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APPROXIMATE METHOD FOR AN INITIAL VALUE PROBLEM FOR IMPULSIVE DIFFERENCE EQUATIONS

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Abstract. An algorithm for constructing two monotone sequences of upper and lower solutions of the initial value problem for nonlinear impulsive difference equations is given. It is proved that both functional sequences are convergent and their limits are minimal and maximal solutions of the considered problem. Theoretical results are ilustrated by an example.

1. Statement of the problem

Let the sequence $\{n_i\}_{i=0}^{p+1}: \ n_i \in \mathbb{Z}_+, \ n_{i+1} \geq n_i + 2$ be given. Denote $\mathbb{Z}_+$ the set of all whole numbers and $\mathbb{Z}[a,b] = \{z \in \mathbb{Z}_+ : a \leq z \leq b\}, \ a, b \in \mathbb{Z}_+, \ a < b$ and $I_k = \mathbb{Z}[n_k, n_{k+1}-2], \ k \in \mathbb{Z}[0, p-1]$ and $I_p = \mathbb{Z}[n_p, n_{p+1}-1]$.

Consider the *initial value problem (IVP)* for the nonlinear *impulsive difference equation (IDE)*

$$\begin{aligned} &\Delta x(n) = f(n, x(n), x(n+1)) \text{ for } n \in I_k, \ k \in \mathbb{Z}[0,p] \\ &x(n_k) = g_k(x(n_k - 1)), \ k \in \mathbb{Z}[1,p] \\ &x(n_0) = x_0, \end{aligned} \tag{1}$$

where $\Delta x(n) = x(n+1) - x(n), \quad x, x_0 \in \mathbb{R}, \ f : \bigcup_{k=0}^{p} I_k \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g_k : \mathbb{R} \to \mathbb{R}, \ k \in \mathbb{Z}[1,p]$.

Definition 1. *The function $\alpha : \mathbb{Z}[n_0, n_{p+1}] \to \mathbb{R}$ is a minimal(maximal) solution of the IVP for IDE (1) in $\mathbb{Z}[n_0, n_{p+1}]$ if it is a solution of (1) and for any solution $u(n), n \in \mathbb{Z}[n_0, n_{p+1}]$ the inequality $\alpha(n) \leq u(n)$ ($\alpha(n) \geq u(n)$) holds on $\mathbb{Z}[n_0, n_{p+1}]$*

Definition 2. *The function α is a lower (upper) solution of (1), if:*

$$\begin{aligned} &\Delta\alpha(n) \leq (\geq) f(n, \alpha(n), \alpha(n+1)), \ \textit{for } n \in I_k, \ k \in \mathbb{Z}[0,p] \\ &\alpha(n_k) \leq (\geq) g_k(\alpha(n_k - 1)), \ k \in \mathbb{Z}[1,p] \\ &\alpha(n_0) \leq (\geq) x_0 \end{aligned}$$

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where $\alpha : \mathbb{Z}[n_0, n_{p+1}] \to \mathbb{R}$.

2. Preliminary results

Consider the linear impulsive difference equation of the type

$$\begin{aligned} &u(n+1) = Q_n u(n) + \sigma_n, \quad n \in I_k, k \in \mathbb{Z}[0,p] \\ &u(n_k) = M_k u(n_k - 1) + \gamma_k, \quad k \in \mathbb{Z}[1,p] \\ &u(n_0) = x_0, \end{aligned} \tag{2}$$

where $u, x_0 \in \mathbb{R}$, $Q_k \neq 0,\ (k \in \mathbb{Z}[n_0, n_{p+1}-1])$, $\sigma_k,\ (k \in \mathbb{Z}[n_0-1, n_{p+1}-1])$ and $M_k \neq 0, \gamma_k,\ (k \in \mathbb{Z}[1,p])$ are given real constants such that $Q_{n_j-1} = 1$ for $j \in \mathbb{Z}[1,p]$, $\sigma_{n_0-1} = x_0$ and $\sigma_{n_j-1} = 0$ for $j \in \mathbb{Z}[1,p]$.

Lemma 1. *The IVP for IDE (2) has an unique solution given by*

$$u(n) = \sum_{j=n_0-1}^{n-1} \Big(\prod_{i=j+1}^{n} N(i)\Big)\sigma_j \prod_{i=j+1}^{n-1} Q_i + \sum_{j=n_0}^{n} \Big(\prod_{i=j+1}^{n} N(i)\Big)\tau(j) \prod_{i=j}^{n-1} Q_i \tag{3}$$

for $n \in \mathbb{Z}[n_0, n_{p+1}]$

where

$$N(n) = \begin{cases} M_j & \textit{for} \ \ n = n_j, \qquad j \in \mathbb{Z}[1,p] \\ 1 & \textit{otherwise} \end{cases} \tag{4}$$

$$\tau(n) = \begin{cases} \gamma_j & \textit{for} \ \ n = n_j, \qquad j \in \mathbb{Z}[1,p] \\ 0 & \textit{otherwise} \end{cases} \tag{5}$$

Proof. We will use an induction with respect to the interval.

