WAVE FRONTS OF ULTRADISTRIBUTIONS VIA FOURIER SERIES COEFFICIENTS

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Abstract. We shall use the properties of the product of periodic ultradistributions and give a new description of the wave front of an ultradistribution $f \in \mathscr{D}^{*\prime}(\mathbb{R}^d)$ in terms of Fourier series coefficients.

1. INTRODUCTION

We have analyzed in [14] the wave front and the Sobolev wave front for distributions through the Fourier coefficients of their appropriate localizations. In this paper we extend the ideas of [14] to ultradistribution spaces. More precisely, we analyze the microlocal properties of an ultradistribution f at $x_0 \in \mathbb{R}^d$ which are determined through the Fourier series expansion of periodizations of φf, where φ is a cut-off function near x_0.

Spaces of periodic ultradistributions have been studied mostly in the last 30 years of the last century. We refer here just a few of the papers and books [5, 6, 7, 15] and also to the papers [18, 17] for the discrete wave fronts which provide another approach to the microlocal analysis. We refer to [3, 4, 10, 11, 16] for a new approach to the analysis of wave fronts and to [1, 2, 12, 19, 20, 22] for the theory of periodic distributions. The main difference between the study of periodic distributions and periodic ultradistributions is in the weight functions we use and the consequences. This will be explained in this paper.

1.1. **Notation.** We follow our notation from [14]. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we write $|x| = (x_1^2 + \ldots + x_d^2)^{1/2}$ and $\langle x \rangle = (1 + |x|^2)^{1/2}$. Let $0 < \eta \leq 1$. We will use the notation

$$I_{\eta,x} = \prod_{j=1}^{d}(x_j - \eta/2, x_j + \eta/2) \text{ and } I_\eta := I_{\eta,0}.$$

We refer to [13] for the spaces of ultradifferentiable functions and their duals, spaces of ultradistributions. In our case, we consider spaces determined by the Gevrey sequence of positive numbers $M_p = p!^s, p \in \mathbb{N}_0$ for some $s > 1$. The sequences of this type satisfy the conditions $(M.1), (M.2), (M.3)$ of Komatsu [13].

Recall that functions or ultradistributions on $\mathbb{R}^d$ are periodic of period 1 in each variable, if $f(x+n) = f(x)$, for all $x \in \mathbb{R}^d, n \in \mathbb{Z}^d$. We put e_y for $e_y(x) = e^{2\pi i y \cdot x}$, $y \in \mathbb{R}^d$. We will consider periodic extensions of localizations of ultradistributions around a point $x_0 \in \mathbb{R}^d$, so if an ultradistribution g is supported by I_{η,x_0}, with $0 < \eta < 1$, we shall write $g_p(x) := \sum_{n \in \mathbb{Z}^d} g(x+n)$ for its periodic extension.

1.2. Spaces of ultradifferentiable functions and their duals.

The space of periodic test functions $\mathscr{P}^* = \mathscr{P}^*(\mathbb{R}^d)$ where $* = (s)$, (respectively $* = \{s\}$) (Beurling and Roumieu cases, respectively) consists of all the ultradifferentiable periodic functions φ of Beurling and Roumieu classes, respectively. It is proved in [7] that φ is periodic ultradifferentiable function of order s of Roumieu type $\mathscr{P}^{\{s\}}(\mathbb{R}^d)$, (resp., Beurling type $\mathscr{P}^{(s)}(\mathbb{R}^d)$) if and only if

$$\sum_{n \in \mathbb{Z}^d} |\varphi_n|^2 \exp(2\alpha^{1/s}|n|^{1/s}) < \infty,$$

for some (resp., any) $\alpha > 0$, where $\varphi_n = \int_{I_1} \varphi(x) e^{-2\pi i n \cdot x} dx = \langle \varphi, e_{-n} \rangle, n \in \mathbb{Z}^d$. The dual space of $\mathscr{P}^*$, the space of periodic ultradistributions, is denoted by $(\mathscr{P}^*)'$. Recall [7]: $f = \sum_{n \in \mathbb{Z}^d} f_n e_n$ is a periodic ultradistribution of order s of Roumieu (Beurling) type if and only if $\sum_{n \in \mathbb{Z}^d} |f_n|^2 \exp(-2\alpha^{1/s}|n|^{1/s}) < \infty$, for any (some) $\alpha > 0$. If $f = \sum_{n \in \mathbb{Z}^d} f_n e_n \in (\mathscr{P}^*)'$ and $\varphi = \sum_{n \in \mathbb{Z}^d} \varphi_n e_n \in \mathscr{P}^*$, then $\langle f, \varphi \rangle = \sum_{n \in \mathbb{Z}^d} f_n \varphi_{-n}$.

