t;\binom{x}{y})=\mathcal{G}_1(\delta_t)\binom{x}{y}$, $t\geq 0$. Arguing as in the Banach space case, we may deduce that $x\in Z_2(A)$ iff $\binom{0}{x}\in Z_1(\mathcal{A})$ iff $\binom{0}{x}\in D(\mathcal{G})$, and $u(t;x)=\pi_2(\mathcal{G}_1(\delta_t)\binom{0}{x})=\pi_1(\mathcal{G}_1(\delta_t)\binom{x}{0})$, $t\geq 0$, where $u(\cdot;x)$ denotes the mild solution of (ACP_2) with $y=0$. Define

$$G\bigl(\delta_t\bigr)x:=\pi_2\left(\mathcal{G}_1(\delta_t)\begin{pmatrix}0\\x\end{pmatrix}\right)=\pi_1\left(\mathcal{G}_1(\delta_t)\begin{pmatrix}x\\0\end{pmatrix}\right),\quad t\geq 0,\ x\in Z_2(A).$$

Then $C(Z_2(A))\subseteq Z_2(A)$, $G(\delta_t)(Z_2(A))\subseteq Z_2(A)$, $t\geq 0$ and for each $x\in Z_2(A)$ one has $G(\delta_t)Cx=CG(\delta_t)x$, $t\geq 0$, $2G(\delta_s)G(\delta_t)x=G(\delta_{t+s})x+G(\delta_{|t-s|})x, t,\ s\geq 0$ and $\mathbf{G}(\varphi)x=\int_0^\infty\varphi(t)CG(\delta_t)x\,dt, \varphi\in\mathcal{D}_0$. Furthermore, if A is a subgenerator of a global n-times integrated C-cosine function $(C_n(t))_{t\geq 0}$, then the solution space $Z_2(A)$ consists exactly of those vectors $x\in E$ such that, for every $t\geq 0$, $C_n(t)x\in R(C)$ and the mapping $t\mapsto C^{-1}C_n(t)x$, $t\geq 0$ is n-times continuously differentiable (hence, $Z_2(A)=Z_2(A')$ if A' is also a subgenerator of $(C_n(t))_{t\geq 0}$). In this case, for any $x\in Z_2(A)$ and $t\geq 0$, we have $G(\delta_t)x=\frac{d^n}{dt^n}C^{-1}C_n(t)x$. A similar statement holds for global n-times integrated C-semigroups ([17]).

For more details about C-distribution semigroups, C-distribution cosine functions, integrated C-semigroups and integrated C-cosine functions, the reader may consult [1], [11], [16]-[23] and [27].

2. Frequently hypercyclic C-distribution cosine functions and frequently hypercyclic fractionally integrated C-cosine functions

Let $P([0,\infty))$ denote the power set of $[0,\infty)$. The following general definition has been recently introduced for C-distribution semigroups in [22]:

Definition 5. *Suppose that* $\mathbf{G}$ *is a* C*-distribution cosine function generated by* A *and* $x\in Z_2(A)$. *Let* $\mathcal{F}\in P(P([0,\infty)))$ *and* $\mathcal{F}\neq\emptyset$. *Then it is said that* x *is an* $\mathcal{F}$*-hypercyclic element of* $\mathbf{G}$ *iff for each open non-empty subset* V *of* E *we have*

$$S(x,V):=\bigl\{t\geq 0: G(\delta_t)x\in V\bigr\}\in\mathcal{F};$$

$\mathbf{G}$ *is said to be* $\mathcal{F}$*-hypercyclic iff there exists an* $\mathcal{F}$*-hypercyclic element of* $\mathbf{G}$.

This definition enables one to consider the notions of q-frequent hypercyclicity, upper q-frequent hypercyclicity, f-frequent hypercyclicity and upper frequent hypercyclicity for C-distribution cosine functions and fractionally integrated C-cosine functions in Fréchet spaces (see [22] for related results established for the abstract differential equations of first order). In the sequel, we will focus our attention to the usually considered frequent hypercyclicity, only:

Definition 6. *Let* $\mathbf{G}$ *be a* C*-distribution cosine function generated by* A*. Then it is said that* $\mathbf{G}$ *is frequently hypercyclic iff there exists* $x \in Z_2(A)$ *(frequently hypercyclic vector of* $\mathbf{G}$*) such that for each open non-empty subset* V *of* E *we have* $\underline{d}_c(\{t \geq 0 : G(\delta_t)x \in V\}) > 0$.

