ABSOLUTE RIESZ SUMMABILITY FACTORS FOR FOURIER SERIES AND CONJUGATE SERIES

by

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1. **Definitions and Notations.** Let $L = L(w)$ be a differentiable, monotonic increasing $(\uparrow)$, function of w tending to infinity with w. For a given infinite series $\sum a_n$*, we write

$$A_r(w) = \sum_{n \leqslant w} (L(w) - L(n))^r a_n \quad (r \geqslant 0).$$

The series $\sum a_n \in |R, L, r|$ $(r > 0)$, if

$$\int_h^\infty \left| d\left\{\frac{A_r(w)}{(L(w))^r}\right\} \right| < \infty,$$

where h is a positive number ([4], [5], aslo see [3]).

Now, for $r > 0$ and $m < w < m + 1$,

$$\frac{d}{dw}\left\{\frac{A_r(w)}{(L(w))^r}\right\} = \frac{r\,L'(w)}{(L(w))^{r+1}} \sum_{n<w} (L(w) - L(n))^{r-1} L(n)\, a_n.$$

Hence, the series $\sum a_n \in |R, L, r|$ $(r > 0)$, if

$$\int_h^\infty \frac{L'(w)}{(L(w))^{r+1}} \left| \sum_{n<w} (L(w) - L(n))^{r-1} L(n)\, a_n \right| dw < \infty.$$

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* Summations are over $1, 2, \ldots, \infty$ when there is no indication to the contrary.

We define the summability $|R, L, 0|$ equivalent to the absolute convergence.

Let $f(t)$ be a periodic function with period 2π, integrable in the sense of Lebesgue over $(-\pi, \pi)$. Without any loss of generality the constant term of the Fourier series of $f(t)$ can be taken to be zero, that is,

$$\int_{-\pi}^{\pi} f(t)\, dt = 0.$$

Now let the Fourier series of $f(t)$ be

$$\Sigma (a_n \cos nt + b_n \sin nt) = \Sigma A_n(t).$$

So that the conjugate series of the Fourier series of $f(t)$ be

$$\Sigma (b_n \cos nt - a_n \sin nt) = \Sigma B_n(t).$$

The Fourier series $\Sigma A_n(t)$ and its conjugate series $\Sigma B_n(t)$, at the point $t = x$, will be denoted, respectively, by $\Sigma A_n(x)$ and $\Sigma B_n(x)$.

Throughout we use the following notations:

(1.1) $\Phi(t) = \dfrac{1}{2}(f(x+t) + f(x-t)).$

(1.2) $\psi(t) = \dfrac{1}{2}(f(x+t) - f(x-t)).$

(1.3) $e(w) = exp(w^a)$ $(0 < a < 1).$

(1.4) $e^{(1)}(w) = \dfrac{d}{dw}(e(w)).$

(1.5) $e^d(w) = (e(w))^d$ (for finite d).

(1.6) $E(w, t) = \sum\limits_{n<w} (e(w) - e(n))^{r-1} e(n)\, n^{b-1} \sin nt.$

$$(r > 0,\ b > 0)$$

(1.7) $K(w, t) = \sum\limits_{n<w} (e(w) - e(n))^{r-1} e(n)\, n^{b-1} \cos nt.$

$$(r > 0,\ b > 0)$$

2. **Introduction.** In 1951, Mohanty [3, Theorems 2 and 3] proved the following:

Theorem A. If $\Phi(t) \in BV(0, \pi)$, then the series $\sum \frac{A_n(x)}{\log(n+1)}$ $\in |R, e(w), 1|$.

Theorem B. Let $c > 0$ and $b = 1 + \frac{1}{c}$. Them $t^{-c}\, \Phi(t) \in BV(0, \pi)$ implies that $\sum A_n(x) \in \left|R, exp\left\{\frac{w}{(\log w)^b}\right\}, 1\right|$.

By taking a hypothesis like that of Theorem *B* and the absolute Riesz summability process of the kind used in Theorem *A*, the present author [1] proved the following:

Theorem C. Let

(2.1) $a > 0 < b,\ 1 > a + b$ and $b = c(1 - a)$.

