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RATIONAL APPROXIMATED MODELS OF CHAOTIC OSCILLATORS

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Abstract. A class of Sprott dynamical systems is considered, described by a third order differential equations. The non-homogenous nonlinearity is (2, 2)-rational approximation function containing free parameters. The aim of this note is to study system behavior upon variation of these parameters. It is shown that the complexity of generated regime changes from quasi-periodic to chaotic dynamics. Adequate numerical examples are provided.

1. Introduction

A very well known Poincarè Bendixon theorem stated that autonomous first-order ODE with continuous coefficients could not have bounded chaotic solution unless it is at least three dimensional. The classic example is the famous Lorenz's model of atmospheric heath conduction (1963)[1]

$$\begin{aligned}&\dot{x} = A(y - x),\\ &\dot{y} = (B - z)x - y,\\ &\dot{z} = xy - Cz, \quad A, B, C \in \mathbb{R}.\end{aligned}$$

Rössler (1976) simplified this model by making use of toroidal topology thereby gained restriction of the number of terms to six and only one nonlinearity. In 1997 Linz showed that systems proposed by Lorenz and Rössler can be transformed in the following form:

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$$\dddot{x} = J(\ddot{x},\, \dot{x},\, x)\,, \tag{1}$$

where J is called *jerk function*, after the third parametric derivative d^3x/dt^3 of the displacement x which is called "jerk". Thus, in order to study different aspects of chaos, the differential equation (1) can be considered instead of 3D system. In 1994, Sprott underwent experiments with further reduction and simplification of chaotic systems, which inspired Gottlieb to pose important question of finding the simplest jerk function that generates chaos. Trying to further simplify J-function, Sprott [8],[9] considered jerk functions of the form

$$J(\ddot{x},\, \dot{x},\, x), = A\ddot{x} + B\dot{x} + f(x)\,,$$

where A, B are real parameters and $f(x)$ is a simple nonlinear function. He suggested the following nonlinearities:

$$\begin{array}{ll} f(x) = b|x| + c\,, & f(x) = b - c\max(x,\, 0)\,, \\ f(x) = bx - c\,\mathrm{sgn}(x)\,, & f(x) = \mathrm{sgn}[\max(x,\, 0)]\,, \\ f(x) = \mathrm{sgn}(x)\min(|x|,\, 1) \in (|x|,\, 1)\,, & \text{etc.} \end{array} \tag{2}$$

where b and c are real constants. The usual physical meaning of $f(x)$ is characteristic function of a nonlinear element that is used in the dynamical system. In these cases, for proper values of constants, Sprott found these circuits having chaotic behavior, with all corresponding features like doubling of periods (bifurcations), phase locking, inverse bifurcations and so on. Our idea is to consider possibility of approximation of such functions by some approximating function containing free parameters that can be used to control the approximation closeness [2]-[5]. Here, for approximation of Sprott's functions, we will use rational approximation function.

2. Rational approximation

A (p, q)-*rational curve*, $p \geq 0$, $q \geq 1$, is the parametric curve given by

$$R(t) = \{(x(t),\, y(t))\,|t \in [\alpha\,, \beta] \subset \mathbb{R}\}\,,$$

where at last one of the coordinates $x(t)$ or $y(t)$ is a (p, q)-rational function of t, i.e. the function given as ratio of polynomials with degrees p and q

respectively, $t \mapsto \dfrac{P_p(t)}{Q_q(t)}$. Then, every conic section can be described as a $(2,2)$-rational Bézier curve.

Let a (proper) triangle, called *control triangle*, is given by its vertices $\mathbf{P}, \mathbf{Q}$ and $\mathbf{R}$ in the plane. Then, a $(2,2)$-rational parametric function is given by

$$(3) \qquad \mathbf{c}(t) = \mathbf{c}(\omega,\, t) = \frac{\mathbf{P}b_0(t) + \omega\mathbf{Q}b_1(t) + \mathbf{R}b_2(t)}{b_0(t) + \omega b_1(t) + b_2(t)}, \quad 0 \le t \le 1,$$

where $b_0(t) = (1-t)^2$, $b_1(t) = 2t(1-t)$, $b_0(t) = t^2$ are elements of quadratic Berstein basis. The real number ω, called *weight*, is associated with the vertex $\mathbf{Q}$ of the control triangle. Then, the parametric curve $\mathbf{c}(t)$ is a segment of a conic section that interpolates vertices $\mathbf{P}$ and $\mathbf{R}$ (Figure 1). The classification of conic sections upon the weight ω is given in Table 1.