In the following definition, we reword this notion for fractionally integrated C-cosine functions (see Lemma 2):

Definition 7. *Suppose that* A *is a subgenerator of a global* α*-times integrated* C*-cosine function* $(C_\alpha(t))_{t\geq 0}$ *for some* $\alpha \geq 0$*. Then we say that an element* $x \in Z_2(A)$ *is a frequently hypercyclic element of* $(C_\alpha(t))_{t\geq 0}$ *iff* x *is a frequently hypercyclic element of the induced* C*-distribution cosine function* $\mathbf{G}$ *defined through* (1)*;* $(C_\alpha(t))_{t\geq 0}$ *is said to be frequently hypercyclic iff* $\mathbf{G}$ *is frequently hypercyclic.*

We continue by stating the following useful extension of [21, Lemma 32], stated here for the operators acting on Fréchet spaces:

Lemma 4. *Let* $\lambda \in \mathbb{C}$*. Then* $\lambda \in \sigma_p(\mathcal{A})$ *iff* $\lambda^2 \in \sigma_p(A)$*; if* $f(\lambda^2)$ *an eigenvector of* A *with the eigenvalue* λ^2*, then* $F(\lambda) = (f(\lambda^2), \lambda f(\lambda^2))^T$ *is an eigenvector of* $\mathcal{A}$ *with the eigenvalue* λ.

We also need the following result, proved recently in [22]:

Lemma 5. *Let* $t_0 > 0$ *and let* A *be a subgenerator of a global* C*-regularized semigroup* $(S_0(t))_{t\geq 0}$ *on* E*. Suppose that* $R(C)$ *is dense in* E*. Set* $T(t)x := C^{-1}S_0(t)x,\ t \geq 0,\ x \in Z_1(A)$*. Assume that there exists a family* $(f_j)_{j\in\Gamma}$ *of twice continuously differentiable mappings* $f_j : I_j \to E$ *such that* I_j *is an interval in* $\mathbb{R}$ *and* $Af_j(t) = itf_j(t)$ *for every* $t \in I_j,\ j \in \Gamma$*. Set* $\tilde{E} := \overline{span\{f_j(t) : j \in \Gamma,\ t \in I_j\}}$*. Then* $A_{|\tilde{E}}$ *is a subgenerator of a global* $C_{|\tilde{E}}$*-regularized semigroup* $(S_0(t)_{|\tilde{E}})_{t\geq 0}$ *on* $\tilde{E}$*,* $(S_0(t)_{|\tilde{E}})_{t\geq 0}$ *is frequently hypercyclic in* $\tilde{E}$ *and the operator* $T(t_0)_{|\tilde{E}}$ *is frequently hypercyclic in* $\tilde{E}$*.*

The following result is closely connected with [21, Theorem 33]:

Theorem 1. *Assume that* $\alpha \geq 0$, A *is a subgenerator of an* α*-times integrated* C*-cosine function* $(C_\alpha(t))_{t\geq 0}$ *on* E *and there exists an open nonempty subset* Ω *of* $\mathbb{C}$ *such that* $\Omega \subseteq \rho_C(A)$ *and the mapping* $\lambda \mapsto (\lambda - A)^{-1}C$, $\lambda \in \Omega$ *is strongly continuous. Let* $D(A)$ *and* $R(C)$ *be dense in* E, *and let* $\lambda_0 \in \mathbb{C}$ *be such that* $\lambda_0^2 \in \Omega$. *Define* $k := (\lceil \alpha \rceil + \chi_{2\mathbb{N}+1}(\lceil \alpha \rceil))/2$.

