Математички Билтен
42(LXVIII) No. 1
2018(27-36)
Скопје, Македонија

ISSN 0351-336X (print)
ISSN 1857-9914 (online)
UDC: 517.938:519.876.5

BIFURCATION ANALYSIS OF FRACTIONAL-ORDER CHAOTIC RÖSSLER SYSTEM

GJORGJI MARKOSKI

Abstract. In the paper Fractional Multistep differential transform method it is used for investigation of the the approximate numerical solutions of the fractional-order Rössler chaotic system.

Numerical results are presented graphically, with bifurcation diagrams and Lyapunov exponents, for different values of the parameters.

1. Introduction

The basic mathematical ideas about fractional calculus were developed long ago by the Leibniz (1695), Liouville (1834), Riemann (1892), and others. First book on the topic was published by Oldham and Spanier in 1974 ([13]). Recently, fractional calculus has application in physics, signal processing, electromagnetics, bioengineering, etc.

In [8], [7] Multistep differential transform method (MDTM) is applied to fractional-order Rössler chaotic and hyperchaotic systems, and in [9] to fractional-order Chua's system.

More generaly about DTM one can see in [2].

Twodimensional generalized DTM is used in [12], [10], [11], [14].

2. Fractional Differential Transform Method (FDTM)

Caputo fractional derivative of order q, $q>0$, of the function $f(x)$ is defined as

$$D_{x_0}^{q} f\left(x\right)=\begin{cases} \frac{1}{\Gamma(m-q)}\int\limits_{x_0}^{x}\frac{f^{(m)}(\tau)}{(t-\tau)^{q+1-m}}d\tau, & m-1<q<m \\ \frac{d^m}{dt^m}f\left(x\right), & q=m \end{cases}$$

where $x>x_0$, m is pozitive integer, $m-1<q\leq m$ and Γ is the Gamma function.

2000 *Mathematics Subject Classification.* 34C28, 34A08, 74H15.

Key words and phrases. Fractional Rössler system, Multistep differential transform method, Lyapunov exponents.

More details about Caputo fractional derivative one can see, for example, in [3], [15], [4].

Further we assume that $q > 0$ is rational number and $m = 1$.

The fractional differential transform method consists in the following.

The function $f(x)$ is expanded in terms of a fractional power series

$$f\left(x\right)=\sum_{k=0}^{\infty}F\left(k\right)\left(x-x_0\right)^{\frac{k}{\alpha}}$$

where $F(k)$ is the fractional differential transform of $f(x)$ and α is pozitive integer such that $q\alpha$ is pozitive integer. Usualy α is choosen as smalest number with that property.

Let consider the equation

$$D_{x_0}^{q}f\left(x\right)=G\left(x,f\right) \tag{1}$$

with initial conditions $f(x_0), f'(x_0), ..., f^{(m-1)}(x_0)$.

In Caputo derivative the initial conditions are implemented to the integer order derivatives, so the transformation of the initial conditions are

$$F\left(k\right)=\begin{cases} \frac{1}{\left(\frac{k}{\alpha}\right)!}\left.\frac{d^{\frac{k}{\alpha}}f(x)}{dx^{\frac{k}{\alpha}}}\right|_{x=x_0}, & \frac{k}{\alpha}\in\mathbb{Z}^+,\ k=0,1,...,q\alpha-1 \\ 0,\ \frac{k}{\alpha}\notin\mathbb{Z}^+ \end{cases}$$

where $\mathbb{Z}^+$ is the set of nonnegative integers.

For properties and details about DTM one can see, for example, [1], [6]. Theorems 1-6 from [1] give the nesesery properties for computing the coefficients $F(k)$, for $k \geq q\alpha$.

