Математички Билтен
49 (LXXV) 2 (2025), 141–161

UDC: 517.965:[512.541+512.62]
DOI: 10.37560/matbil25492141vn

EQUIVARIANT HYERS–ULAM STABILITY OF QUADRATIC FUNCTIONAL EQUATIONS ON ABELIAN GROUPS AND POLYNOMIAL MODULES

LY VAN NGOC

Abstract. The classical dyadic construction gives a quadratic limit for an approximately quadratic map on an Abelian group under summable control. We prove that endomorphism-intertwining identities pass to this limit when the rescaled intertwining defects vanish. For $M = (\mathbb{Z}[X]/(X^n+1))^k$, coefficient norms give explicit multiplier criteria. Relations between multiplier words force operator identities on the span $W = \operatorname{span}_{\mathbb{F}} Q(M)$. This span is finite-dimensional, and $X^n = -1$ implies $U^n|_W = I_W$. Expected defects are treated in $L^1(\Omega; Y)$. McDiarmid's inequality gives bounds at fixed dyadic stages on finite grids, not a global pathwise stability theorem.

1. INTRODUCTION

Consider the quadratic equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

The dyadic method $4^{-m} f(2^m x)$ is classical: see Skof [23], Cholewa [6], Czerwik [8], and Grabiec [12]; see also [24, 14, 21, 15, 9, 11]. For related settings see [22, 7, 19, 4, 2, 3].

We study whether intertwining survives this limit. For an additive $S : G \to G$ and $\Pi(S) \in \mathcal{L}(Y)$, a dyadically negligible defect $f(Sx) - \Pi(S)f(x)$ yields $Q(Sx) = \Pi(S)Q(x)$ (Theorem 3.2). The theorem does not require Π to be a representation.

For $R = \mathbb{Z}[X]/(X^n+1)$ and $M = R^k$, the bound $\|ax\|_2 \le \sqrt{n}\, \|a\|_2 \|x\|_2$ gives multiplier criteria. If $g_u = \pm g_v$ in R, the corresponding ordered operator words agree on $Q(M)$ (Theorem 3.6). One approximate relation for X suffices for all powers of X; since $X^n = -1$,

$$Q(M) \subseteq \ker(U^n - I_Y)$$

2020 *Mathematics Subject Classification.* Primary 39B52, 39B82; Secondary 46B99, 60E15.

Key words and phrases. quadratic functional equation; Hyers–Ulam stability; Abelian groups; polynomial modules; multiplier equivariance; finite-grid concentration.

(Theorem 3.7). These restrictions apply to the stabilized range, not necessarily to all of Y.

Section 4 gives the L^1 formulation for expected defects. Section 5 concerns fixed-stage bounds on finite grids.

2. Preliminaries

Let G be an additive Abelian group. Write $2^m x$ for repeated addition and reserve M for R^k.

2.1. **Discrete polynomial modules.** Let $n, k \geq 1$ be integers, and define

$$R = \mathbb{Z}[X]/(X^n + 1).$$

Since $X^n + 1$ is monic of degree n, every element $a \in R$ has a unique representative

$$a(X) = \sum_{i=0}^{n-1} a_i X^i, \qquad a_i \in \mathbb{Z}.$$

The coefficient embedding

$$a \longmapsto (a_0, \ldots, a_{n-1})$$

identifies the underlying additive group of R with $\mathbb{Z}^n$. We equip R with the Euclidean coefficient norm

$$\|a\|_2 = \left(\sum_{i=0}^{n-1} a_i^2 \right)^{1/2}.$$

Let $M = R^k$. For

$$x = (x_1, \ldots, x_k), \qquad x_j = \sum_{i=0}^{n-1} x_{j,i} X^i,$$

define

$$\|x\|_2 = \left(\sum_{j=1}^{k} \sum_{i=0}^{n-1} x_{j,i}^2 \right)^{1/2}. \tag{2.1}$$

As an additive group, $M \cong \mathbb{Z}^{nk}$. The target Y is a real or complex Banach space with norm $\|\cdot\|_Y$.