In the sequel, we will consider only Roumieu type basic spaces and their duals. The corresponding results for the Beurling type spaces are similar and even easier to prove.

Let ω, ν be positive functions over $\mathbb{Z}^d$. We call the function ν *submultiplicative* if

$$\nu(m+n) \leq \nu(m)\nu(n), \ \ \forall m, n \in \mathbb{Z}^d. \tag{1.1}$$

The function ω is a ν-*moderate* weight if there is a constant C such that

$$\omega(m+n) \leq C\omega(m)\nu(n), \ \ \forall m, n \in \mathbb{Z}^d. \tag{1.2}$$

For a fixed submultiplicative function ν, the set of all ν-moderate weights is $\mathscr{M}_\nu$. Following [11], for $s > 1$,

$$\mathscr{M}_{\{s\}}(\mathbb{Z}^d) = \{\omega \in \mathscr{M}_\nu : (\exists C > 0)(\exists k > 0)(\forall n \in \mathbb{Z}^d)(\nu(n) \leq Ce^{k|n|^{1/s}})\}.$$

For $\omega \in \mathscr{M}_{\{s\}}(\mathbb{Z}^d)$, we define

$$\mathscr{P}l_\omega^q = \{f \in (\mathscr{P}^{\{s\}})' : \{f_n \omega(n)\}_{n \in \mathbb{Z}^d} \in l^q, \text{ where } f_n = \langle f, e_{-n} \rangle\}$$

supplied by the norm $\|f\|_{\mathscr{P}l_\omega^q} = \|\{f_n \omega(n)\}\|_{l^q}$. We consider from now on only values $q \geq 1$. Clearly, $\mathscr{P}l_{\omega_1}^{q_1} \subseteq \mathscr{P}l_{\omega_2}^{q_2}$, if $q_1 \leq q_2$ and $\omega_2 \leq C\omega_1$. We will also consider the local space $\mathscr{P}l_{\omega,loc}^q$ consisting of distributions $f \in (\mathscr{D}^{\{s\}})'(\mathbb{R}^d)$ such that the periodic extensions $(\varphi f)_p \in \mathscr{P}l_\omega^q$, for all $x_0 \in \mathbb{R}^d$ and $\varphi \in \mathscr{D}^{\{s\}}(I_{1,x_0})$. Its topology is defined via the family of seminorms $\|f\|_{x_0,\varphi} = \|(\varphi f)_p\|_{\mathscr{P}l_\omega^q}$, where $x_0 \in \mathbb{R}^d$ and $\varphi \in \mathscr{D}^{\{s\}}(I_{1,x_0})$. For the sake of completeness, we give the following elementary proposition (its proof also shows that the definition of $\mathscr{P}l_{\omega,loc}^q$ is consistent).

Proposition 1.1. $\mathscr{P}l_\omega^q \subset \mathscr{P}l_{\omega,loc}^q.$

Proof. Let $f \in \mathscr{P}l_\omega^q$ and $\varphi \in \mathscr{D}^{\{s\}}(I_{1,x_0})$, then $(\varphi f)_p = \varphi_p f$. Write $f = \sum_n f_n e_n$ and $\varphi_p = \sum_n \varphi_n e_n \in \mathscr{P}^{\{s\}}$. By (1.2) and the generalized Minkowski inequality, we have

$$\|\varphi_p f\|_{\mathscr{P}l_\omega^q} \le C \left(\sum_n \left(\sum_j |\varphi_j| \nu(j) |f_{n-j}| \omega(n-j) \right)^q \right)^{1/q}$$

$$\le C \|\varphi_p\|_{\mathscr{P}l_\nu^1} \|f\|_{\mathscr{P}l_\omega^q} < \infty.$$

$\square$

For fixed $s > 1$, set $\omega_k(n) = e^{k|n|^{1/s}}$, $k \in \mathbb{R}$. For convenience, we write $\mathscr{P}l_k^q := \mathscr{P}l_{\omega_k}^q$ and $\mathscr{P}l_{k,loc}^q := \mathscr{P}l_{\omega_k,loc}^q$. We clearly have

$$\mathscr{P}^{\{s\}} = \bigcap_{k \ge 0} \mathscr{P}l_k^q = \bigcap_{\omega \in \mathscr{M}_{\{s\}}(\mathbb{Z}^d)} \mathscr{P}l_\omega^q$$

and

$$(\mathscr{P}^{\{s\}})' = \bigcup_{k \le 0} \mathscr{P}l_k^q = \bigcup_{\omega \in \mathscr{M}_{\{s\}}(\mathbb{Z}^d)} \mathscr{P}l_\omega^q.$$