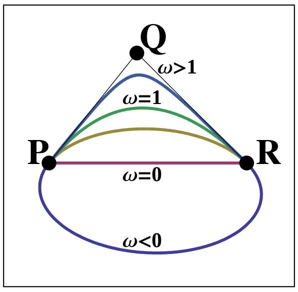


Figure 1.

weight	**conic section**
$\omega > 1$	hyperbola
$\omega = 1$	parabola
$0 < \omega < 1$	ellipse
$\omega = 0$	line segment
$\omega < 0$	complementary segment

Table 1.

Since the "jerk" dynamical system (1) can be described as

$$(4) \qquad \dddot{x} + A\ddot{x} + B\dot{x} = f(x),$$

the function $f(x)$ appears as an inhomogeneity factor. Therefore, it has considerable influence on behavior of the system. This influence is even augmented by the presence of $\mathbf{C}^1$ or $\mathbf{C}^0$ discontinuity of $f(x)$ (one of those given in (2)). The $(2,2)$-rational parametric function (3) can be used to smooth down the discontinuity which, as a result should have milder dynamic and diminishing of chaos degree.

Setting $\mathbf{P} = (p_1, p_2), \mathbf{Q} = (q_1, q_2)$ and $\mathbf{R} = (r_1, r_2)$, the parametric vector form of (3) can be recalculated to bring the univariate form

(5) $$f_\omega = \frac{r_2 m^2(x) + 2q_2\omega m(x)n(x) + p_2 n^2(x)}{[m(x)+n(x)]^2 + 2(\omega-1)m(x)n(x)},$$

where

$$m(x) = p_1 + x(\omega-1) - q_1\omega + \left[(p_1-x)(x-r_1) + \omega^2(x-q_1)^2\right]^{1/2},$$
$$n(x) = r_1 + x(\omega-1) - q_1\omega - \left[(p_1-x)(x-r_1) + \omega^2(x-q_1)^2\right]^{1/2}.$$

To achieve approximation of inhomogeneity function $f(x)$ in the vicinity of the breakpoint x_0, it is advisable to take $p_1 \le q_1 \le r_1$ and $p_2 = f(p_1)$, $q_2 = f(q_1)$, $r_2 = f(r_1)$, and $f(x)$ is then replaced by

(6) $$f^*(x) = \begin{cases} f_\omega(x), & p_1 \le x \le r_1 \\ f(x), & \text{otherwise} \end{cases}.$$

Now, the numerical solution of the modified Cauchy problem

(7) $$\dddot{x} + A\ddot{x} + B\dot{x} = f^*(x), \qquad x(0)=\alpha,\ \dot{x}(0)=\beta,\ \ddot{x}(0)=\gamma,$$

is seeking. Having impact on the degree of smoothing of the discontinuity of $f(x)$, the real parameter ω will have influence on solution of (7) and thereby, on the dynamics of the system itself. But, note that only nonnegative values of ω will be useful, since only $f_\omega(x)$ for $\omega \le 0$ really does the smoothing of $f \in \mathbf{C}^0$ or $f \in \mathbf{C}^{-1}$. By increasing the value of ω one can obtain a closer approximation of the discontinuity. Assuming the symmetric data at $x \in \{-\epsilon, 0, \epsilon\}$, $(\epsilon > 0)$, from the function $f(x) = |x|$ are taken, the approximating function becomes

$$f_\omega(x) = \frac{\omega\sqrt{\epsilon^2 + x^2(\omega^2-1)} - \epsilon}{\omega^2-1},$$

so that

$$\|f_\omega - f\|_\infty = f_\omega(0) = \frac{\epsilon}{\omega+1},$$

which implies $\lim_{\omega\to\infty} \|f_\omega - f\|_\infty = 0$. This is a nice property of parametric rational approximation: the convergence can be controlled just by changing a linear parameter, without increasing the degree of approximand.