Let $n \in I_0 = \mathbb{Z}[n_0, n_1 - 2]$. Then the fuction $u(n)$ satisfies the difference equation

$$u(n+1) = Q_n u(n) + \sigma_n,\ n \in I_0 \tag{6}$$

with initial condition $u(n_0) = x_0$. Let $\sigma_{n_0-1} = x_0$. Then the unique solution of (6) is given by

$$u(n) = \sum_{j=n_0-1}^{n-1} \sigma_j \prod_{i=j+1}^{n-1} Q_i,\ n \in \mathbb{Z}[n_0+1, n_1-1]$$

Let $n = n_1$. Then using $\sigma_{n_1-1} = 0, \quad Q_{n_1-1} = 1$ we get

$$u(n_1) = M_1 u(n_1 - 1) + \gamma_1 = M_1 \sum_{j=n_0-1}^{n_1-1} \sigma_j \prod_{i=j+1}^{n_1-1} Q_i + \gamma_1,$$

We define $N(n_1) = M_1$ and $N(k) = 1,\ k \in \mathbb{Z}[n_0, n_1 - 1]$. Hence,

$$u(n_1) = \sum_{j=n_0-1}^{n_1-1} \Big(\prod_{i=j+1}^{n_1} N(i)\Big)\sigma_j \prod_{i=j+1}^{n_1-1} Q_i + \gamma_1. \tag{7}$$

Let $n \in I_1 = \mathbb{Z}[n_1, n_2 - 2]$. Then the fuction $u(n)$ satisfies the difference equation(6) with initial condition (7) which unique solution is given by

$$\begin{aligned} u(n) = & \sum_{j=n_0-1}^{n_1-1} (\prod_{i=j+1}^{n} N(i))\sigma_j \prod_{i=j+1}^{n-1} Q_i + \sum_{j=n_1}^{n-1} (\prod_{i=j+1}^{n} N(i))\sigma_j \prod_{i=j+1}^{n-1} Q_i \\ & + \gamma_1 \prod_{i=n_1}^{n-1} Q_i \\ = & \sum_{j=n_0-1}^{n-1} (\prod_{i=j+1}^{n} N(i))\sigma_j \prod_{i=j+1}^{n-1} Q_i + \gamma_1 \prod_{i=n_1}^{n-1} Q_i, \quad n \in \mathbb{Z}[n_1+1, n_2-1] \end{aligned}$$

Let $n = n_2$. Then using $\sigma_{n_2-1} = 0, \quad Q_{n_2-1} = 1$ we get

$$\begin{aligned} u(n_2) = & M_2 \sum_{j=n_0-1}^{n_2-1} (\prod_{i=j+1}^{n_2-1} N(i))\sigma_j \prod_{i=j+1}^{n_2-1} Q_i + (\prod_{i=n_1+1}^{n_2} N(i))\gamma_1 \prod_{i=n_1}^{n_2-1} Q_i \\ & + (\prod_{i=n_2+1}^{n_2} N(i))\gamma_2 \prod_{i=n_2}^{n_2-1} Q_i \\ = & \sum_{j=n_0-1}^{n_2-1} (\prod_{i=j+1}^{n_2} N(i))\sigma_j \prod_{i=j+1}^{n_2-1} Q_i + \sum_{j=n_0}^{n_2} (\prod_{i=j+1}^{n_2} N(i))\tau(j) \prod_{i=j}^{n_2-1} Q_i \end{aligned}$$

Let $n \in \mathbb{Z}[n_2 + 1, n_3 - 2]$. Then the fuction $u(n)$ satisfies the difference equation the fuction $u(n)$ satisfies the difference equation(6) with initial value $u(n_2)$ and unique solution given by

$$\begin{aligned} u(n) = & \sum_{j=n_0-1}^{n_2-1} (\prod_{i=j+1}^{n} N(i))\sigma_j \prod_{i=j+1}^{n-1} Q_i + M_2\gamma_1 \prod_{i=n_1}^{n-1} Q_i + \prod_{i=n_2}^{n-1} Q_i\gamma_2 \\ & + \sum_{j=n_2}^{n-1} (\prod_{i=j+1}^{n} N(i))\sigma_j \prod_{i=j+1}^{n-1} Q_i \\ = & \sum_{j=n_0-1}^{n-1} (\prod_{i=j+1}^{n} N(i))\sigma_j \prod_{i=j+1}^{n-1} Q_i + \sum_{j=n_0}^{n} (\prod_{i=j+1}^{n} N(i))\tau(j) \prod_{i=j}^{n-1} Q_i. \end{aligned}$$