Moreover,

$$\mathscr{E}^{\{s\}} = \bigcap_{k \ge 0} \mathscr{P}l_{k,loc}^q = \bigcap_{\omega \in \mathscr{M}_{\{s\}}(\mathbb{Z}^d)} \mathscr{P}l_{\omega,loc}^q$$

and

$$(\mathscr{D}^{\{s\}})'_F = \bigcup_{k \le 0} \mathscr{P}l_{k,loc}^q = \bigcup_{\omega \in \mathscr{M}_{\{s\}}(\mathbb{Z}^d)} \mathscr{P}l_{\omega,loc}^q,$$

where $\mathscr{E}^{\{s\}}$ is the space of all ultradifferentiable functions of order s of Roumieu type and $(\mathscr{D}^{\{s\}})'_F$ is the space of ultradistributions of finite order of Roumieu type on $\mathbb{R}^d$, that is $g \in (\mathscr{D}^{\{s\}})'_F$ if for some ultradifferential operator P of class $\{s\}$ and some G, continuous on $\mathbb{R}^d$, there holds $g = P(D)G$,[13].

1.3. **Multiplication within periodic test spaces.** We fix two weight functions $\omega \in \mathscr{M}_\nu, \nu \in \mathscr{M}_{\{s\}}(\mathbb{Z}^d)$ (cf. (1.2)). Let $f_1 = \sum_{n \in \mathbb{Z}^d} f_{1,n} e_n \in \mathscr{P}l_\omega^{q_1}$ and $f_2 = \sum_{n \in \mathbb{Z}^d} f_{2,n} e_n \in \mathscr{P}l_\nu^{q_2}$. Then we define their product as $f := f_1 f_2 := \sum_{n \in \mathbb{Z}^d} f_n e_n$, where

$$f_n = \sum_{j \in \mathbb{Z}^d} f_{1,n-j} f_{2,j}, \ n \in \mathbb{Z}^d.$$

This definition allows us to introduce multiplication in the local versions of these spaces. Let $f_1 \in \mathscr{P}l_{\omega,loc}^{q_1}$ and $f_2 \in \mathscr{P}l_{\nu,loc}^{q_2}$. Then the product $f := f_1 f_2$ is defined as follows. Let $x_0 \in \mathbb{R}^d$, $0 < \eta < 1$ and let $\phi \in \mathscr{D}^{\{s\}}(I_{1,x_0})$ be such that $\phi(x) = 1$ for $x \in I_{\eta,x_0}$. We define $f_{I_{\eta,x_0}} \in (\mathscr{D}^{\{s\}})'(I_{\eta,x_0})$ as the restriction of $(\phi f_1)_p (\phi f_2)_p$ to I_{η,x_0}. Note that different choices of ϕ lead to different Fourier coefficients but, by Proposition 1.1, we have $f_{I_{\eta,x_0}} = f_{I_{\eta',x_0'}}$ on $I_{\eta,x_0} \cap I_{\eta',x_0'}$. Thus, by $\{f_{I_{\eta,x_0}}\}$ we have a distribution $f \in \mathscr{P}l_{\omega,loc}^q$ as the product of $f_1 f_2 := f$.

Proposition 1.2. *Let $q, q_1, q_2 \in [1, \infty]$ are such that $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q} + 1$. Then the mappings*

$$\mathscr{P}l_\omega^{q_1} \times \mathscr{P}l_\nu^{q_2} \ni (f_1, f_2) \mapsto f_1 f_2 \in \mathscr{P}l_\omega^q \tag{1.3}$$

and

$$\mathscr{P}l_{\omega,loc}^{q_1} \times \mathscr{P}l_{\nu,loc}^{q_2} \ni (f_1, f_2) \mapsto f_1 f_2 \in \mathscr{P}l_{\omega,loc}^q \tag{1.4}$$

are continuous.

Proof. The continuity of (1.4) follows from (1.3), while (1.3), Young's inequality and (1.2) imply

$$\|f_1 f_2\|_{\mathscr{P}l_\omega^q} \leq C \|f_1\|_{\mathscr{P}l_\omega^{q_1}} \|f_2\|_{\mathscr{P}l_\nu^{q_2}} .$$

$\square$

As a corollary, we have:

Corollary 0.1. *Let $k, k_1, k_2 \in \mathbb{R}$,*

$$k_1 + k_2 \geq 0, \qquad k \leq \min\{k_1, k_2\}. \tag{1.5}$$

Then, the mappings $\mathscr{P}l_{k_1}^{q_1} \times \mathscr{P}l_{k_2}^{q_2} \ni (f_1, f_2) \mapsto f_1 f_2 \in \mathscr{P}l_k^q$ and $\mathscr{P}l_{k_1,loc}^{q_1} \times \mathscr{P}l_{k_2,loc}^{q_2} \ni (f_1, f_2) \mapsto f_1 f_2 \in \mathscr{P}l_{k,loc}^q$ are continuous.