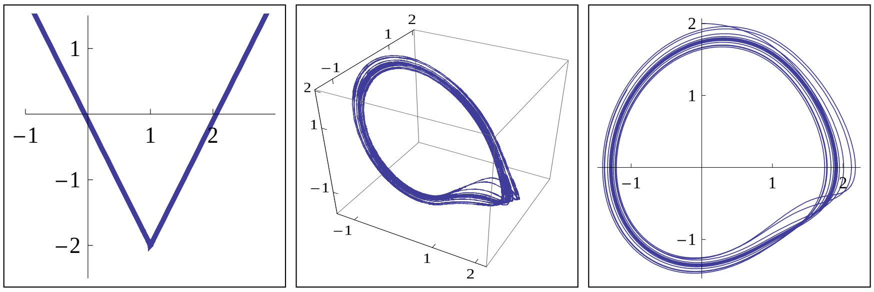
3. Numerical examples

According to the peculiarity of rational parametric approximation, as it is shown above, the function $f_\omega(x)$ approximates discontinuity of inhomogeneity $f(x)$ in the "jerk" dynamical system (4) by smoothing it down. The lower values of ω yields a more parabolic-like functions. So, it is to be expected that the chaotic dynamics for lower values of should be less chaotic. In order to test it, some numerical experiments will be carried out. All examples have been done by employing the Wolfram's MATHEMATICA 6.0 package.

Example 1. Let us consider a circuit described by the initial value problem

$$\dddot{x} + a\ddot{x} - \dot{x} = b|x-1| + c, \quad x(0) = \ddot{x}(0) = 0,\ \dot{x}(0) = 2, \tag{8}$$

where $a = 1.1$, $b = 1.9$, $c = -2$. The jerk function is $f(x) = 1.9|x-1| - 2$. A numerical solution of (8) shown in Figure 2 suggests chaotic behavior of the circuit. The range of time variable is [0, 150]. Here, part a) represents a graph of jerk function while parts b) and c) give the attractor i.e. $(x, \dot{x})$ projection of the attractor in $(x, \dot{x}, \ddot{x})$- phase space respectively.



a) b) c)

Figure 2.

For the purpose of approximation, we choose three sampling knots: $x_1 = 0$, $x_2 = 1$, $x_3 = 2$. Thus, the control polygon will be

$$\{(0, -0.1), (1, -2), (2, -0.1)\}.$$

Choosing $\omega = 1.05$, the parametric rational function is obtained

$$\mathbf{c}(1.05, t) = \left\{ \frac{2.1(1-t)t + 2t^2}{1 + 0.1(1-t)t}, \frac{-0.1(1-t)^2 - 4.2(1-t)t - 0.1t^2}{1 + 0.1(1-t)t} \right\}.$$

Elimination of parameter t yields the function

$$f_\omega(x) = \frac{10^{17}\left(106.47 - 10.1159x + 5.05793x^2 - 32.4677h(x)\right)}{-2.5353 \cdot 10^{17} + 5.6259x + 2.81475x^2 + 8.11692 \cdot 10^{16}h(x)},$$

where $h(x) = \sqrt{10.7561 - 2x + x^2}$. Function $f_\omega(x)$ approximates $f(x)$ over the interval [0, 2]. Now, the right hand side of the modified Cauchy problem (7) is replaced by

$$f^*(x) = \begin{cases} f_\omega(x), & 0 \le x \le 2 \\ f(x), & \text{otherwise} \end{cases}$$

and which, for $0 < x < 2$, have the following asymptotic behavior:

$$f_\omega(x) \sim \frac{1064.7 - 101.159x + 50.5793x^2 - 324.677\sqrt{10.7561 - 2x + x^2}}{-25.353 + 8.11692\sqrt{10.7561 - 2x + x^2}}.$$

Now, the approximation function $f^*(x)$, given by (6) is well defined, and one can look for the numeric solution. Our solution is displayed in Figure 3 for $\omega = 1.05$, 2.0, 3.0, 4.0 and 8.0.

The graph of the corresponding $\mathbf{C}^1$ function is shown in Figure 3 (a), while parts b) and c) give the attractor i.e. $(x, \dot{x})$ projection of the attractor in $(x, \dot{x}, \ddot{x})$- phase space respectively. As it is expected, the attractors obtained exhibit typical chaotic features: non-periodicity, laminar structure and bifurcation grouping of trajectories. The large value of ω ($\omega = 8$) gives approximation function that is closest to the original modular function $f(x)$, and thus, the attractor is close to the original Sprott one. As ω decreases, the attractors becomes more calm and regular, because the derivative of the new inhomogeneity function increasingly differs from the discontinuous derivative of $f(x)$.