By induction it proves (3) for all $n \in \mathbb{Z}[n_0, n_{p+1}]$. □

Lemma 2. *Assume that* $m : \mathbb{Z}[n_0, n_{p+1}] \to \mathbb{R}$ *satisfies*

$$\begin{aligned} & m(n+1) \leq Q_n m(n), \ \textit{for } n \in I_k, \ k \in \mathbb{Z}[0,p] \\ & m(n_k) \leq M_k m(n_k - 1), \ k \in \mathbb{Z}[1,p] \\ & m(n_0) \leq 0, \end{aligned} \tag{8}$$

where $Q_n > 0, \ n \in I_k, \ k \in \mathbb{Z}[0,p]$ *and* $M_k > 0, \ k \in \mathbb{Z}[1,p]$.

Then $m(n) \leq 0$ *for every* $n \in \mathbb{Z}[n_0, n_{p+1}]$.

The proof is based on an induction with respect to the interval and we omit it.

3. Main results

For any pair of function $\alpha, \beta : \mathbb{Z}[n_0, n_{p+1}] \to \mathbb{R}$ such that $\alpha(n) \leq \beta(n)$ for $n \in \mathbb{Z}[n_0, n_{p+1}]$ we define the sets

$$S(\alpha,\beta) = \{u : \mathbb{Z}[n_0, n_{p+1}] \to \mathbb{R} : \alpha(n) \leq u(n) \leq \beta(n), \quad n \in \mathbb{Z}[n_0, n_{p+1}]\}$$
$$\Gamma(\alpha,\beta) = \{x \in \mathbb{R} : \alpha(n_k - 1) \leq x \leq \beta(n_k - 1), \quad k \in \mathbb{Z}[1, p]\}$$
$$\Omega(\alpha,\beta) = \{u \in \mathbb{R} : \alpha(n) \leq u(n) \leq \beta(n), \quad n \in I_k, \ k \in \mathbb{Z}[0, p]\}$$

Theorem 1. *Let the following conditions be fulfilled:*

(1) *The functions $\alpha, \beta : \mathbb{Z}[n_0, n_{p+1}] \to \mathbb{R}$ are lower and upper solutions of the IVP for IDE (1), respectively, and $\alpha(n) \leq \beta(n)$ for $n \in \mathbb{Z}[n_0, n_{p+1}]$.*

(2) *The functions $f : \bigcup_{k=0}^{p} I_k \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous in $\Omega(\alpha,\beta)$ and there exist constants $L_n < 1$ and $P_n > -1$ such that for any $n \in I_k, k \in \mathbb{Z}[0, p]$ and $x_1, x_2, y_1, y_2 \in \Omega(\alpha,\beta) :\ x_1 \leq x_2,\ y_1 \leq y_2$*
$$f(n, x_1, y_1) - f(n, x_2, y_2) \leq -L_n(x_1 - x_2) - P_n(y_1 - y_2).$$

(3) *The functions $g_k : \mathbb{R} \to \mathbb{R},\ k \in \mathbb{Z}[1, p]$ are continuous in $\Gamma(\alpha,\beta)$ and there exist constants $M_k > 0, k \in \mathbb{Z}[1, p]$ such that for any $n = n_k, k \in \mathbb{Z}[1, p]$ and $z_1, z_2 \in \Gamma(\alpha,\beta)$ with $z_1 \leq z_2$*
$$g_k(z_1) - g_k(z_2) \leq M_k(z_1 - z_2).$$

Then there exist two sequences of functions $\{\alpha^{(j)}(n)\}_0^\infty$ and $\{\beta^{(j)}(n)\}_0^\infty$ with $\alpha^{(0)} = \alpha$ and $\beta^{(0)} = \beta$ such that:

a)
$$\alpha(n) \leq \alpha^{(j)}(n) \leq \alpha^{(j+1)}(n) \leq \beta^{(j+1)}(n) \leq \beta^{(j)}(n) \leq \beta(n), \ n \in \mathbb{Z}[n_0, n_{p+1}];$$

b) $\alpha^{(j)}(n)$ and $\beta^{(j)}(n)$ are lower and upper solutions of (1), respectively;

c) Both sequences are convergent on $\mathbb{Z}[n_0, n_{p+1}]$;

d) The limits $\lim_{j\to\infty} \alpha^{(j)}(n) = A(n)$, $\lim_{j\to\infty} \beta^{(j)}(n) = B(n)$ are the minimal and maximal solutions of IVP for IDE (1) in $S(\alpha,\beta)$, respectively;

e) If (1) has an unique solution $u(n) \in S(\alpha,\beta)$, then $A(n) \equiv u(n) \equiv B(n)$ for $n \in \mathbb{Z}[n_0, n_{p+1}]$.