Proof. Assume $k_1 \geq 0$ and $k = k_2$ (it is not a restriction). Clearly $k_1 \geq |k_2|$ has to hold in order to have $k_1 + k_2 \geq 0$. The continuity then follows from Proposition 1.2 with $\omega(n) = e^{k_2|n|^{1/s}}$ and $\nu(n) = e^{k_1|n|^{1/s}}$ because (1.2) holds in this case. $\square$

2. Wave Front

As in the case of distributions, our aim is to describe the wave front of an $f \in (\mathscr{D}^{\{s\}})'(\mathbb{R}^d)$ via the Fourier series coefficients of the periodic extension of an appropriate localization of f around $x_0 \in \mathbb{R}^d$. Recall that $(x_0, \xi_0) \notin WF(f)$ if there exist $\psi \in (\mathscr{D}^{\{s\}})'(\mathbb{R}^d)$ with $\psi \equiv 1$ in a neighborhood of x_0 and an open cone $\Gamma \subset \mathbb{R}^d \setminus \{0\}$ containing ξ_0 such that

$$(\exists N > 0)(\exists C_N > 0))(\forall \xi \in \Gamma) \; |\widehat{\psi f}(\xi)| \leq C_N e^{-N|\xi|^{1/s}}. \tag{2.1}$$

Theorem 1. *Let $f \in (\mathscr{D}^{\{s\}})'(\mathbb{R}^d)$ and $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$. The following conditions are equivalent:*

  (i) *There exist $\phi \in \mathscr{D}^{\{s\}}(I_{\varepsilon, x_0})$, with $\varepsilon \in (0, 1)$ and $\phi \equiv 1$ in a neighborhood of x_0, and an open cone Γ containing ξ_0 such that*

$$(\exists N \in \mathbb{N})(\exists C_N > 0)(\forall n \in \Gamma \cap \mathbb{Z}^d) \; |\widehat{\psi f}(n)| \leq C_N e^{-N|n|^{1/s}}. \tag{2.2}$$

  (ii) *$(x_0, \xi_0) \notin WF(f)$.*

Proof. The proof is similar to the one given in [14] for distributions, but for the sake of completeness we give all the details. By shrinking the conic neighborhood of ξ_0, we can choose ψ in (2.1) with arbitrarily small support around x_0. With this (ii) implies (i). So, now it is enough to show that (i) implies (ii). Assume (i). We will prove that there are ε' and an open cone $\xi_0 \in \Gamma_1$ such that

$$(\forall B \text{ bounded set in } \mathscr{D}^{\{s\}}(I_{\varepsilon', x_0}))(\forall N > 0)(\exists C_N' > 0)$$

$$(\forall n \in \Gamma_1 \cap \mathbb{Z}^d) \quad \sup_{\varphi \in B} |\widehat{\varphi f}(n)| \leq C'_N e^{-N|n|^{1/s}}. \tag{2.3}$$

Let ε' be such that $\phi(x) = 1$ for every $x \in I_{\varepsilon', x_0}$. Let Γ_1 be an open cone with $\xi_0 \in \Gamma_1$ and $\overline{\Gamma}_1 \subset \Gamma \cup \{0\}$. Let $0 < c < 1$ be a constant smaller than the distance between $\partial\Gamma$ and the intersection of $\overline{\Gamma}_1$ with the unit sphere. Then $\{y \in \mathbb{R}^d : (\exists \xi \in \Gamma_1)(|\xi - y| \leq c|\xi|)\} \subset \Gamma$. Let $B \subset \mathscr{D}^{\{s\}}(I_{\varepsilon', x_0})$ be a bounded set. Since $\phi\varphi = \varphi$, $\forall \varphi \in B$ we have that $\widehat{f\varphi}(n)$ are the Fourier coefficients of the periodic distribution $(\phi f)_p(\varphi)_p$. Thus, for $\varphi \in B$ and $n \in \Gamma_1 \cap \mathbb{Z}^d$,

$$\left|\widehat{\varphi f}(n)\right| = \left|\sum_{j \in \mathbb{Z}^d} \widehat{\phi f}(n-j)\widehat{\varphi}(j)\right| \leq \left(\sum_{|j| \leq c|n|} + \sum_{|j| > c|n|}\right)|\widehat{\phi f}(n-j)\widehat{\varphi}(j)|$$
$$=: I_1(n) + I_2(n)$$