(i) *Let there exist a family* $(f_j)_{j\in\Gamma}$ *of twice continuously differentiable mappings* $f_j : I_j \to E$ *such that* I_j *is an interval in* $\mathbb{R}$ *and* $Af_j(-t^2) = -t^2 f_j(-t^2)$ *for every* $t \in I_j$, $j \in \Gamma$. *Set* $F_j(t) := (f_j(-t^2), itf_j(-t^2))^T$, $t \in I_j$, $j \in \Gamma$ *and* $\tilde{E} := \overline{span\{F_j(t) : j \in \Gamma,\ t \in I_j\}}$. *Set*

$$\mathrm{C} := \begin{pmatrix} (\lambda_0^2 - A)^{-k}C & 0 \\ 0 & (\lambda_0^2 - A)^{-k}C \end{pmatrix}. \tag{2}$$

Then the operator $\mathcal{A}_{|\tilde{E}}$ *is a subgenerator of a global* $((\lambda_0 - \mathcal{A})^{-1}\mathrm{C})_{|\tilde{E}}$*-regularized semigroup* $(S(t))_{t\geq 0} \subseteq L(\tilde{E})$ *on* $\tilde{E}$, $(S(t))_{t\geq 0}$ *is frequently hypercyclic in* $\tilde{E}$ *and the operator* $(((\lambda_0 - \mathcal{A})^{-1}\mathrm{C})^{-1})_{|\tilde{E}}S(t_0)$ *is frequently hypercyclic in* $\tilde{E}$ *for any* $t_0 > 0$.

(ii) *Let there exist an open connected subset* Ω *of* $\mathbb{C}$ *which satisfies* $\sigma_p(A) \supseteq \{\lambda^2 : \lambda \in \Omega\}$ *and* $\Omega \cap i\mathbb{R} \neq \emptyset$. *Let* $f : \{\lambda^2 : \lambda \in \Omega\} \to E$ *be an analytic mapping satisfying* $f(\lambda^2) \in N(A - \lambda^2) \setminus \{0\}$, $\lambda \in \Omega$, *let* $F(\lambda) := (f(\lambda^2), \lambda f(\lambda^2))^T$, $\lambda \in \Omega$ *and let* $\tilde{E} := \overline{span\{F(\lambda) : \lambda \in \Omega\}}$. *Define* C *through* (2). *Then the operator* $\mathcal{A}_{|\tilde{E}}$ *is a subgenerator of a global* $((\lambda_0 - \mathcal{A})^{-1}\mathrm{C})_{|\tilde{E}}$*-regularized semigroup* $(S(t))_{t\geq 0} \subseteq L(\tilde{E})$ *on* $\tilde{E}$, $(S(t))_{t\geq 0}$ *is frequently hypercyclic in* $\tilde{E}$ *and the operator* $(((\lambda_0 - \mathcal{A})^{-1}\mathrm{C})^{-1})_{|\tilde{E}}S(t_0)$ *is frequently hypercyclic in* $\tilde{E}$ *for any* $t_0 > 0$.

Proof. We will prove only (i). Since [16, Proposition 2.3.12] holds for integrated C-cosine functions in Fréchet spaces, the prescribed assumptions imply that the operator A is a subgenerator of a global $((\lambda_0^2 - A)^{-k}C)$-regularized cosine function on E. Due to Lemma 3, we get that the operator $\mathcal{A}$ is a subgenerator of a global once integrated C-semigroup on $E \times E$. Since [16, Proposition 2.3.13] holds for integrated C-semigroups in Fréchet spaces, we may conclude from the above and the obvious facts $R(\mathrm{C}) \subseteq D(\lambda_0 - \mathcal{A})$, $\lambda_0 \in \rho_{\mathrm{C}}(\mathcal{A})$ that the operator $\mathcal{A}$ is a subgenerator of a global $(\lambda_0 - \mathcal{A})^{-1}\mathrm{C}$-regularized semigroup $(W_0(t))_{t\geq 0}$ on $E \times E$. Using the assumptions that $D(A)$ and $R(C)$ are dense in E, it readily follows that the operator $\mathcal{A}$ is densely defined as well as that $R((\lambda_0^2 - A)^{-k}C)$ is dense in E, which clearly implies that $R((\lambda_0 - \mathcal{A})^{-1}\mathrm{C})$ is dense in $E \times E$. Now the final conclusion follows immediately by applying Lemma 4 and Lemma 5. $\square$