Proof. For any arbitrary fixed function $\eta \in S(\alpha,\beta)$, we consider the IVP for the linear IDE

$$\begin{aligned}
&u(n+1) = (1 - L_n)u(n) - P_n u(n+1) + \psi(n, \eta(n), \eta(n+1)) \\
&\qquad n \in I_k, k \in \mathbb{Z}[0, p] \\
&u(n_k) = M_k u(n_k - 1) + \xi_k(\eta(n_k - 1)), \quad k \in \mathbb{Z}[1, p] \\
&u(n_0) = x_0,
\end{aligned} \tag{9}$$

where $u, x_0 \in \mathbb{R}$, the functions $\psi \in C(I_k \times S(\alpha,\beta) \times S(\alpha,\beta), \mathbb{R}), k \in \mathbb{Z}[0,p]$ and $\xi_k \in C(\Gamma(\alpha,\beta), \mathbb{R}), k \in \mathbb{Z}[1,p]$ are defined by equalities:

$$\psi(n,\eta(n),\eta(n+1)) = f(n,\eta(n),\eta(n+1)) + L_n\eta(n)$$
$$+ P_n\eta(n+1), n \in I_k, k \in \mathbb{Z}[0,p]$$
$$\xi_k(\eta(n_k-1)) = g_k(\eta(n_k-1)) - M_k\eta(n_k-1), k \in \mathbb{Z}[1,p].$$

According to Lemma 1 the IVP for linear IDE (9) has an unique solution given by (3) with $\sigma_n = \frac{\psi(n,\eta(n),\eta(n+1))}{1+P_n}$, $Q_n = \frac{1-L_n}{1+P_n}$ and $\tau(n)$ is given by (5) for $\gamma_j = \xi_j(\eta(n_j-1)),\ j \in \mathbb{Z}[1,p]$.

Consider the operator $Q : S(\alpha,\beta) \to S(\alpha,\beta)$ by $Q\eta = u$, where u is the unique solution of (9) for the function η.

Set $Q\alpha = \alpha^{(1)}$, where $\alpha^{(1)}$ is the unique solution of (9) with $\eta = \alpha$ and let $m(n) = \alpha(n) - \alpha^{(1)}(n)$. Then for any $n \in I_k, k \in \mathbb{Z}[0,p]$ we get

$$m(n+1) \le \alpha(n) + f(n,\alpha(n),\alpha(n+1)) - \alpha^{(1)}(n) + L_n\alpha^{(1)}(n)$$
$$+ P_n\alpha^{(1)}(n+1) - f(n,\alpha(n),\alpha(n+1)) - L_n\alpha(n) - P_n\alpha(n+1)$$
$$= (1-L_n)m(n) - P_n m(n+1)$$

Hence the inequality $m(n+1) \le \frac{1-L_n}{1+P_n}\ m(n)$ holds for $n \in I_k,\ k \in \mathbb{Z}[0,p]$.

For any $n = n_k$ we obtain

$$m(n_k) \le g_k(\alpha(n_k-1)) - M_k\alpha^{(1)}(n_k-1) - g_k(\alpha(n_k-1)) + M_k\alpha(n_k-1)$$
$$= M_k m(n_k-1).$$

Therefore, the function $m(n)$ satisfies the inequalities (8) with $Q_n = \frac{1-L_n}{1+P_n} > 0$. According to Lemma 2 the function $m(n)$ is non-positive in $\mathbb{Z}[n_0,n_{p+1}]$, i.e. $\alpha \le Q\alpha$. Analogously it can be proved that the inequality $\beta \ge Q\beta$ holds.

Let η_1 and $\eta_2 \in S(\alpha,\beta)$ be arbitrary functions such that $\eta_1(n) \le \eta_2(n)$ for $n \in \mathbb{Z}[n_0,n_{p+1}]$ and $u^{(1)} = Q\eta_1$ and $u^{(2)} = Q\eta_2$. Denote $m(n) = u^{(1)}(n) - u^{(2)}(n),\ n \in \mathbb{Z}[n_0,n_{p+1}]$.