Now we estimate $I_1(n)$:

$$I_1(n) = \sum_{|n-j| \leq c|n|} |\widehat{\phi f}(j)||\widehat{\varphi}(n-j)| \leq C \sup_{|n-j| \leq c|n|} |\widehat{\phi f}(j)|,$$

where C only depends on B. Since $|n-j| \leq c|n|$ implies $|j| \geq (1-c)|n|$, we have

$$\sup_{\varphi \in B, \, n \in \Gamma_1} e^{-N|n|^{1/s}} I_1(n) \leq C \sup_{n \in \Gamma_1} e^{-N|n|^{1/s}} \sup_{|n-j| \leq c|n|} |\widehat{\phi f}(j)|$$
$$\leq C \sup_{j \in \Gamma_{\xi_0}} (1-c)^{-N} e^{-N|j|^{1/s}} |\widehat{\phi f}(j)| = C(1-c)^{-N} C_N. \tag{2.4}$$

Next we estimate I_2. For this we use that $|n-j| \leq (1+c^{-1})|j|$ if $|j| \geq c|n|$. By the Paley-Wiener theorem, there are $M > 0, D > 0$ such that

$$|\widehat{\phi f}(n-j)| \leq D e^{-M|n-j|^{1/s}}, \quad n, j \in \mathbb{Z}^d.$$

The boundedness of $B \subset \mathscr{D}^{\{s\}}(\mathbb{R}^d)$, implies

$$\sup_{\varphi \in B} \sum_{j \in \mathbb{Z}^d} e^{(M+N)|j|^{1/s}} |\widehat{\varphi}(j)| =: K_N < \infty.$$

Moreover, we have

$$\sup_{n \in \Gamma_1} e^{-N|n|^{1/s}} I_2(n) \leq D \sup_{n \in \Gamma_1} e^{-N|n|^{1/s}} \sum_{|j| \geq c|n|} e^{M|n-j|^{1/s}} |\widehat{\varphi}(j)|$$
$$\leq D C^{-N} (1 + c^{-1})^M K_N, \quad \forall \varphi \in B. \tag{2.5}$$

By (2.4) and (2.5), we obtain (2.3).

Now we show that $(x_0, \xi_0) \notin WF(f)$ by (2.3). Let $\psi \in \mathscr{D}^{\{s\}}(I_{\varepsilon', x_0})$ be equal to 1 in a neighborhood of x_0. Then, the set $B = \{\varphi_t := \psi e_{-t} : t \in [0, 1)^d\}$ is a bounded subset of $\mathscr{D}^{\{s\}}(I_{\varepsilon', x_0})$. This implies

$$\sup_{t \in [0,1)^d} |\widehat{\psi f}(n+t)| = \sup_{t \in [0,1)^d} |\widehat{\varphi_t f}(n)| \leq \frac{C'_N}{e^{N|n|^{1/s}}}, \quad \forall n \in \Gamma_1 \cap \mathbb{Z}^d,$$

$$\sup_{\xi \in (\Gamma_1 \cap \mathbb{Z}^d) + [0,1)^d} |e^{N|\xi|^{1/s}} \widehat{\psi f}(\xi)| \leq (1 + 4d)^{N/2} C'_N. \tag{2.6}$$

Let Γ_2 be an open conic neighborhood of ξ_0 such that $\overline{\Gamma}_2 \subset \Gamma_1 \cup \{0\}$ and let c' be such that $\{y \in \mathbb{R}^d : (\exists \xi \in \Gamma_2)(|\xi - y| \leq c'|\xi|)\} \subset \Gamma_1$. Then $\Gamma_2 \cap \{\xi \in \mathbb{R}^d : |\xi|c' \geq 1\} \subset (\Gamma_1 \cap \mathbb{Z}^d) + [0, 1)^d$. Thus

$$\sup_{\xi \in \Gamma_2} e^{N|\xi|^{1/s}} |\widehat{\psi f}(\xi)| \leq \max\{C_N'', (1 + 4d)^{N/2} C_N'\} = C_N < \infty,$$

where $C_N'' = \sup_{\xi \in \Gamma_2, |\xi| < 1/c'} e^{N|\xi|^{1/s}} |\widehat{\psi f}(\xi)|$. This shows that $(x_0, \xi_0) \notin WF(f)$. The assertion is proved. $\qquad\square$

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