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ASYMMETRIC INNER PRODUCT AND THE ASYMMETRIC QUASI NORM FUNCTION

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Abstract. This paper attempts to generalize the semi scalar product concept according to G. Lumer by replacing Cauchy inequality with another inequality which is more generalized. Based on this attempt of generalization it is built a function which fulfils the conditions which are changed. In this paper it is also generalized quasi norm function by replacing homogeneity condition with a more restricted condition by producing this time a more generalized asymmetric semi norm function. As a result, in this paper it is defined the asymmetric inner product function and the asymmetric quasi norm function. Moreover, it is even given relation between these two.

1. INTRODUCTION

Semi-inner products, that can be naturally defined in general Banach spaces over the real or complex number field, play an important role in describing the geometric properties of these spaces.

Starting from its axiomatic, many researchers have made various modifications passing in its generalization. Semi-scalar products mark the very first generalizations of the scalar product function. The strong bond between these functions with the norm function has made it possible to obtain a lot of interesting results which are connected with the orthogonality and convexity [1],[2].

In [3],[4] it is also generalized the quasi norm function by replacing homogeneity condition with a more restricted condition by producing this time a more generalized asymmetric semi norm function.

Let be $p_0 : \mathbb{R} \to \mathbb{R}^+$ a function defined by:

$$p_0(x) = \begin{cases} |x|, & x < 0 \\ 2|x|, & x \geq 0 \end{cases}.$$

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Definition 1. *The* $p_0 : \mathbb{R} \to \mathbb{R}^+$ *function is called an asymmetric semi norm if:*

a) $p_0(x) \geq 0$ *for* $\forall x \in \mathbb{R}$.

b) $p_0(\lambda x) = \lambda p_0(x)$ *for* $\lambda > 0, \forall x \in \mathbb{R}$

c) $p_0(x+y) \leq p_0(x) + p_0(y)$ $\forall x, y \in \mathbb{R}$.

For every $x(x_1, x_2) \in \mathbb{R}^2$, we define the function $p(x) = p_0(x_1) + p_0(x_2)$, where $p_0(x_1), p_0(x_2)$ are asymmetric semi norms in $\mathbb{R}$.

Proposition 1. *The function* $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *such that* $p(x) = p_0(x_1) + p_0(x_2)$ *it is also an asymmetric semi norm in* $\mathbb{R}^2$.

Proof. a) We have

$$p(x) = p_0(x_1) + p_0(x_2) \geq 0, \forall (x_1, x_2) \in \mathbb{R}^2 \text{ and}$$
$$p(x) = 0 \Rightarrow p_0(x_1) = 0 \wedge p_0(x_2) = 0 \Rightarrow x_1 = x_2 = 0$$

b) We have

$$\begin{aligned} p(\lambda x) &= p_0(\lambda x_1) + p_0(\lambda x_2) = \lambda p_0(x_1) + \lambda p_0(x_2) \\ &= \lambda [p_0(x_1) + p_0(x_2)] = \lambda p(x), \quad \text{for } \lambda > 0. \end{aligned}$$

c) We have

$$\begin{aligned} p(x+y) &= p_0(x_1 + y_1) + p_0(x_2 + y_2) \\ &\leq p_0(x_1) + p_0(y_1) + p_0(x_2) + p_0(y_2) \\ &= [p_0(x_1) + p_0(x_2)] + [p_0(y_1) + p_0(y_2)] \\ &= p(x) + p(y). \end{aligned}$$

So $p(x+y) \leq p(x) + p(y)$, $\forall x, y \in \mathbb{R}^2$.

For every two points $x(x_1, x_2)$ and $y(y_1, y_2)$ in $\mathbb{R}^2$ we build the function $(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that:

$$(x, y) = \begin{cases} p(y)[\frac{x_1 y_1}{p_0(y_1)} + \frac{x_2 y_2}{p_0(y_2)}], & \text{for } y_1 \neq 0 \text{ and } y_2 \neq 0, \\ p(y)\frac{x_1 y_1}{p_0(y_1)}, & \text{for } y_1 \neq 0 \text{ and } y_2 = 0, \\ p(y)\frac{x_2 y_2}{p_0(y_2)}, & \text{for } y_1 = 0 \text{ and } y_2 \neq 0, \\ 0, & \text{for } y_1 = 0 \text{ and } y_2 = 0. \end{cases}.$$

The function defined above have the following properties:

1) $(x, x) \geq 0, \forall (x_1, x_2) \in \mathbb{R}^2$.