For any $n \in I_k, k \in \mathbb{Z}[0,p]$ we get the inequality

$$m(n+1) = (1-L_n)(u^{(1)}(n) - u^{(2)}(n)) - P_n(u^{(1)}(n+1) - u^{(2)}(n+1))$$
$$+ L_n(\eta_1(n) - \eta_2(n)) + P_n(\eta_1(n+1) - \eta_2(n+1)) + f(n,\eta_1(n),\eta_1(n+1))$$
$$- f(n,\eta_2(n),\eta_2(n+1)) \le (1-L_n)m(n) - P_n m(n+1)$$

Hence the inequality $m(n+1) \le \frac{1-L_n}{1+P_n}\ m(n)$ holds for $n \in I_k,\ k \in \mathbb{Z}[0,p]$.

For any $n = n_k, k \in \mathbb{Z}[1,p]$ we obtain

$$m(n_k) = M_k(u^{(1)}(n_k-1) - u^{(2)}(n_k-1)) - M_k(\eta_1(n_k-1) - \eta_2(n_k-1))$$
$$+ g_k(\eta_1(n_k-1)) - g_k(\eta_2(n_k-1))$$
$$\le M_k m(n_k-1) - M_k(\eta_1(n_k-1) - \eta_2(n_k-1)) + M_k(\eta_1(n_k-1)$$
$$- \eta_2(n_k-1)) = M_k m(n_k-1)$$

According to Lemma 2 with $Q_n = \frac{1-L_n}{1+P_n} > 0$, the function $m(n) \leq 0$, i.e. $Q\eta_1 \leq Q\eta_2$, for $\eta_1(n) \leq \eta_2(n), n \in \mathbb{Z}[n_0, n_{p+1}]$.

Let $\eta \in S(\alpha, \beta)$ be a lower solution of (1).

We consider the function $Q\eta = m$. According to above $\eta(n) \leq m(n), n \in \mathbb{Z}[n_0, n_{p+1}]$.

For any $n \in I_k, k \in \mathbb{Z}[0, p]$ we get the inequality

$$\begin{aligned}
&\Delta m(n) \\
&= f(n, m(n), m(n+1)) - f(n, m(n), m(n+1)) - L_n m(n) \\
&- P_n m(n+1) + f(n, \eta(n), \eta(n+1)) + L_n \eta(n) + P_n \eta(n+1) \\
&\leq f(n, m(n), m(n+1))
\end{aligned} \tag{10}$$

For any $n = n_k, k \in \mathbb{Z}[1, p]$ we obtain

$$\begin{aligned}
m(n_k) &= g_k(m(n_k - 1)) - g_k(m(n_k - 1)) + M_k m(n_k - 1) \\
&+ g_k(\eta(n_k - 1)) - M_k \eta(n_k - 1) \\
&\leq g_k(m(n_k - 1)) + M_k(m(n_k - 1) - \eta(n_k - 1)) \\
&- M_k(m(n_k - 1) - \eta(n_k - 1)) \\
&= g_k(m(n_k - 1))
\end{aligned} \tag{11}$$

Inequalities (10) and (11) prove the function m is a lower solution of IDE (1). Similarly, if $\eta \in S(\alpha, \beta)$ is an upper solution of IDE (1) then the function $m = Q\eta$ is an upper solution of (1).

We define the sequences of functions $\{\alpha^{(j)}(n)\}_0^\infty$ and $\{\beta^{(j)}(n)\}_0^\infty$ by the equalities

$$\begin{aligned}
\alpha^{(0)} &= \alpha \qquad & \beta^{(0)} &= \beta \\
\alpha^{(j)} &= Q\alpha^{(j-1)} \qquad & \beta^{(j)} &= Q\{\beta^{(j-1)}
\end{aligned}$$

Therefore, the functions $\alpha^{(s)}(n)$ and $\beta^{(s)}(n)$ satisfy the initial value problem

$$\begin{aligned}
&\alpha^{(s)}(n+1) = (1 - L_n)\alpha^{(s)}(n) - P_n \alpha^{(s)}(n+1) \\
&\qquad + \psi(n, \alpha^{(s-1)}(n), \alpha^{(s-1)}(n+1)), \quad n \in I_k, k \in \mathbb{Z}[0, p] \\
&\alpha^{(s)}(n_k) = M_k \alpha^{(s)}(n_k - 1) + \xi_k(\alpha^{(s-1)}(n_k - 1)), \quad k \in \mathbb{Z}[1, p] \\
&\alpha^{(s)}(n_0) = x_0,
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
&\beta^{(s)}(n+1) = (1 - L_n)\beta^{(s)}(n) - P_n \beta^{(s)}(n+1) \\
&\qquad + \psi(n, \beta^{(s-1)}(n), \beta^{(s-1)}(n+1)), \quad n \in I_k, k \in \mathbb{Z}[0, p] \\
&\beta^{(s)}(n_k) = M_k \beta^{(s)}(n_k - 1) + \xi_k(\beta^{(s-1)}(n_k - 1)), \quad k \in \mathbb{Z}[1, p] \\
&\beta^{(s)}(n_0) = x_0,
\end{aligned} \tag{13}$$