Lemma 1 (Coefficient-norm control of multiplication)**.** *For all $a, b \in R$,*

$$\|ab\|_2 \leq \sqrt{n}\, \|a\|_2 \, \|b\|_2 . \tag{2.2}$$

For every $a \in R$ and every $x \in M$,

$$\|ax\|_2 \leq \sqrt{n}\, \|a\|_2 \, \|x\|_2 . \tag{2.3}$$

Proof. Write

$$a(X) = \sum_{i=0}^{n-1} a_i X^i, \qquad b(X) = \sum_{i=0}^{n-1} b_i X^i.$$

Reduction modulo $X^n + 1$ gives

$$ab = \sum_{\ell=0}^{n-1} c_\ell X^\ell,$$

where

$$c_\ell = \sum_{i=0}^{\ell} a_i b_{\ell-i} - \sum_{i=\ell+1}^{n-1} a_i b_{n+\ell-i}.$$

The sum defining c_ℓ pairs the coefficients of a with a signed permutation of those of b. Cauchy–Schwarz gives

$$|c_\ell| \le \|a\|_2 \|b\|_2 .$$

Summing over ℓ, we obtain

$$\|ab\|_2^2 = \sum_{\ell=0}^{n-1} c_\ell^2 \le n \|a\|_2^2 \|b\|_2^2 ,$$

which proves (2.2).

For $x = (x_1, \dots, x_k) \in M$, applying (2.2) coordinatewise yields

$$\begin{aligned} \|ax\|_2^2 &= \sum_{j=1}^{k} \|ax_j\|_2^2 \\ &\le n \|a\|_2^2 \sum_{j=1}^{k} \|x_j\|_2^2 \\ &= n \|a\|_2^2 \|x\|_2^2 . \end{aligned}$$

Taking square roots proves (2.3). □

For $a \in R$, define the multiplication map

$$T_a : M \to M, \qquad T_a(x) = ax.$$

The map T_a is additive. Moreover, by Lemma 1,

$$\|T_a(x) - T_a(y)\|_2 \le \sqrt{n} \|a\|_2 \|x - y\|_2 , \qquad x, y \in M. \tag{2.4}$$

Thus T_a is Lipschitz in the coefficient norm.

Lemma 2 (Dyadic compatibility)**.** *For every $a \in R$, $x \in M$, and integer $m \ge 0$,*

$$a(2^m x) = 2^m (ax).$$

Equivalently,

$$T_a(2^m x) = 2^m T_a(x).$$

Proof. The module action is distributive and $2^m x$ is repeated addition. □

2.2. **Quadratic mappings and defect operators.**

Definition 2.1. *A mapping $Q : G \to Y$ is called quadratic if*

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y), \qquad x, y \in G. \tag{2.5}$$

Equation (2.5) implies

$$Q(0) = 0, \qquad Q(-x) = Q(x), \qquad Q(2x) = 4Q(x).$$

By iteration,

$$Q(2^m x) = 4^m Q(x), \qquad m \geq 0, \quad x \in G. \tag{2.6}$$

For any mapping $h : G \to Y$, define its quadratic defect by

$$\Delta_h(x,y) = h(x+y) + h(x-y) - 2h(x) - 2h(y), \qquad x, y \in G. \tag{2.7}$$

Thus h is quadratic exactly when $\Delta_h \equiv 0$.

For $f : G \to Y$ set

$$\Delta_f(x,y) = f(x+y) + f(x-y) - 2f(x) - 2f(y). \tag{2.8}$$

In particular,

$$\Delta_f(x,x) = f(2x) + f(0) - 4f(x). \tag{2.9}$$

For $f(0) = 0$, the right-hand side is $f(2x) - 4f(x)$.