2) For $\lambda > 0$

$$(x, \lambda y) = p(\lambda y)[\frac{x_1(\lambda y_1)}{p_0(\lambda y_1)} + \frac{x_2(\lambda y_2)}{p_0(\lambda y_2)}] = \lambda^2 p(y)[\frac{x_1 y_1}{\lambda p_0(y_1)} + \frac{x_2 y_2}{\lambda p_0(y_2)}]$$

$$= \lambda p(y)[\frac{x_1 y_1)}{p_0(y_1)} + \frac{x_2 y_2}{p_0(y_2)}].$$

$$(\lambda x, y) = p(y)[\frac{(\lambda x_1) y_1}{p_0(y_1)} + \frac{(\lambda x_2) y_2}{p_0(y_2)}] = \lambda p(y)[\frac{x_1 y_1}{p_0(y_1)} + \frac{x_2 y_2}{p_0(y_2)}] = \lambda (x, y),\ \lambda \in \mathbb{R}.$$

3) $(x + x', y) = (x, y) + (x', y)$

<u>*Case 1*</u>. $x = (x_1, x_2)$, $x' = (x_1', x_2')$ and $y = (y_1, y_2)$ where $x_1 \neq 0, x_2 \neq 0,$ $x_1' \neq 0,\ x_2' \neq 0:$

$$(x + x', y) = p(y)[\frac{(x_1 + x_1') y_1}{p_0(y_1)} + \frac{(x_2 + x_2') y_2}{p_0(y_2)}]$$

$$= p(y)[\frac{x_1 y_1}{p_0(y_1)} + \frac{x_1' y_1}{p_0(y_1)} + \frac{x_2 y_2}{p_0(y_2)} + \frac{x_2' y_2}{p_0(y_2)}]$$

$$= p(y)[\frac{x_1 y_1}{p_0(y_1)} + \frac{x_2 y_2}{p_0(y_2)}] + p(y)[\frac{x_1' y_1}{p_0(y_1)} + \frac{x_2' y_2}{p_0(y_2)}]$$

$$= (x, y) + (x', y).$$

<u>*Case 2*</u>. $x = (x_1, x_2)$, $x' = (x_1', 0)$ and $y = (y_1, y_2)$ where $x_1 \neq 0, x_2 \neq 0,$ $x_1' \neq 0:$

$$(x, y) = p(y)[\frac{x_1 y_1}{p_0(y_1)} + \frac{x_2 y_2}{p_0(y_2)}] \ \ \textit{and} \ \ (x', y) = p(y)\frac{x_1' y_1}{p_0(y_1)}$$

In this case $x + x' = (x_1 + x_1', x_2)$ therefore:

$$(x + x', y) = p(y)[\frac{(x_1 + x_1') y_1}{p_0(y_1)} + \frac{x_2 y_2}{p_0(y_2)}]$$

$$= p(y)[\frac{x_1 y_1}{p_0(y_1)} + \frac{x_2 y_2}{p_0(y_2)}] + p(y)\frac{x_1' y_1}{p_0(y_1)}$$

$$= (x, y) + (x', y).$$

The reconciliation $(x + x', y) = (x, y) + (x', y)$ goes equally in these cases:

a) $x = (x_1, x_2), x' = (0, x_2')$ and $y = (y_1, y_2)$ where $x_1 \neq 0, x_2 \neq 0, x_2' \neq 0$

b) $x = (x_1, 0), x' = (x_1', x_2')$ and $y = (y_1, y_2)$ where $x_1 \neq 0, x_1' \neq 0, x_2' \neq 0$

c) $x = (0, x_2), x' = (x_1', x_2')$ and $y = (y_1, y_2)$ where $x_2 \neq 0, x_1' \neq 0, x_2' \neq 0$

<u>*Case 3:*</u> $x = (x_1, x_2), x' = (x'_1, x'_2)$ and $y = (y_1, y_2)$ where $x_1 \neq 0, x_2 \neq 0,$ $x_1' \neq 0, x_2' \neq 0$ but $x_1 + x_1' = 0$ and $x_2 + x_2' = 0$ so $x_1 = -x_1'$ and $x_2 = -x_2'$.