Therefore, the following reprezantations are valid

$$\alpha^{(s)}(n) = \sum_{j=n_0-1}^{n-1} \Big(\prod_{i=j+1}^{n} N(i) \Big) \frac{\psi(j, \alpha^{(s-1)}(j), \alpha^{(s-1)}(j+1))}{1+P_j} \prod_{i=j+1}^{n-1} \frac{1-L_i}{1+P_i}$$
$$+ \sum_{j=n_0}^{n} \Big(\prod_{i=j+1}^{n} N(i) \Big) \tau(j) \prod_{i=j}^{n-1} \frac{1-L_i}{1+P_i}, \text{for } n \in \mathbb{Z}[n_0, n_{p+1}], \tag{14}$$

where $\tau(n)$ is given by (5) for $\gamma_j = \xi_j(\alpha^{(s-1)}(n_j - 1)),\ j \in \mathbb{Z}[1, p]$.

$$\beta^{(s)}(n) = \sum_{j=n_0-1}^{n-1} \Big(\prod_{i=j+1}^{n} N(i) \Big) \frac{\psi(j, \beta^{(s-1)}(j), \beta^{(s-1)}(j+1))}{1+P_j} \prod_{i=j+1}^{n-1} \frac{1-L_i}{1+P_i}$$
$$+ \sum_{j=n_0}^{n} \Big(\prod_{i=j+1}^{n} N(i) \Big) \tau(j) \prod_{i=j}^{n-1} \frac{1-L_i}{1+P_i}, \text{for } n \in \mathbb{Z}[n_0, n_{p+1}], \tag{15}$$

where $\tau(n)$ is given by (5) for $\gamma_j = \xi_j(\beta^{(s-1)}(n_j - 1)),\ j \in \mathbb{Z}[1, p]$.

According to the above proved, functions $\alpha^{(s)}(n)$ and $\beta^{(s)}(n)$ are lower and upper solutions of IDE (1), respectively and they satisfy for $n \in \mathbb{Z}[n_0, n_{p+1}]$ the following inequalities

$$\alpha^{(0)}(n) \le \alpha^{(1)}(n) \le \ldots \le \alpha^{(s)}(n) \le \beta^{(s)}(n) \le \ldots \le \beta^{(1)}(n) \le \beta^{(0)}(n) \tag{16}$$

Both sequences of functions being monotonic and bounded are convergent on $\mathbb{Z}[n_0, n_{p+1}]$.

Let $A(n) = \lim_{s\to\infty} \alpha^{(s)}(n), B(n) = \lim_{s\to\infty} \beta^{(s)}(n)$.

Take a limit in (14) for $s \to \infty$ we obtain

$$A(n) = \sum_{j=n_0-1}^{n-1} \Big(\prod_{i=j+1}^{n} N(i) \Big) \frac{\psi(j, A(j), A(j+1))}{1+P_j} \prod_{i=j+1}^{n-1} \frac{1-L_i}{1+P_i}$$
$$+ \sum_{j=n_0}^{n} \Big(\prod_{i=j+1}^{n} N(i) \Big) \tau(j) \prod_{i=j}^{n-1} \frac{1-L_i}{1+P_i},\ \text{for } n \in \mathbb{Z}[n_0, n_{p+1}], \tag{17}$$

where $\tau(n)$ is given by (5) for $\gamma_j = \xi_j(A(n_j - 1)),\ j \in \mathbb{Z}[1, p]$

From (17) it follows for $n \in I_k, k \in \mathbb{Z}[0, p]$ the equality $\Delta A(n) = f(n, A(n), A(n+1)),\quad k \in \mathbb{Z}[0, p]$ holds and for $n = n_k,\quad k \in \mathbb{Z}[1, p]$ the equality $A(n_k) = g_k(A(n_k - 1)),\quad k \in \mathbb{Z}[1, p]$ is satisfied.

Therefore, the function $A(n)$ is a solution of IDE (1), $n \in \mathbb{Z}[n_0, n_{p+1}]$

Similarly, we prove the function $B(n)$ is a solution of IDE (1), $n \in \mathbb{Z}[n_0, n_{p+1}]$

We will prove that the functions $A(n)$ and $B(n)$ are minimal and maximal solutions of the initial value problem (1) in $S(\alpha,\beta)$.