3. The Multistep Fractional Differential Transform Method (MFDTM)

The FDTM is not suitable for large intervals ([8]), so we divide the working interval $[x_0, T]$ into subintervals $[t_0, t_1], [t_1, t_2], \ldots, [t_{N-1}, t_N]$, where $t_0 < t_1 < \ldots t_N$, $t_0 = x_0$, $t_N = T$ and $t_i - t_{i-1} = h > 0$, for $i = 1, \ldots N$. Then FDTM is applied on first interval with initial condition $c_1 = f(x_0)$ and it is obtained the approximate solution $f_1(x)$. At the second interval again FDTM is applied with initial condition $c_2 = f_1(t_1)$ and it is obtained solution $f_2(x)$. This procedure is applied to all intervals $[t_i, t_{i-1}]$, $i = 1, \ldots, N$.

So the final approximate solution is

$$f(x)=\begin{cases} f_1\left(x\right),\ x\in\left[t_0,t_1\right] \\ f_2\left(x\right),\ x\in\left[t_1,t_2\right] \\ . \\ . \\ . \\ f_N\left(x\right),\ x\in\left[t_{N-1},t_N\right]. \end{cases}$$

Further We assume that $x_0 = 0$.

4. Bifurcation analysis of fractional-order Rössler chaotic system

We consider the fractional-order Rössler chaotic system

$$\begin{aligned} D^{q_1}x &= -y - z \\ D^{q_2}y &= x + ay \\ D^{q_3}z &= b + z\,(x - c) \end{aligned}$$

wit initial conditions $x_0 = x(0), y_0 = y(0)$ and $z_0 = z(0)$, where x, y, z are unknown funcions of t, $q_1 = q_2 = q_3 = q$, $0 < q \le 1$ and $a, b, c > 0$.

Wrking interval $[0, T]$, it is divided into subintervals $[0, t_1], [t_1, t_2], \ldots, [t_{N-1}, t_N]$, where $0 < t_1 < t_2 < \ldots t_N$, $t_N = T$ and $t_i - t_{i-1} = h > 0$, for $i = 1, \ldots, N$.

When DTM is applied to this system at the interval $[0, t_1]$ are obtained approximations x_1, y_1 and z_1 of the functions $x(t)$, $y(t)$ and $z(t)$, respectively.

We apply DTM at the interval $[t_{i-1}, t_i]$, with initial conditions $x_{i-1}(t_{i-1})$, $y_{i-1}(t_{i-1})$ and $z_{i-1}(t_{i-1})$, and obtain the approximations x_i, y_i and z_i:

$$\begin{aligned} x_i\,(t) &= \sum_{k=0}^{M_i} X_i\,(k)\,(t - t_i)^{\frac{k}{\alpha}} \\ y_i\,(t) &= \sum_{k=0}^{M_i} Y_i\,(k)\,(t - t_i)^{\frac{k}{\alpha}} \\ z_i\,(t) &= \sum_{k=0}^{M_i} Z_i\,(k)\,(t - t_i)^{\frac{k}{\alpha}} \end{aligned}$$

for every $i = 1, \ldots, N$.

Finaly, in this way we obtain approxtimations x_N, y_N and z_N at the interval $[t_{n-1}, t_N]$.

So, the approximations of functions x, y and z at $[0, T]$ are

$$X(t) = \begin{cases} x_1\,(t)\,,\ t \in [0, t_1] \\ x_2\,(t)\,,\ t \in [t_1, t_2] \\ . \\ . \\ . \\ x_N\,(t)\,,\ t \in [t_{N-1}, t_N] \end{cases},$$

$$Y(t) = \begin{cases} y_1\,(t)\,,\ t \in [0, t_1] \\ y_2\,(t)\,,\ t \in [t_1, t_2] \\ . \\ . \\ . \\ y_N\,(t)\,,\ t \in [t_{N-1}, t_N] \end{cases}$$

and

$$Z(t)=\begin{cases} z_1(t), & t\in[0,t_1] \\ z_2(t), & t\in[t_1,t_2] \\ . \\ . \\ . \\ z_N(t), & t\in[t_{N-1},t_N] \end{cases}$$

Through this paper the initial conditions are choosen $x_0 = 2$, $y_0 = 3$ and $z_0 = 4$, and $M_1 = M_2 = \cdots = M_N$.