In this case $x + x' = (0, 0)$ therefore $(x + x', y) = 0$ while:

$$(x', y) = p(y)[\frac{x_1' y_1}{p_0(y_1)} + \frac{x_2' y_2}{p_0(y_2)} = p(y)[\frac{-x_1 y_1}{p_0(y_1)} + \frac{-x_2 y_2}{p_0(y_2)}]$$

$$= -p(y)[\frac{x_1 y_1}{p_0(y_1)} + \frac{x_2 y_2}{p_0(y_2)}] = -(x, y)$$

from where: $(x, y) + (x', y) = 0 = (x + x', y)$.

Case 4: $x=(x_1,x_2)$, $x'=(x_1',x_2')$ and $y=(y_1,y_2)$ where $x_1\neq 0, x_2\neq 0$, $x_1'\neq 0$, $x_2'\neq 0$ but $x_1+x_1'=0$ so $x_1=-x_1'$.

In this case $x+x'=(0,x_2+x_2')$ therefore:

$$(x+x',y)=p(y)\frac{(x_2+x_2')y_2}{p_0(y_2)}=p(y)\frac{x_2y_2}{p_0(y_2)}+p(y)\frac{x_2'y_2}{p_0(y_2)}$$

while:

$$(x,y)=p(y)[\frac{x_1y_1}{p_0(y_1)}+\frac{x_2y_2}{p_0(y_2)}]$$

and

$$(x',y)=p(y)[\frac{x_1'y_1}{p_0(y_1)}+\frac{x_2'y_2}{p_0(y_2)}]=p(y)[\frac{-x_1y_1}{p_0(y_1)}+\frac{x_2'y_2}{p_0(y_2)}]$$

Since, from $(x,y)+(x',y)=p(y)\frac{x_2y_2}{p_0(y_2)}+p(y)\frac{x_2'y_2}{p_0(y_2)}=(x+x',y)$.

It is equally demonstrated when: $x=(x_1,x_2), x'=(x_1',x_2')$ and $y=(y_1,y_2)$ where $x_1\neq 0, x_2\neq 0, x_1'\neq 0, x_2'\neq 0$ but $x_2+x_2'=0$ so $x_2=-x_2'$.

4) From the definition of the function

$$p_0(x)=\begin{cases}|x|, & x<0\\ 2|x|, & x\geq 0\end{cases}$$

we obtain the inequality: $|x|\leq p_0(x)$, $\forall x\in\mathbb{R}^2$, from where:

$$|x_1|\leq p_0(x_1)\wedge|x_2|\leq p_0(x_2), |y_1|\leq p_0(y_1)\wedge|y_2|\leq p_0(y_2)$$

brings:

$$\begin{aligned}|(x,y)|&\leq p(y)[|x_1|\frac{|y_1|}{p_0(y_1)}+|x_2|\frac{|y_2|}{p_0(y_2)}]\\ &=p(y)[|x_1|+|x_2|]\leq p(y)\left[p_0(x_1)+p_0(x_2)\right]\\ &=p(y)p(x)=p(x)p(y).\end{aligned}$$

So $|(x,y)|\leq p(x)p(y)$, from where $(x,x)=|(x,x)|\leq p^2(x)$.