Then

(2.2) $t^{-c}\, \Phi(t) \in BV(0, \pi)$

implies that $\sum A_n(x)\, n^b \in |R, e(w), 1|$.

In this context we prove the following theorem which replaces unity, the order of absolute Riesz summability of Theorem *C*, by $r > c > 0$:

THEOREM 1. Let (2.1 **hold. Then** (2.2) **implies that** $\sum A_n(x)\, n^b \in |R, e(w), r|\ (r > c)$.

We also prove the following analogue of Theorem 1 for the conjugate series:

THEOREM 2. Let (2.1) **hold. Then** $t^{-c}\, \psi(t) \in BV(0, \pi)$ **implies that** $\sum B_n(x)\, n^b \in |R, e(w), r|\ (r > c)$.

3. We shall use the following order-estimates tor $1 \geqslant r > 0,\ b > 0$ and large w and uniformly in $0 < t \leqslant \pi$.

(3.1) $\sum\limits_{n<w} (e(w) - e(n))^{r-1}\, e(n)\, n^{b-1} = 0\left\{e^r(w)\, w^{b-a}\right\}$.

$$(3.2)\quad \begin{Bmatrix} E(w,\,t) \\ K(w,\,t) \end{Bmatrix} = O\left\{t^{-r}\, e^r(w)\, w^{b-a+r(a-1)}\right\} (w > t^{-1}).$$

Proof of (3.1). Since $e(n) \sim e(n-1)$, we have

$$\sum_{n<w} (e(w) - e(n))^{r-1}\, e(n)\, n^{b-1} =$$

$$= O\left\{\int_1^w (e(w) - e(x))^{r-1}\, e(x)\, x^{b-1}\, dx\right\}$$

$$= O(1) + O\{I\}, \text{ say.}$$

For w_1, determired by the relation: $w^a - w_1{}^a = 1$, we write

$$I = \int_1^w (e(w) - e(x))^{r-1}\, e(x)\, x^{b-1}\, dx$$

$$= \left(\int_1^{w_1} + \int_{w_1}^w\right)\left((e(w) - e(x))^{r-1}\, e(x)\, x^{b-1}\, dx\right)$$

$$= I_1 + I_2, \text{ say.}$$

Now

$$I_1 = \int_1^{w_1} (e(w) - e(x))^{r-1}\, e(x)\, x^{b-1}\, dx$$

$$= O\left\{(e(w) - e(w_1))^{r-1} \int_1^{w_1} e(x)\, x^{b-1}\, dx\right\}$$

$$= O\left\{e^{r-1}(w) \int_1^w e^{(1)}(x)\, x^{b-a}\, dx\right\}$$

$$= O\left\{e^r(w)\, w^{b-a}\right\},$$

since

$$\int_1^w e^{(1)}(x)\, x^{b-a}\, dx = \Big[e(x)\, x^{b-a}\Big]_1^w - (b-a)\int_1^{\cdot} e(x)\, x^{b-a-1}\, dx$$

$$< e(w)\, w^{b-a} - \frac{b-a}{a} \int_1^w e^{(1)}(x)\, x^{b-2a}\, dx$$

$$< e(w)\, w^{b-a} + \int_1^w e^{(1)}(x)\, x^{b-2a}\, dx$$

and hence we easily follow that

$$\int_1^w e^{(1)}(x)\, x^{b-a}\, dx = O\left\{w^{b-a}\, e(w)\right\}.$$

And

$$I_2 = \int_{w_1}^w (e(w) - e(x))^{r-1}\, e(x)\, x^{b-1}\, dx$$

$$= O\left\{\int_{w_1}^w (e(w) - e(x))^{r-1}\, e^{(1)}(x)\, x^{b-a}\, dx\right\}$$

$$= O\left\{w^{b-a} \int_{w_1}^w (e(w) - e(x))^{r-1}\, e^{(1)}(x)\, dx\right\}$$

$$= O\left\{e^r(w)\, w^{b-a}\right\}.$$

On combining the results we follow the proof of (3.1).