In this section we choose $\alpha = 10$, so tranformation of the initial conditions are $X_i(k) = 0$, $Y_i(k) = 0$ and $Z_i(k) = 0$, for each interval $[t_{i-1}, t_i]$, $i = 1, \ldots, N$, and for $k = 1, \ldots, 10q - 1$.

For $k = 0$ initial conditions are $X_1(0) = 2$, $Y_1(0) = 3$, $Z_1(0) = 4$, and $X_i(0) = X_{i-1}(t_i)$, $Y_i(0) = Y_{i-1}(t_i)$, $Z_i(0) = Z_{i-1}(t_i)$ for $i = 1, \ldots, N$.

Using Theorems 1-6 from [1] we obtain that fractional-order Rössler chaotic system, at the interval $[t_i, t_{i-1}]$ transforms to

$$\begin{aligned}
X_i(k+10q) &= \frac{\Gamma\left(1+\frac{k}{10}\right)}{\Gamma\left(q+1+\frac{k}{10}\right)}\left(-Y_i(k)-Z_i(k)\right)\\
Y_i(k+10q) &= \frac{\Gamma\left(1+\frac{k}{10}\right)}{\Gamma\left(q+1+\frac{k}{10}\right)}\left(X_i(k)+aY_i(k)\right)\\
Z_i(k+10q) &= \frac{\Gamma\left(1+\frac{k}{10}\right)}{\Gamma\left(q+1+\frac{k}{10}\right)}\left(b\delta(k)-cZ_i(k)+\sum_{p=0}^{k}Z_i(p)\,X_i(k-p)\right)
\end{aligned}$$

where Γ is the Gamma function.

Trough the paper we choose $h = \frac{1}{100}$, $T = 400$ (so $N = 40000$, $0 \le t \le 400$), $a = 0.2$ and $c = 5.7$. The parameter b is changing with stepsize of 0.005 and $0 < b \le 4$.

All computations are made for $0 \le t \le 400$, and the diagrams are ploted for $200 \le t \le 400$, using software Mathematica. With maxx (maxy, maxz) are denoted graphs of maximal peaks of $\{X(k)|\, 20001 \le k \le 39999\}$ ($\{Y(k)|\, 20001 \le k \le 39999\}$, $\{Z(k)|\, 20001 \le k \le 39999\}$), where

$$X(k) = x_k(t_k),\ Y(k) = y_k(t_k),\ Z(k) = z_k(t_k)$$

for $k = 1, \ldots, N$.

Respectively are denoted minimal peaks with minx (miny, minz).

Graphs for maximal Lyapunov exponents are denoted by maxlyapunov.

For $q = 0.1, q = 0.2$ and $q = 0.3$ occurs owerflow in the computations.

- $q = 0.4$

Overflow in computation is occured for $b < 0.93$. It is choosen $M = 10q\alpha$. Rezulting diagrams are showed in Figure 1.