Remark. For (x,x) where $x(x_1,x_2)\in\mathbb{R}^2$ we have:

1) $x_1\neq 0, x_2\neq 0\Rightarrow p(x_1)\neq 0, p(x_2)\neq 0\Rightarrow$

$$\begin{aligned}(x,x)&=p(x)[\frac{x_1^2}{p_0(x_1)}+\frac{x_2^2}{p_0(x_2)}]=p(x)[\frac{|x_1|^2}{p_0(x_1)}+\frac{|x_2|^2}{p_0(x_2)}]\\ &\leq p(x)\left[p_0(x_1)+p_0(x_2)\right]=p^2(x).\end{aligned}$$

2) $x_1\neq 0, x_2=0\Rightarrow p(x_1)\neq 0, p(x_2)=0\Rightarrow$

$$(x,x)=p(x)\frac{x_1^2}{p_0(x_1)}=|x_1|^2\leq p_0^2(x)=p^2(x)$$

3) $x_1=0, x_2\neq 0\Rightarrow p(x_1)=0, p(x_2)\neq 0$

$$(x,x)=p(x)\frac{x_2^2}{p_0(x_2)}=|x_2|^2\le p_0^2(x)=p^2(x)$$

4) $x_1=0, x_2=0 \Rightarrow p(x_1)=p(x_2)=0 \Rightarrow p(x)=0 \ \Rightarrow (x,x)=0=p^2(x)$

Finally: $|(x,x)|\le p^2(x)$.

Remark. Frankly, $(x,x)=p^2(x)$ every time is not true. Because for $x=(-1,2)$ we have $p(x)=|-1|+2|2|=5 \Rightarrow p^2(x)=25$ and other side:

$$(x,x)=p(x)[\frac{x_1^2}{p_0(x_1)}+\frac{x_2^2}{p_0(x_2)}]=5[\frac{(-1)^2}{|-1|}+\frac{2^2}{2|2|}]=5[1+1]=10\ .$$

In this case $(x,x)\ne p^2(x)$.

Record 1. *Also we can prove that* $p^2(x)\le 2(x,x)$.

Proof. <u>Case 1</u>: For $x(x_1,x_2)\in\mathbb{R}^2$ where $x_1<0\wedge x_2<0$ we have:

$$(x,x)=p(x)[\frac{x_1^2}{p_0(x_1)}+\frac{x_2^2}{p_0(x_2)}]=p(x)[\frac{|x_1|^2}{|x_1|}+\frac{|x_2|^2}{|x_2|}]=p(x)[|x_1|+|x_2|]=p(x)p(x)$$

or $p^2(x)=(x,x)\le 2(x,x)$

<u>Case 2</u>: For $x(x_1,x_2)\in\mathbb{R}^2$ where $x_1>0\wedge x_2>0$ we have:

$$(x,x)=p(x)[\frac{x_1^2}{p_0(x_1)}+\frac{x_2^2}{p_0(x_2)}]=p(x)[\frac{|x_1|^2}{2|x_1|}+\frac{|x_2|^2}{2|x_2|}]=p(x)[\frac{|x_1|}{2}+\frac{|x_2|}{2}]=\frac{p^2(x)}{2}.$$

So $p^2(x)=2(x,x)$.

<u>Case 3</u>: For $x(x_1,x_2)\in\mathbb{R}^2$ and $x_1>0\wedge x_2<0$ $[x_1<0\wedge x_2>0]$ we have:

$$(x,x)=p(x)[\frac{x_1^2}{p_0(x_1)}+\frac{x_2^2}{p_0(x_2)}]=p(x)[\frac{|x_1|^2}{2|x_1|}+\frac{|x_2|^2}{|x_2|}]$$

$$=p(x)[\frac{|x_1|}{2}+|x_2|]\ge p(x)[\frac{|x_1|}{2}+\frac{|x_2|}{2}]=\frac{p^2(x)}{2}.$$

So $p^2(x)\le 2(x,x)$.

Record 2. *The function* (x,y) *defined as above provides the benefit of the function* $\overline{p}:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *such that:* $\overline{p}(x)=\sqrt{(x,x)}$.