If $f(0) \neq 0$, define

$$\widetilde{f}(x) = f(x) - f(0).$$

Then $\widetilde{f}(0) = 0$, and

$$\Delta_{\widetilde{f}}(x,y) = \Delta_f(x,y) + 2f(0). \tag{2.10}$$

Consequently, a bound

$$\left\|\Delta_f(x,y)\right\|_Y \leq \varphi(x,y)$$

implies

$$\left\|\Delta_{\widetilde{f}}(x,y)\right\|_Y \leq \varphi(x,y) + 2\left\|f(0)\right\|_Y.$$

The results below assume $f(0) = 0$. Otherwise apply them to $\widetilde{f}$ with the control

$$\widetilde{\varphi}(x,y) = \varphi(x,y) + 2\left\|f(0)\right\|_Y.$$

If φ satisfies the summability and dyadic-vanishing conditions (3.2) and (3.3) (with $\varepsilon = \varphi$), then so does $\widetilde{\varphi}$. Put

$$\Phi_\varphi(x) := \sum_{j=0}^{\infty} 4^{-(j+1)} \varphi(2^j x, 2^j x).$$

Theorem 3.1 applied to $\widetilde{f}$ gives the quadratic limit Q for $\widetilde{f}$ and f, since $4^{-m} f(0) \to 0$. From (2.9),

$$\left\|f(x) - Q(x)\right\|_Y \leq \Phi_\varphi(x) + \frac{1}{3}\left\|f(0)\right\|_Y.$$

Indeed, $\left\|f(2u) - 4f(u)\right\|_Y \leq \varphi(u,u) + \left\|f(0)\right\|_Y$. Telescoping with $u = 2^j x$ and weights $4^{-(j+1)}$ gives the factor $1/3$. For an additive S and $\Pi(S) \in \mathcal{L}(Y)$,

$$\widetilde{f}(Sx) - \Pi(S)\widetilde{f}(x) = f(Sx) - \Pi(S)f(x) + (\Pi(S) - I_Y)f(0).$$

Thus a control $\|f(Sx) - \Pi(S)f(x)\|_Y \le \rho_S(x)$ gives the valid normalized control

$$\widetilde{\rho}_S(x) = \rho_S(x) + \|(\Pi(S) - I_Y)f(0)\|_Y .$$

The added term is independent of m; hence dyadic decay of ρ_S also gives dyadic decay of $\widetilde{\rho}_S$.

3. Deterministic Stability and Equivariance

The argument below uses only the Abelian-group law and completeness of Y.

3.1. **Quadratic stability on Abelian groups.**

Theorem 3.1 (Quadratic stability under summable dyadic control)**.** *Let G be an Abelian group, let Y be a Banach space, and let $f : G \to Y$ satisfy $f(0) = 0$. Suppose that there exists a control function*

$$\varepsilon : G \times G \to [0, \infty)$$

such that

$$\left\|\Delta_f(x, y)\right\|_Y \le \varepsilon(x, y), \qquad x, y \in G. \tag{3.1}$$

Assume that, for every $x \in G$,

$$\Phi(x) := \sum_{j=0}^{\infty} 4^{-(j+1)}\varepsilon(2^j x, 2^j x) < \infty, \tag{3.2}$$

and that, for all $x, y \in G$,

$$\lim_{m\to\infty} 4^{-m}\varepsilon(2^m x, 2^m y) = 0. \tag{3.3}$$

Then, for every $x \in G$, the limit

$$Q(x) = \lim_{m\to\infty} 4^{-m} f(2^m x) \tag{3.4}$$

exists in Y. The mapping $Q : G \to Y$ is quadratic and satisfies

$$\|f(x) - Q(x)\|_Y \le \Phi(x), \qquad x \in G. \tag{3.5}$$

This Q is the unique quadratic mapping satisfying (3.5)*.*

Proof. Putting $y = x$ in (3.1) and using $f(0) = 0$, we get

$$\|f(2x) - 4f(x)\|_Y \le \varepsilon(x, x).$$

Hence, for every $j \ge 0$,

$$\left\|4^{-(j+1)} f(2^{j+1}x) - 4^{-j} f(2^j x)\right\|_Y \le 4^{-(j+1)}\varepsilon(2^j x, 2^j x).$$