Remark 1. (i) *As in the Banach space case, an element x is a hypercyclic vector of $(C_\alpha(t))_{t\geq 0}$ if $\binom{x}{0}$ ($\binom{0}{x}$) is a hypercyclic vector of the induced $\mathcal{C}$-distribution semigroup $\mathcal{G}$. Observing that for each non-empty open subset V of E one has $\{t \geq 0 : G(\delta_t)x \in V\} = \{t \geq 0 : \mathcal{G}_1(\delta_t)\binom{0}{x} \in E \times V\} = \{t \geq 0 : \mathcal{G}_1(\delta_t)\binom{x}{0} \in V \times E\}$, the same holds for frequent hypercyclicity.*

(ii) *It is not clear how we can neglect the condition on the existence of open non-empty subset $\Omega \subseteq \rho_C(A)$ such that the mapping $\lambda \mapsto (\lambda - A)^{-1}C$, $\lambda \in \Omega$ is strongly continuous. Speaking-matter-of-factly, in the present situation, we do not know whether the assertion of Lemma 5 can be extended to fractionally integrated C-semigroups.*

(iii) *Assume that $\binom{x}{y}$ is a frequently hypercyclic vector for $(S(t))_{t\geq 0} \subseteq L(\tilde{E})$. Then*

$$\mathcal{G}_1(\delta_t)\begin{pmatrix} x \\ y \end{pmatrix} = \left(\pi_1\left(\mathcal{G}_1(\delta_t)\begin{pmatrix} x \\ y \end{pmatrix}\right), \frac{d}{dt}\pi_1\left(\mathcal{G}_1(\delta_t)\begin{pmatrix} x \\ y \end{pmatrix}\right)\right)^T, \quad t \geq 0$$

and

$$u(t) = \pi_1\left(\mathcal{G}_1(\delta_t)\begin{pmatrix} x \\ y \end{pmatrix}\right), \quad t \geq 0$$

is a mild solution of (ACP_2). This simply implies that for each pair of open non-empty sets V_1, V_2 in E the set $\{t \geq 0 : (u(t), u'(t))^T \in (V_1 \times V_2) \cap \tilde{E}\}$ has positive lower density. Since

$$\Big\{t \geq 0 : (u(t), u'(t))^T \in (V_1 \times V_2) \cap \tilde{E}\Big\} \subseteq \big\{t \geq 0 : u(t) \in V_1 \cap \pi_1(\tilde{E})\big\} \cap \big\{t \geq 0 : u'(t) \in V_2 \cap \pi_2(\tilde{E})\big\},$$

we also have that the set $\{t \geq 0 : u(t) \in V_1 \cap \pi_1(\tilde{E})\} \cap \{t \geq 0 : u'(t) \in V_2 \cap \pi_2(\tilde{E})\}$ has positive lower density.

We continue by stating the following result (see also [21, Theorem 35]):

Theorem 2. *Suppose that $\theta \in (0, \frac{\pi}{2})$ and $-A$ generates an exponentially equicontinuous, analytic strongly continuous semigroup of angle θ. Assume $n \in \mathbb{N}$, $a_n > 0$, $a_{n-i} \in \mathbb{C}$, $1 \leq i \leq n$, $D(p(A)) = D(A^n)$, $p(A) = \sum_{i=0}^n a_i A^i$ and $n(\frac{\pi}{2} - \theta) < \frac{\pi}{2}$. Then there exists $\omega \in \mathbb{R}$ such that, for every $\alpha \in (1, \frac{\pi}{n\pi - 2n\theta})$, $p(A)$ generates an entire $C \equiv e^{-(p(A)-\omega)^\alpha}$-regularized group $(T(t))_{t\in\mathbb{C}}$. Set $C(z) := \frac{1}{2}(T(z) + T(-z))$, $z \in \mathbb{C}$ and*

$$\tilde{E} := \overline{span\{(f(\lambda), p(-\lambda)f(\lambda))^T : \lambda \in \Omega\}}.$$