Let $u \in S(\alpha,\beta)$ be a solution of IVP for IDE (1). From inequalities (16) it follows there exists a natural number p such that $p \in \mathbb{N}:$

$$\alpha^{(p)}(n) \le u(n) \le \beta^{(p)}(n) \quad \text{for} \quad n \in \mathbb{Z}[n_0, n_{p+1}].$$

We introduce the notation $m(n) = \alpha^{(p+1)}(n) - u(n), \quad n \in \mathbb{Z}[n_0, n_{p+1}]$.

For any $n \in I_k, \quad k \in \mathbb{Z}[0,p]$ we get the inequality

$$\begin{aligned}
&m(n+1) = m(n) + L_n u(n) - L_n u(n) + P_n u(n+1) - P_n u(n+1)\\
&- L_n \alpha^{(p+1)}(n) - P_n \alpha^{(p+1)}(n+1) + f(n, \alpha^{(p)}(n), \alpha^{(p)}(n+1))\\
&+ L_n \alpha^{(p)}(n) + P_n \alpha^{(p)}(n+1) - f(n, u(n), u(n+1))\\
&\le m(n) - L_n(\alpha^{(p+1)}(n) - u(n)) - P_n(\alpha^{(p+1)}(n+1) - u(n+1))\\
&= (1 - L_n)m(n) - P_n m(n+1)
\end{aligned}$$

Hence the inequality $m(n+1) \le \frac{1-L_n}{1+P_n}\, m(n)$ holds for $n \in I_k,\ k \in \mathbb{Z}[0,p]$.

For any $n = n_k, \quad k \in \mathbb{Z}[0,p]$ we obtain

$$\begin{aligned}
&m(n_k) = M_k \alpha^{(p+1)}(n_k - 1) + M_k u(n_k - 1) - M_k u(n_k - 1)\\
&+ g_k(\alpha^{(p)}(n_k - 1)) - M_k \alpha^{(p)}(n_k - 1) - g_k(u(n_k - 1))\\
&\le M_k(\alpha^{(p+1)}(n_k - 1) - u(n_k - 1)) - M_k(\alpha^{(p)}(n_k - 1) - u(n_k - 1))\\
&+ M_k(\alpha^{(p)}(n_k - 1) - u(n_k - 1))\\
&= M_k m(n_k - 1)
\end{aligned}$$

According to Lemma 2 with $Q_n = \frac{1-L_n}{1+P_n}$, the function $m(n)$ is non-positive, i.e. $\alpha^{(p+1)}(n) \le u(n), \quad n \in \mathbb{Z}[n_0, n_{p+1}]$. Similarly $\beta^{(p+1)}(n) \ge u(n), \quad n \in \mathbb{Z}[n_0, n_{p+1}]$, and hence $\alpha^{(j+1)} \le u(n) \le \beta^{(j+1)}, \quad n \in \mathbb{Z}[n_0, n_{p+1}]$. Since $\alpha^{(0)}(n) \le u(n) \le \beta^{(0)}(n)$ this proves by induction that $\alpha^{(j)}(n) \le u(n) \le \beta^{(j)}(n), \quad n \in \mathbb{Z}[n_0, n_{p+1}]$,for every j.

Taking the limit as $j \to \infty$ we conclude $A(n) \le u(n) \le B(n), \quad n \in \mathbb{Z}[n_0, n_{p+1}]$. Hence $A(n)$ and $B(n)$ are minimal and maximal solutions of IVP for IDE (1), respectively.

Let the IVP for IDE (1) has an unique solution $u(n) \in S(\alpha,\beta)$.

Then from above it follows $A(n) \equiv u(n) \equiv B(n), \quad n \in \mathbb{Z}[n_0, n_{p+1}]$. □

4. Applications

Consider the nonlinear IVP for IDE

$$\begin{aligned}
&\Delta x(n) = a_n x(n)(x(n) + 0.9) + b_n x(n+1)(x(n+1) + 2),\\
&\qquad n \in I_k,\ k \in \mathbb{Z}[0,3]\\
&x(5k) = b_k x(5k-1)(x(5k-1) + 6),\ k \in \mathbb{Z}[1,3]\\
&x(0) = 0,
\end{aligned} \tag{18}$$