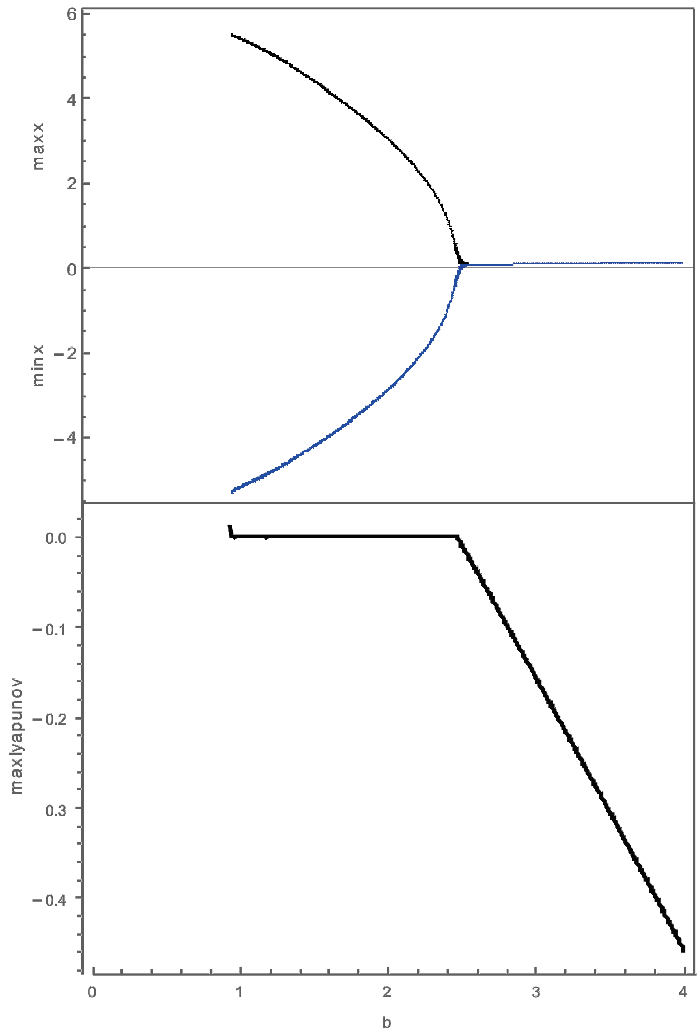

Figure 1: Diagram for $q = 0.4, a = 0.2, c = 5.7$ and $0 < b < 4$ with stepsize 0.005. Overflow in computation is occured for $b < 0.93$

- $q = 0.5$
$M = 5q\alpha$ (Figure 2)

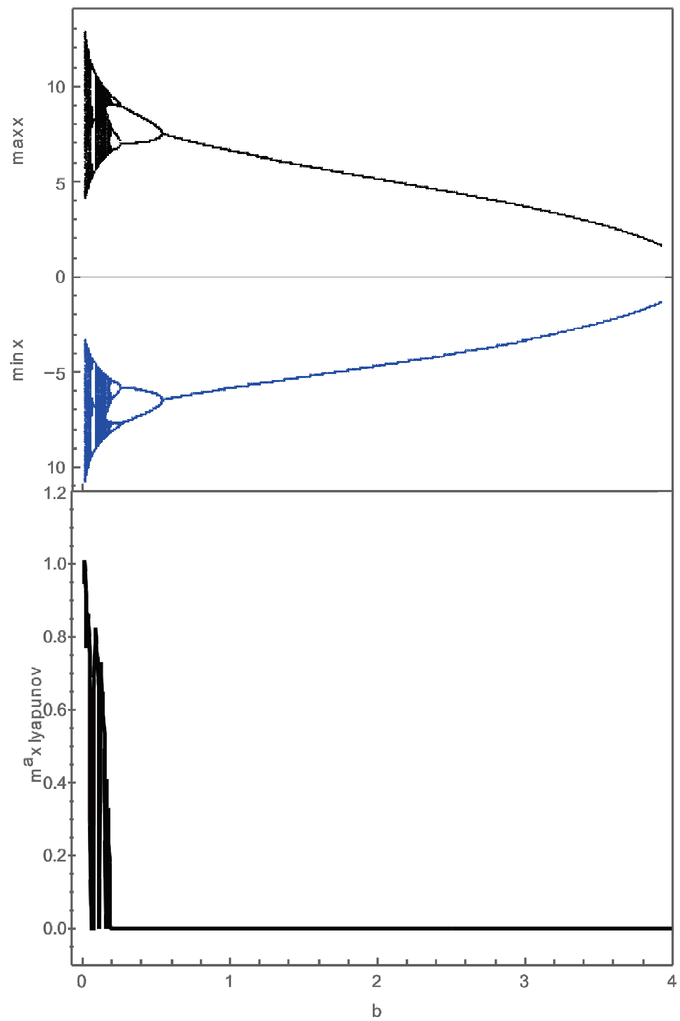

Figure 2: Diagram for $q = 0.5$, $a = 0.2, c = 5.7$ and $0 < b < 4$ with stepsize 0.005.