Then $(C(t))_{t\geq 0}$ is a C-regularized cosine function generated by $p^2(A)$, the mapping $z\mapsto C(z)$, $z\in\mathbb{C}$ is entire and there exists a pair $(x,y)^T\in\tilde{E}$ such that the abstract Cauchy problem $(ACP)_2$, with the operator A replaced with the operator $p^2(A)$ therein, has a unique mild solution $u(\cdot)$ satisfying that the mapping $t\mapsto Cu(t)$, $t\geq 0$ can be extended to an entire function and that for each pair of open non-empty sets $V_1,\ V_2$ in E the set $\{t\geq 0: u(t)\in V_1\cap\pi_1(\tilde{E})\}\cap\{t\geq 0: u'(t)\in V_2\cap\pi_2(\tilde{E})\}$ has positive lower density.

Proof. The required conclusions immeditaly follow from the argumentation employed in the proof of above-mentioned theorem, the equalities $p^2(A)f(\lambda)=p^2(-\lambda)f(\lambda)$, $\lambda\in\Omega$ and

$$\begin{pmatrix} 0 & I \\ p^2(A) & 0 \end{pmatrix}\begin{pmatrix} f(\lambda) \\ p(-\lambda)f(\lambda)\end{pmatrix}=p(-\lambda)\begin{pmatrix} f(\lambda) \\ p(-\lambda)f(\lambda)\end{pmatrix},\quad \lambda\in\Omega,$$

as well as Theorem 1(ii) and Remark 1(iii). $\square$

As thoroughly explained in [21], there is a substantially large class of abstract second order differential equations which cannot be treated by integrated cosine functions. In what follows, we examine frequently hypercyclic properties for certain types of abstract second order differential equations by using the theory of C-regularized cosine functions:

Example 1. (i) *(see e.g.* [12, Example 4.12] *and* [21, Example 36(i)]*) Suppose that $a,\ b,\ c>0$ and $c<\frac{b^2}{2a}<1$. Of concern is the equation*

$$\begin{cases} u_t=au_{xx}+bu_x+cu:=-Au, \\ u(0,t)=0,\ \ t\geq 0, \\ u(x,0)=u_0(x),\ \ x\geq 0. \end{cases}$$

It is well known that the operator $-A$, with domain $D(-A)=\{f\in W^{2,2}([0,\infty)): f(0)=0\}$, generates an analytic strongly continuous semigroup of angle $\frac{\pi}{2}$ in the space $E=L^2([0,\infty))$; the same assertion holds in the case that the operator $-A$ acts on $E=L^1([0,\infty))$ with domain $D(-A)=\{f\in W^{2,1}([0,\infty)): f(0)=0\}$. Set

$$\Omega:=\left\{\lambda\in\mathbb{C}: \left|\lambda-\left(c-\frac{b^2}{4a}\right)\right|\leq\frac{b^2}{4a},\ \Im\lambda\neq 0 \textit{ if } \Re\lambda\leq c-\frac{b^2}{4a}\right\}$$

and suppose that $p(x)=\sum_{i=0}^n a_ix^i$ is a non-constant polynomial such that $a_n>0$ and $p(-\Omega)\ \cap\ i\mathbb{R}\neq\emptyset$ (this, in particular, holds if $a_0\in i\mathbb{R}$). Define $\tilde{E}:=\overline{span\{(f_\lambda(\cdot),p(-\lambda)f_\lambda(\cdot))^T:\lambda\in\Omega\}}$, with the function $f_\lambda(\cdot)$ being given in [12, Example 4.12]. *An application of Theorem 2 gives that there exists an injective operator $C\in L(E)$ such that $p^2(A)$ generates a global C-regularized cosine function*

$(C(t))_{t\geq 0}$ satisfying that there exists a pair $(x,y)^T \in \tilde{E}$ such that the abstract Cauchy problem (ACP_2), with the operator A replaced with the operator $p^2(A)$ therein, has a unique mild solution $u(\cdot)$ satisfying that the mapping $t \mapsto Cu(t)$, $t \geq 0$ can be extended to an entire function and that for each pair of open non-empty sets $V_1,\ V_2$ in E the set $\{t \geq 0 : u(t) \in V_1 \cap \pi_1(\tilde{E})\} \cap \{t \geq 0 : u'(t) \in V_2 \cap \pi_2(\tilde{E})\}$ has positive lower density.