From the inequality: $p^2(x)\le 2(x,x)$ we have

$$p^2(x)\le 2\overline{p}^2(x) \text{ or } p\ (x)\le\sqrt{2}\,\overline{p}\ (x)\ ,$$

and from the inequality $|(x,y)|\le p(x)p(y)$ we have:

$$|(x,y)|\le p(x)p(y)\le\sqrt{2}\,\overline{p}\ (x)\sqrt{2}\,\overline{p}(y)=2\overline{p}(x)\overline{p}(y)\ .$$

So for the function $\overline{p}:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ these properties hold:

1) $\overline{p}(x) \geq 0, \overline{p}(x) = 0 \Rightarrow x = 0$ for $x \in \mathbb{R}^2$
2) $\overline{p}(\lambda x) = \lambda \overline{p}(x)$, for $\lambda > 0$, $x \in \mathbb{R}^2$
3) for $x, y \in \mathbb{R}^2$:

$$\begin{aligned}\overline{p}^2(x+y) &= |(x+y, x+y)| \leq |(x, x+y)| + |(y, x+y)| \\ &\leq 2\overline{p}(x)\overline{p}(x+y) + 2\overline{p}(y)\overline{p}(x+y) \\ &= 2\overline{p}(x+y)[\overline{p}(x) + \overline{p}(y)]\end{aligned}$$

So $\overline{p}(x+y) \leq 2[\overline{p}(x) + \overline{p}(y)]$, for $x, y \in \mathbb{R}^2$.

2. MAIN RESULTS

Definition 2. The function $(\cdot,\cdot): X \times X \to \mathbb{R}$, where X is a vectorial space, it is called *the asymmetric quasi inner product* if:
a) $(x, x) \geq 0$, $\forall x \in X$
b) $(\lambda x, y) = \lambda(x, y)$, $\forall (x, y) \in X^2$ and $\forall \lambda \in \mathbb{R}$
$(x, \lambda y) = \lambda(x, y)$, $\forall (x, y) \in X^2$ and $\lambda > 0$
c) $(x + x', y) = (x, y) + (x' + y)$, $\forall x, x', y \in X$
d) $|(x, y)|^2 \leq k(x, x)(y, y)$, for $k \geq 1$.

Definition 3. The function $p : X \to \mathbb{R}^+$ it is called *the asymmetric quasi norm function* if:
a) $p(x) \geq 0$, $\forall x \in X$
b) $p(\lambda x) = \lambda p(x)$, $\forall x \in X$ and $\lambda > 0$
c) $p(x + y) \leq k[p(x) + p(y)]$, $\forall (x, y) \in X^2$ and $k \geq 1$.

Proposition 2. *If* (x, y) *is the asymmetric quasi inner product function on* X *, then the function* $\overline{p} : X \to \mathbb{R}$ *such that* $\overline{p}(x) = \sqrt{(x, x)}$ *is an asymmetric quasi norm function.*
Proof. 1) We have

$$\overline{p}(x) = \sqrt{(x, x)} \geq 0, \forall x \in X$$

2) We have

$$\overline{p}(\lambda x) = \sqrt{(\lambda x, \lambda x)} = \sqrt{\lambda^2 (x, x)}, \text{ for } \lambda > 0.$$

Therefore

$$\overline{p}(\lambda x) = |\lambda| \sqrt{(x, x)} = \lambda \overline{p}(x).$$

3) We have

$$\begin{aligned}\overline{p}^2(x+y) &= |(x+y,x+y)| \\ &= |(x,x+y)+(y,x+y)| \\ &\le |(x,x+y)|+|(y,x+y)| \\ &\le \sqrt{k'(x,x)(x+y,x+y)}+\sqrt{k'(y,y)(x+y,x+y)} \\ &= \sqrt{k'}\,\overline{p}(x)\overline{p}(x+y)+\sqrt{k'}\,\overline{p}(y)\overline{p}(x+y) \\ &= \sqrt{k'}\left[\overline{p}(x)+\overline{p}(y)\right]\overline{p}(x+y).\end{aligned}$$

From where: $\overline{p}(x+y)\le\sqrt{k'}\left[\overline{p}(x)+\overline{p}(y)\right]$ and if we denote $\sqrt{k'}=k\ge 1$ we have:

$$\overline{p}(x+y)\le k[\overline{p}(x)+\overline{p}(y)].$$

3. Conclusions

An asymmetric quasi norm function can be obtained by an asymmetric inner product function, and the link between them is the function: $\overline{p}:X\to\mathbb{R}$, so that $\overline{p}(x)=\sqrt{(x,x)}$.

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