Thus, for $m > \ell \ge 0$,

$$\left\|4^{-m} f(2^m x) - 4^{-\ell} f(2^\ell x)\right\|_Y \le \sum_{j=\ell}^{m-1} 4^{-(j+1)}\varepsilon(2^j x, 2^j x).$$

The summable tail makes $4^{-m} f(2^m x)$ Cauchy. Letting $m \to \infty$ with $\ell = 0$ gives (3.5).

For $x, y \in G$, continuity of addition and scalar multiplication in Y gives

$$\Delta_Q(x,y) = \lim_{m\to\infty} 4^{-m}\Delta_f(2^m x, 2^m y).$$

Using (3.1) and (3.3),

$$\left\|\Delta_Q(x,y)\right\|_Y \le \limsup_{m\to\infty} 4^{-m}\varepsilon(2^m x, 2^m y) = 0.$$

Thus Q is quadratic.

Let $P : G \to Y$ be another quadratic mapping satisfying (3.5). Quadratic homogeneity gives

$$\|Q(x) - P(x)\|_Y \le 2\,4^{-m}\Phi(2^m x).$$

The remaining term is the tail of (3.2):

$$4^{-m}\Phi(2^m x) = \sum_{\ell=m}^{\infty} 4^{-(\ell+1)}\varepsilon(2^\ell x, 2^\ell x) \longrightarrow 0,$$

Therefore $P = Q$. □

Corollary 3.1.1 (Power-type control on the polynomial module)**.** *Let $M = R^k$ be equipped with the coefficient norm $\|\cdot\|_2$. Let $0 < p < 2$ and $\theta \ge 0$. Suppose that $f : M \to Y$ satisfies $f(0) = 0$ and*

$$\left\|\Delta_f(x,y)\right\|_Y \le \theta\left(\|x\|_2^p + \|y\|_2^p\right), \qquad x, y \in M.$$

Then there exists a unique quadratic mapping $Q : M \to Y$ such that

$$\|f(x) - Q(x)\|_Y \le \frac{\theta}{2(1-2^{p-2})}\,\|x\|_2^p, \qquad x \in M. \tag{3.6}$$

Moreover $Q(x) = \lim_{m\to\infty} 4^{-m} f(2^m x)$.

Proof. Apply Theorem 3.1 to the Abelian group underlying M, with

$$\varepsilon(x,y) = \theta\left(\|x\|_2^p + \|y\|_2^p\right).$$

Since $\left\|2^j x\right\|_2 = 2^j \|x\|_2$,

$$\Phi(x) = \sum_{j=0}^{\infty} 4^{-(j+1)} 2\theta \left\|2^j x\right\|_2^p = \frac{\theta}{2(1-2^{p-2})}\,\|x\|_2^p.$$

The dyadic vanishing condition follows from $p < 2$. □

Corollary 3.1.2 (Uniform control)**.** *Let $f : M \to Y$ satisfy $f(0) = 0$ and*

$$\left\|\Delta_f(x,y)\right\|_Y \le 2\theta, \qquad x, y \in M,$$

where $\theta \ge 0$. Then there exists a unique quadratic mapping $Q : M \to Y$ such that

$$\|f(x) - Q(x)\|_Y \le \frac{2\theta}{3}, \qquad x \in M.$$

Proof. Use Theorem 3.1 with $\varepsilon(x,y) = 2\theta$. Then $\Phi(x) = \sum_{j=0}^{\infty} 4^{-(j+1)} 2\theta = 2\theta/3$, and dyadic vanishing is immediate. □

3.2. **Intertwining under additive endomorphisms.** We pass the endomorphism defect to the dyadic limit.