(ii) *(*[14]*) Theorem 2 can be applied in the analyis of Laplace-Beltrami type operators considered by L. Ji and A. Weber in* [14, Theorem 3.1(a), Theorem 3.2, Corollary 3.3]. *For instance, let us assume that E is a symmetric space of non-compact type (of rank one) and $p > 2$. Then there exist a closed linear subspace $\tilde{E}$ of $E \times E$, a number $c_p > 0$ and an injective operator $C \in L(L^p_\natural(E))$ such that for any $c > c_p$ the operator $(-\Delta^\natural_{X,p}+c)^2$ generates a global C-regularized cosine function $(C(t))_{t\geq 0}$ in $L^p_\natural(E)$ satisfying additionally that there exists a pair $(x,y)^T \in \tilde{E}$ such that the abstract Cauchy problem (ACP_2), with the operator $A = (-\Delta^\natural_{X,p} + c)^2$ therein, has a unique mild solution $u(\cdot)$ satisfying that the mapping $t \mapsto Cu(t)$, $t \geq 0$ can be extended to an entire function and that for each pair of open non-empty sets $V_1,\ V_2$ in E the set $\{t \geq 0 : u(t) \in V_1 \cap \pi_1(\tilde{E})\} \cap \{t \geq 0 : u'(t) \in V_2 \cap \pi_2(\tilde{E})\}$ has positive lower density. Observe, finally, that Theorem 2 can be applied to the operators examined in* [21, Example 36(ii)]*, as well.*

In the former example, we have used a well known procedure of converting the abstract differential equation of second order into the system of two abstract differential equations of first order. In our approach, the matricial operator $\mathcal{A}$ generates a $\mathbf{C}$-regularized semigroup on the product space $E \times E$, for a certain injective operator $\mathbf{C} \in L(E \times E)$. Concerning hypercyclic and chaotic behaviour of abstract (complete) differential equations of second order, the situation in which the matricial operator obtained after the above described procedure generates a strongly continuous semigroup on the product space $E\times E$ has been analyzed by a great number of authors so far, almost always with analytic function spaces of Herzog type acting as pivot spaces (see e.g. [8]-[10]). Since the Desch-Schappacher-Webb criterion [12] has been essentially employed in these papers and since its validity also implies the frequent hypercyclicity of strongly continuous semigroup under consideration [24], we can also clarify certain results about frequent hypercyclicity of obtained solutions of second order equations. For example, J. A. Conejero, C. Lizama and M. Murilllo-Arcila [10] have analyzed the chaotic and hypercyclic properties of the following abstract differential equation of second order

$$\frac{\partial^2}{\partial t^2}u(t,x)+\gamma\frac{\partial}{\partial t}u(t,x)+\theta u(t,x)=\alpha\frac{\partial^2}{\partial x^2}u(t,x),\quad t\geq 0,\ x\in\mathbb{R};$$
$$u(0,x)=\varphi_1(x),\ \left(\frac{\partial}{\partial t}u(t,x)\right)_{t=0}=\varphi_2(x),\qquad (3)$$

on the Herzog space

$$E_\rho:=\left\{f:\mathbb{R}\to\mathbb{C}\ ;\ f(x)\equiv\sum_{n=0}^{\infty}\frac{a_n\rho^n}{n!}x^n,\quad (a_n)_{n\in\mathbb{N}_0}\in c_0(\mathbb{N}_0)\right\},$$

where $\gamma,\ \alpha,\ \theta\in\mathbb{R}$. In [10, Theorem 3.1], the authors have shown that the corresponding matricial operator $\mathcal{A}_{\gamma,\alpha,\theta}\equiv\begin{pmatrix}0 & I\\ \alpha\frac{\partial^2}{\partial x^2}-\theta I & -\gamma I\end{pmatrix}$, which is bounded and continuous on E_ρ, generates a chaotic and frequently hypercyclic strongly continuous semigroup on the space $E_\rho\times E_\rho$, provided that $\gamma,\ \alpha,\ \theta\in\mathbb{R}$ are real positive numbers, $\gamma^2=4\theta$ and $\rho>\gamma/2\sqrt{\alpha}$. This implies that there exist two functions $\varphi_1,\ \varphi_2\in E_\rho$ such that the eqaution (3) has a unique solution $u(\cdot,\cdot)$ satisfying that for each pair of open non-empty subsets $V_1,\ V_2$ of E_ρ we have that the set $\{t\geq 0: u(t,\cdot)\in V_1\}\cap\{t\geq 0:\frac{\partial}{\partial t}u(t,\cdot)\in V_2\}$ has positive lower density.