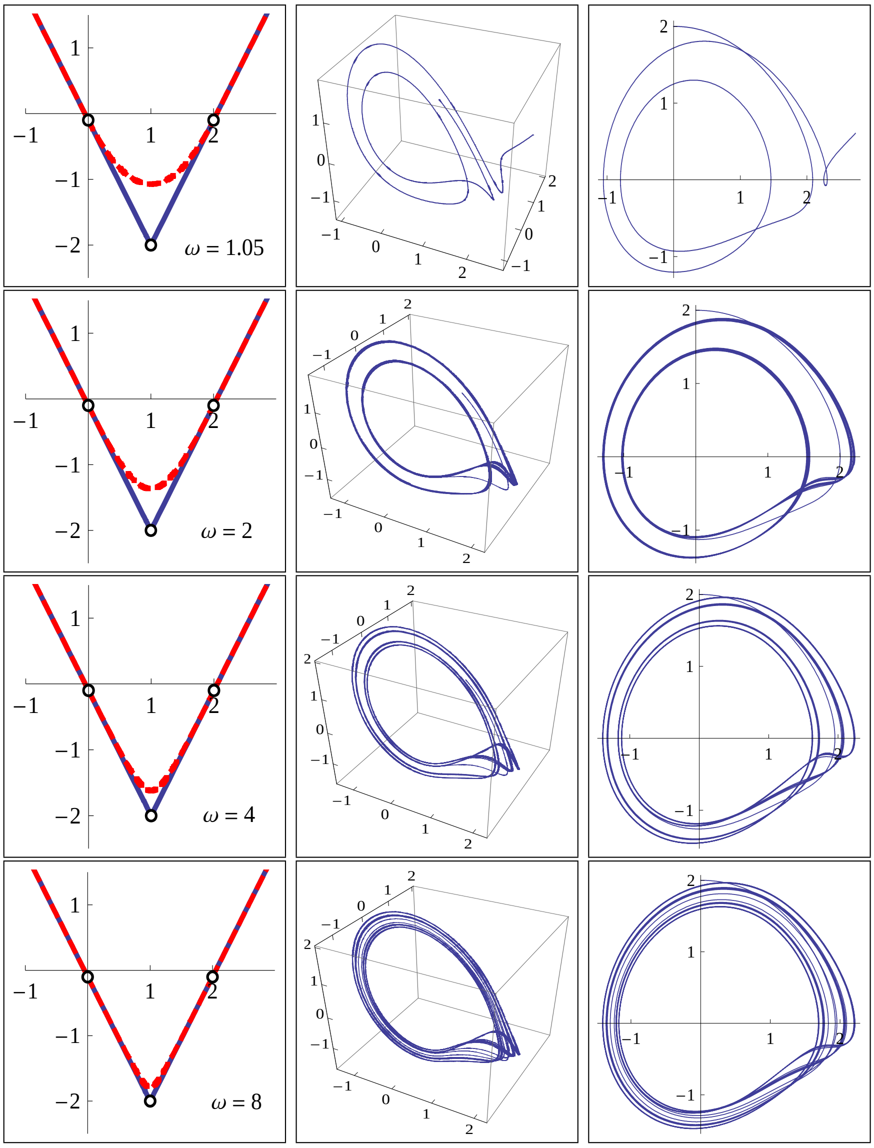


Figure 3.

The next example repeats the pattern of the Example 1.

Example 2. We consider a circuit described by the initial value problem

$$\dddot{x} + 0.6\ddot{x} + \dot{x} = f(x), \quad x(0) = \ddot{x}(0) = 0,\ \dot{x}(0) = 0.1,$$

where the jerk function is $f(x) = \text{sgn}(\max(x, 0))$. Its numerical solution with the range of time variable [0, 150] is shown in the first row in Figure 4. In the second and third row in Figure 4, the graph of approximation function $f^*(x)$, the attractor and its $(x, \dot{x})$ projection for $\omega = 1.05$ and 8.0 are presented.