Proof of (3.2). Since the proof of $E(w, t)$ is exactly the same as that of $K(w, t)$, we simply sketch the proof of $K(w, t)$.

Let w_1 and m stand respectively for the integral part of $\left(w - \frac{1}{t}\right)$ and w. Then

$$K(w, t) = \sum_{n=1}^{w_1} + \sum_{w_1+1}^{m} = \sum\nolimits_1 + \sum\nolimits_2, \text{ say.}$$

Now, since $\left\{e(n)\, n^{b-1}\right\} \uparrow$ with $n > p$, where p be the integral part of $\left(\frac{1-b}{a}\right)^{\frac{1}{a}}$, we write

$$\sum\nolimits_1 = \sum_{n=1}^{p} (e(w) - e(n))^{r-1} e(n)\, n^{b-1} \cos nt$$

$$+ \sum_{n=p+1}^{w_1} (e(w) - e(n))^{r-1} e(n)\, n^{b-1} \cos nt$$

$$= O\left\{e^{r-1}(w)\right\} + P, \text{ say.}$$

By Abel's lemma

$$P = O\left\{\left(e(w) - e\left(w - \frac{1}{t}\right)\right)^{r-1} e(w_1)\, w^{b-1} \max_{1+p \leqslant p' \leqslant w_1} \left|\sum_{n=p'}^{w_1} \sin nt\right|\right\}$$

$$= O\left\{t^{-1}\left(t^{-1} e^{(1)}\left(w - \frac{1}{t}\right)\right)^{r-1} e^{(1)}\left(w - \frac{1}{t}\right) w^{b-a}\right\}$$

$$= O\left\{t^{-r}\left(e^{(1)}\left(w - \frac{1}{t}\right)\right)^{r} w^{b-a}\right\}$$

$$= O\left\{t^{-r}\, w^{b-a+r(a-1)}\, e^{r}(w)\right\},$$

uniformly in $0 < t \leqslant \pi$. Thus

$$\sum\nolimits_1 = O\left\{t^{-r}\, w^{b-a+r(a-1)}\, e^{r}(w)\right\}.$$

And

$$\sum\nolimits_2 = O\left\{\int_{w-\frac{1}{t}}^{w} (e(w) - e(x))^{r-1} e(x)\, x^{b-1}\, dx\right\}$$

$$= O\left\{\int_{w-\frac{1}{t}}^{w} (e(w) - e(x))^{r-1} e^{(1)}(x)\, x^{b-a}\, dx\right\}$$

$$= O\left\{w^{b-a}\int_{w-\frac{1}{t}}^{w} (e(w) - e(x))^{r-1} e^{(1)}(x)\, dx\right\}$$

$$= O\left\{w^{b-a}\left(e(w) - e\left(w - \frac{1}{t}\right)\right)^{r}\right\}$$

$$= O\left\{wb^{-a}\left(t^{-1}\,e^{(1)}(w)\right)^{r}\right\}$$

$$= O\left\{t^{-r}\,wb^{-a+r(a-1)}\,e^{r}(w)\right\},$$

uniformly in $0 < t \leqslant \pi$.

On combining $\sum_1$ and $\sum_2$, we follow the proof.

4. We use the foolowing in this paper:

LEMMA 1. $\sum a_n \in |R, L, r|\ (r \geqslant 0)$ **implies that** $\sum a_n \in |R, L, r'|$ $(r' > r)$.

This is due to Obrechkoff [4, 5].

LEMMA 2. **Let** $0 < \beta < \infty$ **and** $0 < c \leqslant 1$. **Then, uniformly in** $0 < t \leqslant \beta$,

$$\int_0^t u^c \cos nu\ du = t^c\,\frac{\sin nt}{n} + O\left(n^{-1-c}\right) \tag{4.1}$$

and

$$\int_0^t u^c \sin nu\ du = -\,t^c\,\frac{\cos nt}{n} + O\left(n^{-1-c}\right). \tag{4.2}$$

For the proof of (4.1), see Chandra [1], Lemma 2. And for the proof of (4.2), reference can be made of (2,2) of Chandra [2].