- $q = 0.7$

Here $M = 5q\alpha$. Rezulting diagrams are showed in Figure 3.

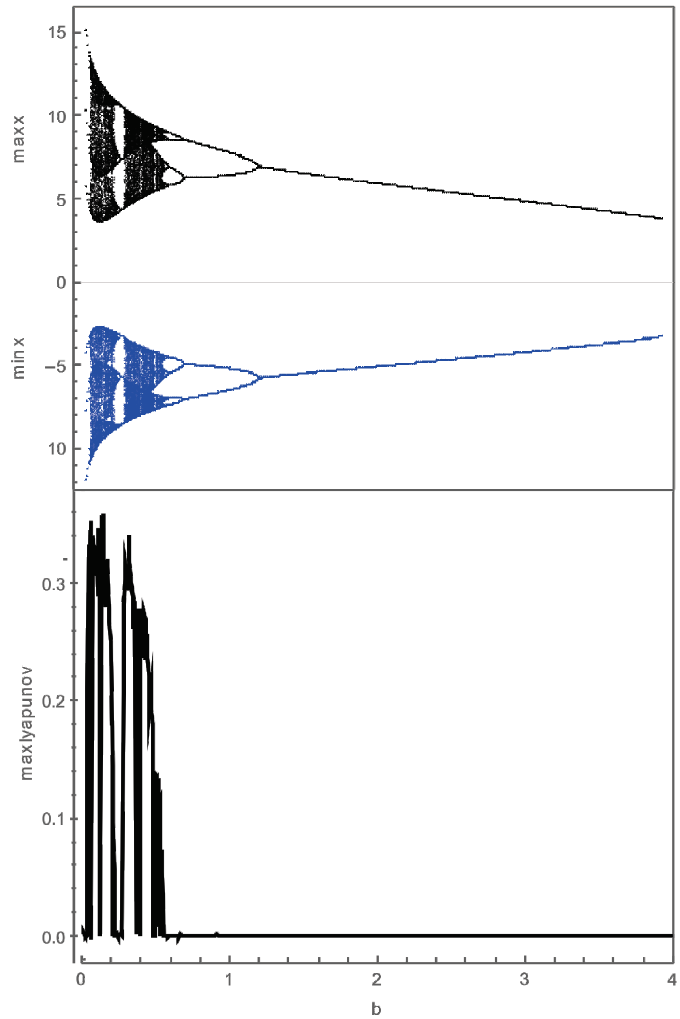

Figure 3: Diagram for $q = 0.7$, $a = 0.2, c = 5.7$ and $0 < b < 4$ with stepsize 0.005.

- $q = 0.9$
$M = 5q\alpha$ (Figure 4)