Theorem 3.2 (Persistence of intertwining under additive endomorphisms)**.** *Let G be an Abelian group, let the assumptions of Theorem 3.1 hold, and let Q be defined by* (3.4)*. Let*

$$\Gamma \subset \mathrm{End}_{\mathbb{Z}}(G)$$

be a family of additive endomorphisms. Suppose that to each $S \in \Gamma$ there is associated a bounded linear operator $\Pi(S) \in \mathcal{L}(Y)$. Assume that, for every $S \in \Gamma$, there exists $\rho_S : G \to [0,\infty)$ such that

$$\|f(Sx) - \Pi(S)f(x)\|_Y \le \rho_S(x), \qquad x \in G, \tag{3.7}$$

and

$$\lim_{m\to\infty} 4^{-m}\rho_S(2^m x) = 0, \qquad x \in G. \tag{3.8}$$

Then

$$Q(Sx) = \Pi(S)Q(x), \qquad S \in \Gamma, \quad x \in G. \tag{3.9}$$

Proof. Since S is additive, $S(2^m x) = 2^m S(x)$. Therefore

$$Q(Sx) = \lim_{m\to\infty} 4^{-m} f(S(2^m x)).$$

Bounded linearity gives

$$\Pi(S)Q(x) = \lim_{m\to\infty} 4^{-m}\Pi(S)f(2^m x).$$

Using (3.7) and (3.8),

$$\|Q(Sx) - \Pi(S)Q(x)\|_Y \le \limsup_{m\to\infty} 4^{-m}\rho_S(2^m x) = 0.$$

□

If Γ is a composition semigroup and Π is a representation, the resulting identities respect composition:

$$Q(S_1S_2x) = \Pi(S_1)\Pi(S_2)Q(x), \qquad S_1, S_2 \in \Gamma,\ x \in G.$$

The representation assumption is not needed for Theorem 3.2.

3.3. **Polynomial-module multiplier criteria.** For $a \in R$, apply the preceding theorem to $T_a(x) = ax$.

Theorem 3.3 (Single multiplier intertwining)**.** *Let the assumptions of Theorem 3.1 hold on the Abelian group underlying M, and let Q be defined by* (3.4)*. Fix $a \in R$, and let $\pi(a) \in \mathcal{L}(Y)$. Suppose that there exists $\rho_a : M \to [0,\infty)$ such that*

$$\|f(ax) - \pi(a)f(x)\|_Y \le \rho_a(x), \qquad x \in M, \tag{3.10}$$

and

$$\lim_{m\to\infty} 4^{-m}\rho_a(2^m x) = 0, \qquad x \in M. \tag{3.11}$$

Then

$$Q(ax) = \pi(a)Q(x), \qquad x \in M. \tag{3.12}$$

Proof. This is Theorem 3.2 with $G = M$, $S = T_a$, and $\Pi(T_a) = \pi(a)$. Lemma 2 gives $T_a(2^m x) = 2^m T_a(x)$. □

Theorem 3.4 (Coefficient-norm multiplier criterion)**.** *Let $M = R^k$, $R = \mathbb{Z}[X]/(X^n + 1)$, and let Q be obtained from Theorem 3.1. Let $\mathcal{A} \subset R$, and assign to each $a \in \mathcal{A}$ a bounded linear operator $\pi(a) \in \mathcal{L}(Y)$. Assume that there are numbers $0 < q < 2$ and $C_a, D_a \geq 0$ such that, for all $a \in \mathcal{A}$ and $x \in M$,*

$$\|f(ax) - \pi(a)f(x)\|_Y \leq C_a \|ax\|_2^q + D_a \|x\|_2^q. \tag{3.13}$$

Then

$$Q(ax) = \pi(a)Q(x), \qquad a \in \mathcal{A}, \quad x \in M. \tag{3.14}$$

By Lemma 1, the decay is controlled by

$$4^{-m} \|f(a2^m x) - \pi(a)f(2^m x)\|_Y \leq \big(C_a(\sqrt{n}\ \|a\|_2)^q + D_a\big)2^{m(q-2)} \|x\|_2^q. \tag{3.15}$$