5. **Proof of the theorems.** In view of Lemma 1, we take $0 < r \leqslant 1$ for the proof of both the theorems.

5.1. **Proof of Theorem 1.** We have

$$A_n(x) = \frac{2}{\pi}\int_0^\pi \Phi(t)\cos nt\ dt$$

$$= \frac{2}{\pi}\int_0^\pi t^{-c}\,\Phi(t)\,t^c \cos nt\ dt$$

$$= \frac{2}{\pi}\,\pi^{-c}\,\Phi(\pi)\int_0^\pi t^c \cos nt\ dt$$

$$-\frac{2}{\pi}\int_0^{\pi} d\,(t^{-c}\,\Phi\,(t))\int_0^{t} u^c \cos nu\, du,$$

integrating by parts.

The series $\sum A_n(x)\, n^b \in |R, e(w), r|\ (r > c)$, if

$$I = \int_1^{\infty} \frac{e^{(1)}(w)}{e^{1+r}(w)} \Big| \sum_{n<w} (e(w) - e(n))^{r-1}\, e(n)\, A_n(x)\, n^b \Big|\, dw$$

is convergent. Now

$$I \leqslant \frac{2\,|\,\Phi(\pi)\,|}{\pi^{1+c}} \int_1^{\infty} \frac{e^{(1)}(w)}{e^{1+r}(w)} \Big| \sum_{n<w} (e(w) - e(n))^{r-1}\, e(n)\, n^b \,\cdot$$

$$\cdot \int_0^{\pi} u^c \cos nu\, du \Big|\, dw$$

$$+ \frac{2}{\pi}\int_0^{\pi} |\, d\,(t^{-c}\,\Phi\,(t))\,| \int_1^{\infty} \frac{e^{(1)}(w)}{e^{1+r}(w)} \Big| \sum_{n<w} (e(w) - e(n))^{r-1} \,\cdot$$

$$\cdot\, e(n)\, n^b \int_0^{t} u^c \cos nu\, du \Big|\, dw.$$

Since, by (2.2), $\pi^{-c}\,|\,\Phi\,(\pi)\,|$ and $\int_0^{\pi} |\, d\,(t^{-c}\,\Phi\,(t))\,|$ are finite therefore for the proof of Theorem 1 we only require to prove that

$$J = \int_1^{\infty} \frac{e^{(1)}(w)}{e^{1+r}(w)} \Big| \sum_{n<w} (e(w) - e(n))^{r-1}\, e(n)\, n^b \int_0^{t} u^c \cos nu\, du \Big|\, dw = O(1),$$

uniformly in $0 < t \leqslant \pi$.

By (4.1) of Lemma 2, we have

$$J = t^c \int_1^\infty \frac{e^{(1)}(w)}{e^{1+r}(w)} \,|E(w, t)|\, dw$$

$$+ O\left\{\int_1^\infty \frac{e^{(1)}(w)}{e^{1+r}(w)} \left| \sum_{n<w} (e(w) - e(n))^{r-1} e(n)\, n^{b-c-1} \right| dw\right\}$$

$$= t^c J_1 + O\{J_2\}, \text{ say.}$$

Now, since $\sum n^{b-c-1} \in |R, e(w), 0|$ therefore the convergence of J_2 follows by Lemma 1. Thus, for the proof of Theorem 1, we only require to prove

$$J_1 = O(t^{-c}),$$

uniformly in $0 < t < \pi$.

For $T = t^{-(1-a)^{-1}}$, we write

$$J_1 = \left(\int_1^T = \int_T^\infty\right)\left(\frac{e^{(1)}(w)}{e^{1+r}(w)} \,|E(w, t)|\, dw\right)$$

$$= I_1 + I_2, \text{ say.}$$

By (3.1), we have

$$I_1 = O\left\{\int_1^T \frac{e^{(1)}(w)}{e(w)}\, w^{b-a}\, dw\right\}$$

$$= O\left\{\int_1^T w^{b-1}\, dw\right\}$$

$$= O(t^{-c}),$$

by (2.1), uniformly in $0 < t < \pi$.