Due to Remark 1(i), an element x is a frequently hypercyclic vector of a $(C-DCF)$ $\mathbf{G}$ if $\binom{x}{0}$ $(\binom{0}{x})$ is a frequently hypercyclic vector of the induced $\mathcal{C}$-distribution semigroup $\mathcal{G}$. The question when $\mathcal{G}$ will have a frequently hypercyclic vector $\binom{x}{y}$ belonging the union of subspaces $\{0\}\times E$ and $E\times\{0\}$ is very non-trivial and, becuase of that, the notion of frequent hypercyclicity of C-distribution cosine functions and integrated C-cosine functions introduced in Definition 6-Definition 7, obeying the approach following the existence of a frequently hypercyclic vector of $\mathbf{G}$, seems to be a little bit strong. Keeping in mind Remark 1(iii), Theorem 2, Example 1 and the consideration from the previous paragraph, it is much better to analyze the frequently hypercyclic vectors of induced $\mathcal{C}$-distribution semigroups on the product space $E\times E$. In our forthcoming paper [7], we will follow this approach for the abstract higher-order differential equations.

Concerning frequently hypercyclic properties of integrated cosine functions, we would like to raise the following issue:

Example and problem. Suppose that $n\in\mathbb{N}$, $\rho(t):=\frac{1}{t^{2n}+1}$, $t\in\mathbb{R}$, $Af:=f'$, $D(A):=\{f\in C_{0,\rho}(\mathbb{R}):f'\in C_{0,\rho}(\mathbb{R})\}$, $E_n:=(C_{0,\rho}(\mathbb{R}))^{n+1}$, $D(A_n):=D(A)^{n+1}$ and $A_n(f_1,\cdots,f_{n+1}):=(Af_1+Af_2,Af_2+Af_3,\cdots,Af_n+$

$Af_{n+1}, Af_{n+1})$, $(f_1, \cdots, f_{n+1}) \in D(A_n)$. Set, for every $\varphi_1, \cdots, \varphi_{n+1} \in \mathcal{D}$,

$$G_{\pm}(\delta_t)\big(\varphi_1, \cdots, \varphi_{n+1}\big)^T := \big(\psi_1, \cdots, \psi_{n+1}\big)^T,$$

where $\psi_i(\cdot) := \sum_{j=0}^{n+1-i} \frac{(\pm t)^j}{j!} \varphi_{i+j}^{(j)}(\cdot \pm t)$, $1 \leq i \leq n+1$. Let $\mathbf{G}_n$ and $(C_n(t))_{t\geq 0}$ denote the (DCF) and global polynomially bounded n-times integrated cosine function generated by A_n^2. In [21, Example 38], we have shown that $\mathbf{G}_n$ and $(C_n(t))_{t\geq 0}$ are topologically mixing as well as that A_n^2 cannot be the integral generator of any (local) $(n-1)$-times integrated cosine function. Furthermore, we have $\lim_{|t|\to\infty} G(\delta_t)(\varphi_1, \cdots, \varphi_{n+1})^T = 0$ for every $\varphi_1, \cdots, \varphi_{n+1} \in \mathcal{D}$. Since the statements of [5, Theorem 1.2, Corollary 1.3, Theorem 1.4] and [21, Theorem 25] cannot be so easily reexamined for frequent hypercyclicity, we would like to ask finally whether $\mathbf{G}_n$ and $(C_n(t))_{t\geq 0}$ are frequently hypercyclic or not?

Frequently hypercyclic properites of cosine operator functions on weighted function spaces will be considered somewhere else.

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