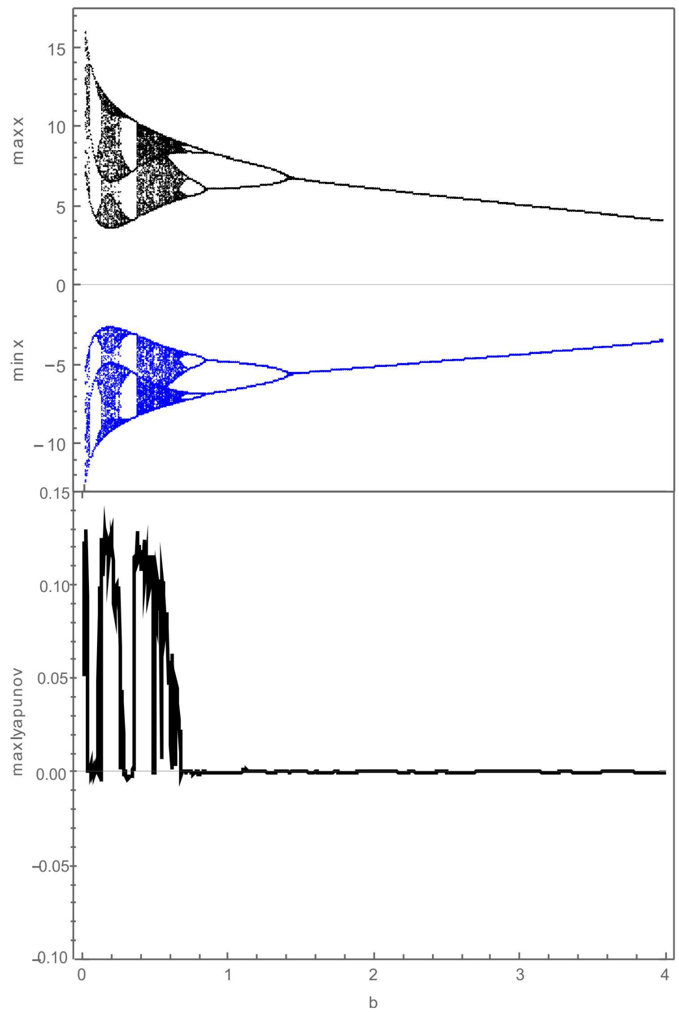

Figure 4: Diagram for $q = 0.9$,$a = 0.2, c = 5.7$ and $0 < b < 4$ with stepsize 0.005.

Now we change q from $q = 0.01$ to $q = 0.99$ with stepsize $h = 0.01$, and for $a = 0.2$, $b = 0.2$ and $c = 5.7$. Notations maxx, maxy, minx and miny remains same as before.

In the Figure 5 it is shown dependence maxx and minx from q, and in the figure 6 dependence maxy and miny from q.

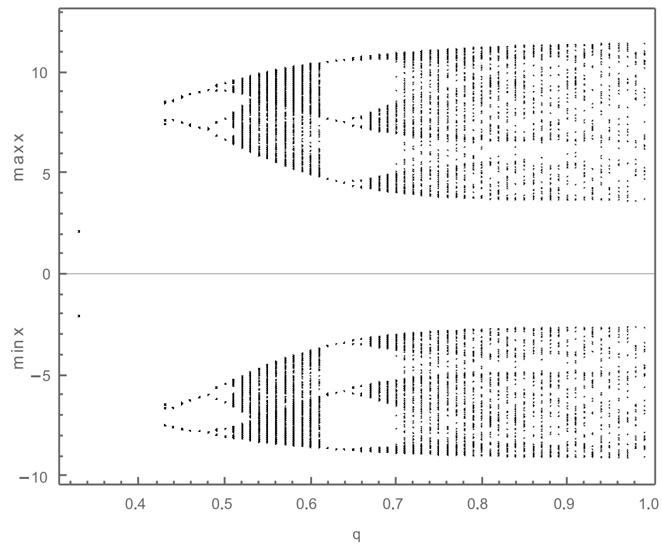

Figure 5: Diagram for maxx and minx, $a = 0.2, b = 0.2, c = 5.7$ and $0.01 \leq q \leq 0.99$ with stepsize 0.01.