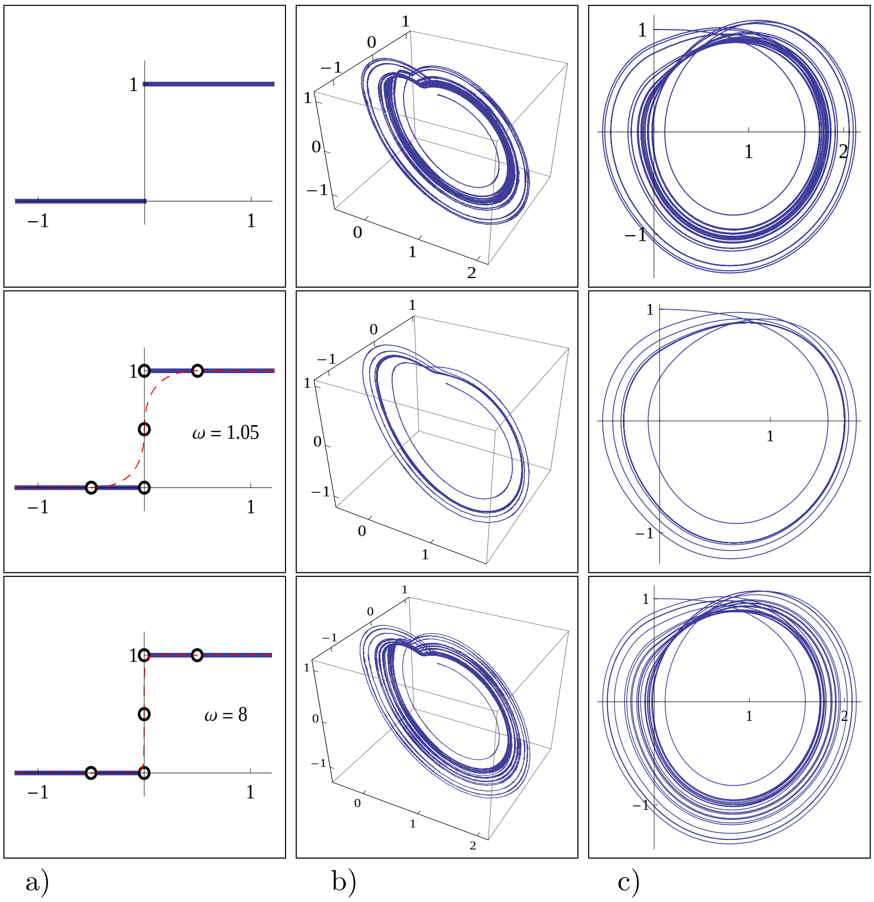


Figure 4.

4. Conclusion

In this paper the dynamics of "jerk" dynamical systems is studied upon variation of inhomogeneity function that appear in the third order differential equation describing the system. The primary inhomogeneity functions, used by Sprott [8], [9], [10] are simple piecewise-linear functions chosen so that the chaotic dynamics they generate is "minimal". Here, the behavior of Sprott "minimally" chaotic systems is examined upon having the inhomogeneity function approximated by a $(2, 2)$-rational function with the shape parameter. The role of parameter is to control approximation closeness. In this sense, rational approximation is considered to be more than high order polynomials or other transcendent functions. The whole palette of Sprott's $\mathbf{C}^1$ or $\mathbf{C}^0$ discontinuous functions like $|x|$, $\mathrm{sgn}(x)$, Heaviside (step) function etc.- functions that might not even have a uniform polynomial approximation at all. Some illustrative examples have been presented.

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РАЦИОНАЛНИ АПРОКСИМАТИВНИ МОДЕЛИ НА ХАОТИЧНИ ОСЦИЛАТОРИ

Соња Геговска-Зајкова, Љубиша Коцић

Апстракт

Во трудот е разгледувана класа Спротови динамички системи опишани со диференцијална равенка од трет ред. Нехомогената нелинеарност е (2, 2)-рационална функција која содржи слободни параметри. Целта на овој труд е да се проучи однесувањето на системот во зависност од овие параметри. Покажано е дека комплексноста на генерираниот режим се

менува од квази-периодична до хаотична динамика. Теоретските резултати се илустрирани со соодветни нумерички примери.

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