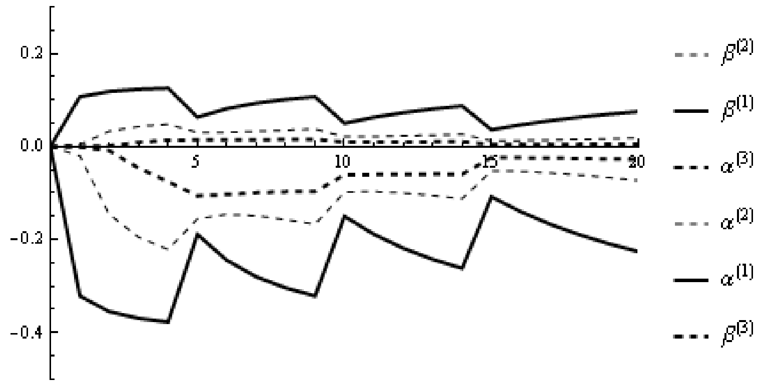

FIGURE 1. Graphs of the succesive lower/upper approximations $\alpha^{(s)}(n)$ and $\beta^{(s)}(n),\ s = 0, 1, 2, 3$ of IDE (18).

where$I_k = [5k, 5(k+1) - 2],\ k \in \mathbb{Z}[0, 2],\ I_3 = \mathbb{Z}[15, 19]$ and $x \in \mathbb{R},\ a_n = -\frac{1}{n+2},\ b_n = -\frac{1}{(n+2)^2},\ b_k = 0.1$. The IDE (18) has a zero solution. We will use the given above algorithm to construct succesive aproximations to the zero solution.

The function $\alpha(n) = -0.5,\ n \in \mathbb{Z}[0, 20]$ is a lower solution of IDE (18) and the function $\beta(n) = 0.5,\ n \in \mathbb{Z}[0, 20]$ is a upper solution of IDE (18).

The condition 2 of Theorem 1 is satisfied for $f(n, x, y) = a_n x(x + 0.9) + b_n y(y + 2)$ with $L_n = -1.9a_n < 1$ and $P_n = -3b_n > -1,\ n \in \mathbb{Z}[0, 20]$.

The condition 3 of Theorem 1 is satisfied for the functions $g_k(z) = b_k z(z+6)$ with $M_k = 7b_k > 0$. According to Theorem 1 there exist a solution of IDE (18) in $S(-0.5, 0.5)$ and we can construct two sequences of successive approximations of the exact solution of (18).

The lower approximations $\alpha^{(s)}(n), s = 1, 2, 3,\ n \in \mathbb{Z}[0, 20]$ are given by (14) with

$\psi(j, \alpha^{(s-1)}(j), \alpha^{(s-1)}(j+1)) = a_j \alpha^{(s-1)}(j)(\alpha^{(s-1)}(j)+0.9)+b_j \alpha^{(s-1)}(j+1)(\alpha^{(s-1)}(j+1)+2)+L_j \alpha^{(s-1)}(j)+P_j \alpha^{(s-1)}(j+1)$ and $\alpha^{(0)}(n) = -0.5,\ n \in \mathbb{Z}[0, 20]$. The functions $N(n)$ and $\tau(n)$ are given by (4) and (5), respectively for $M_j = 7bj,\ j \in \mathbb{Z}[1, 3],\ \gamma_j = b_j \alpha^{(s-1)}(5j - 1)(\alpha^{(s-1)}(5j - 1) + 6) - M_j \alpha^{(s-1)}(n_j - 1),\ j \in \mathbb{Z}[1, 3]$.

Similarly, the upper approximations $\beta^{(s)}(n),\ s = 1, 2, 3,\ n \in \mathbb{Z}[0, 20]$ are given by (14) with $\psi(j, \beta^{(s-1)}(j), \beta^{(s-1)}(j + 1)) = a_j \beta^{(s-1)}(j)(\beta^{(s-1)}(j) + 0.9) + b_j \beta^{(s-1)}(j + 1)(\beta^{(s-1)}(j + 1) + 2) + L_j \beta^{(s-1)}(j) + P_j \beta^{(s-1)}(j + 1)$ and $\beta^{(0)}(n) = 0.5,\ n \in \mathbb{Z}[0, 20]$. The functions $N(n)$ and $\tau(n)$ are given by (4) and (5), respectively for $M_j = 7bj,\ j \in \mathbb{Z}[1, 3],\ \gamma_j = b_j \beta^{(s-1)}(5j - 1)(\beta^{(s-1)}(5j - 1) + 6) - M_j \beta^{(s-1)}(n_j - 1),\ j \in \mathbb{Z}[1, 3]$.

The graphs of the lower approximations $\alpha^{(s)}(n), s = 1, 2, 3$ and the upper approximations $\beta^{(s)}(n),\ s = 1, 2, 3$ are given on the Figure 1. It can be seen the sequence of lower approximations is an increasing, the sequence of upper approximations is a decreasing and both approach to exact solution of (18) $x(n) = 0,\ n \in \mathbb{Z}[0, 20]$.

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