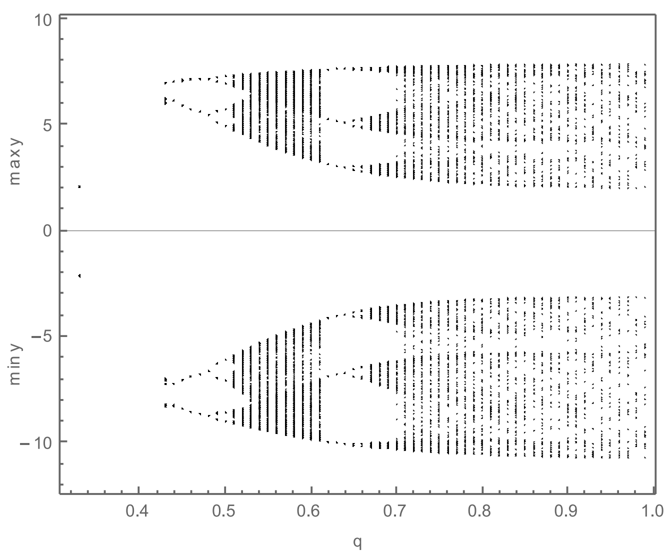

Figure 6: Diagram for maxy and miny, $a = 0.2, b = 0.2, c = 5.7$ and $0.01 \leq q \leq 0.99$ with stepsize 0.01.

References

[1] A. Arikoglu, I. Ozkol, *Solution of fractional differential equations by using differential transform method*, Chaos, Solitons and Fractals 34 (2007) 14731481

[2] C. Bervillier, *Status of the differential transformation method*, Applied Mathematics and Computation, 218 (2012) 1015810170.

[3] M. Caputo, F. Mainard, *Linear models of dissipation in anelastic solid*, Riv. Nuovo Cimento (ser. II), **1** (1971), 161-198.

[4] S. Das, *Functional Fractional Calculus*, Springer-Verlag Berlin Heidelberg (2011).

[5] K. Diethelm, *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo*, Lecture Notes in Mathematics, Springer-Verlag Berlin Heidelberg, 2010.

[6] V. S. Erturk, Sh. Momani, *Solving systems of fractional differential equations using differential transform method*, Journal of Computational andApplied Mathematics 215 (2008) 142151

[7] M. Al-Smadi, A. Freihat, O. Abu Arqub, N. Shawagfeh, *A Novel Multistep Generalized Differential Transform Method for Solving Fractional-order Lu Chaotic and Hyperchaotic Systems*,J. Computational Analysis And Applications, Vol. 19, No.4, 201, 713-724.

[8] A. Freihat, S. Momani, *Adaptation of Differential Transform Method for the Numeric-Analytic Solution of Fractional-Order Rössler Chaotic and Hyperchaotic Systems* Abstr. Appl. Anal., Vol. 2012, Special Issue (2012), Article ID 934219.

[9] A. Freihat, S. Momani, *Application of Multistep Generalized Differential Transform Method for the Solutions of the Fractional-Order Chua's System* , Discrete Dynamics in Nature and Society, Volume 2012, Article ID 427393, Hindawi Publishing Corporation.

[10] M. Mizradesh, M. Pramana, *A novel approach for solving fractional Fisher equation using differential transform method*, Pramana - J Phys (2016), Volume 86, Issue5, pp 957963.

[11] Sh. Momani, Z. Odibat, V. Suat Erturk, *Generalized differential transform method for solving a spaceand time-fractional diffusion-wave equation*, Physics Letters A 370 (2007) 379387.

[12] Z. Odibat, Sh. Momani, A. Alawneh, *Analytic Study on Time-Fractional Schrdinger Equations:Exact Solutions by GDTM*, Journal of Physics: Conference Series 96 (2008) 012066.

[13] K. B. Oldham, J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Mathematics in science and engineering 111, Academic Press, New York, 1974.

[14] S. Saha Ray, *Numerical Solutions and Solitary Wave Solutions of Fractional KdV Equations using Modified Fractional Reduced Differential Transform Method*, Computational Mathematics and Mathematical Physics, 2013, Vol. 53, No. 12, pp. 18701881.

[15] V. V. Uchaikin, *Fractional Derivatives for Physicists and Engineers, Volume I Background and Theory*, Nonlinear Physical Science, Higher Education Press, Beijing and Springer-Verlag Berlin Heidelberg (2013).

Institute of Mathematics,
Faculty of Natural Sciences and Mathematics,
Sts. Cyril and Methodius University, Skopje, Macedonia
E-mail address: `gorgim@pmf.ukim.mk`