And, by (3.2) for $E(w, t)$, we have

$$I_2 = O\left\{t^{-r} \int_T^\infty \frac{e^{(1)}(w)}{e(w)}\, w^{b-a+r(a-1)}\, dw\right\}$$

$$= O\left\{t^{-r}\int_{T}^{\infty} w^{b+r(a-1)-1}\,dw\right\}$$

$$= O\left\{t^{-r}\,T^{b+r(a-1)}\right\}$$

$$= O(t^{-c}),$$

by (2.1), uniformly in $0 < t < \pi$.

On combining I_1 and I_2 we get, uniformly in $0 < t < \pi$,

$J_1 = O(t^{-c})$

This terminates the proof of Theorem 1.

5.2. **Proof of Theorem 2.** We have

$$B_n(x) = \frac{2}{\pi}\int_0^{\pi} t^{-c}\,\psi(t)\,t^c \sin nt\,dt$$

$$= \frac{2}{\pi}\,\frac{\psi(\pi)}{\pi^c}\int_0^{\pi} u^c \sin nu\,du$$

$$-\frac{2}{\pi}\int_0^{\pi} d\left(t^{-c}\,\psi(t)\right)\int_0^{t} u^c \sin nu\,du,$$

integrating by parts.

The series $\sum B_n(x)\,n^b \in |R, e(w)\,r|\ (r > c)$, if

$$J = \int_1^{\infty} \frac{e^{(1)}(w)}{e^{1+r}(w)}\left|\sum_{n<w}(e(w)-e(n))^{r-1}\,e(n)\,n^b\,B_n(x)\right|dw$$

s convergent. Now

$$J \leqslant \frac{2}{\pi}\,\frac{|\psi(\pi)|}{\pi^c}\int_1^{\infty}\frac{e^{(1)}(w)}{e^{1+r}(w)}\left|\sum_{n<w}(e(w)-e(n))^{r-1}\,e(n)\,n^b\,\cdot\right.$$

$$\left.\cdot\int_0^{\pi} u^c \sin nu\,du\right|dw$$

$$+\frac{2}{\pi}\int_0^{\pi}|d(t^{-c}\psi(t))|\int_1^{\infty}\frac{e^{(1)}(w)}{e^{1+r}(w)}\Big|\sum_{n<w}(e(w)-e(n))^{r-1}e(n)\cdot$$

$$\cdot\, nb\int_0^t u^c \sin nu\, du\Big|\, dw$$

Since, $t^{-c}\psi(t)\in BV(0,\pi)$, $\pi^{-c}|\psi(\pi)|$ and $\int_0^{\pi}|d(t^{-c}\psi(t))|$ are finite herefore the theorem will be proved, if

$$I=\int_1^{\infty}\frac{e^{(1)}(w)}{e^{1+r}(w)}\Big|\sum_{n<w}(e(w)-e(n))^{r-1}e(n)\,n^c\int_0^t u^c\sin nu\,du\Big|\,dw$$

$$=O(1),$$

uniformly in $0<t\leqslant\pi$.

Now, by (4.2) of Lemma 2, we have

$$I=t^c\int_1^{\infty}\frac{e^{(1)}(w)}{e^{1+r}(w)}|K(w,t)|\,dw$$

$$+O\left\{\int_1^{\infty}\frac{e^{(1)}(w)}{e^{1+r}(w)}\Big|\sum_{n<w}(e(w)-e(n))^{r-1}e(n)\,n^{b-c-1}\Big|\,dw\right\}$$

$$=t^c I_1+O\{I_2\},\ \text{say}.$$

The proof of

$$I_1=O(t^{-c}),$$

uniformly in $0<t\leqslant\pi$, runs very to the proof of J_1 of Theorem 1 by using (3.1) and (3.2) for $K(w,t)$. And the boundedness of I_2 immediately follows by the use of Lemma 1 since

$$\sum n^{b-c-1}\in|C,0|.$$

This terminates the proof of Theorem 2.

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