If $\mathcal{A}$ is coefficient-bounded and the constants C_a, D_a are bounded uniformly on $\mathcal{A}$, then the right-hand side of (3.15) *is uniform in $a \in \mathcal{A}$.*

Proof. Since $a(2^m x) = 2^m(ax)$, condition (3.13) gives

$$\begin{aligned} 4^{-m} \|f(a2^m x) - \pi(a)f(2^m x)\|_Y &\leq 4^{-m} \left(C_a \|a2^m x\|_2^q + D_a \|2^m x\|_2^q\right) \\ &= 2^{m(q-2)} \left(C_a \|ax\|_2^q + D_a \|x\|_2^q\right). \end{aligned}$$

Apply Lemma 1 to get (3.15). Since $q < 2$, Theorem 3.3 gives the assertion for each $a \in \mathcal{A}$. □

Remark 3.1 (Intertwining and equivariance)**:** The identities $Q(ax) = \pi(a)Q(x)$ give semigroup equivariance only if $\mathcal{A}$ is multiplicative and π is a representation.

Corollary 3.4.1 (Uniform criterion on coefficient-bounded families)**.** *Suppose the hypotheses of Theorem 3.4 hold, with $\mathcal{A} \subset R$, $L \geq 0$, $\|a\|_2 \leq L$, and $C, D \geq 0$ such that $C_a \leq C$, $D_a \leq D$ for all $a \in \mathcal{A}$. Then $Q(ax) = \pi(a)Q(x)$ for $a \in \mathcal{A}$, and, uniformly in a,*

$$4^{-m} \|f(a2^m x) - \pi(a)f(2^m x)\|_Y \leq \big(C(\sqrt{n}\, L)^q + D\big)2^{m(q-2)} \|x\|_2^q.$$

If $\mathcal{A}$ is a multiplicative subsemigroup and $\pi : \mathcal{A} \to \mathcal{L}(Y)$ is a representation, then also $Q(abx) = \pi(ab)Q(x) = \pi(a)\pi(b)Q(x)$.

Proof. Take the uniform upper bounds in (3.15); the last identity is the representation property. □

Since $R \cong \mathbb{Z}^n$, a coefficient-bounded set $\mathcal{A}$ is finite, with $|\mathcal{A}| \leq (2\lfloor L \rfloor + 1)^n$.
The next criterion combines Corollaries 3.1.1 and 3.4.1.

Theorem 3.5 (Consolidated quadratic stability and multiplier intertwining)**.** *Let $M = R^k$, $R = \mathbb{Z}[X]/(X^n + 1)$, let $L \geq 0$, and let $\mathcal{A} \subset R$ satisfy $\|a\|_2 \leq L$ for all $a \in \mathcal{A}$. To each $a \in \mathcal{A}$, assign a bounded linear operator $\pi(a) \in \mathcal{L}(Y)$. Assume that $f : M \to Y$, $f(0) = 0$, satisfies*

$$\Big\|\Delta_f(x, y)\Big\|_Y \leq \theta\big(\|x\|_2^p + \|y\|_2^p\big), \qquad x, y \in M, \tag{3.16}$$

for some $\theta \geq 0$ and $0 < p < 2$. Assume also that

$$\|f(ax) - \pi(a)f(x)\|_Y \leq C\,\|ax\|_2^q + D\,\|x\|_2^q, \qquad a \in \mathcal{A}, \quad x \in M, \tag{3.17}$$

for some $C, D \geq 0$ and $0 < q < 2$. Then there exists a unique quadratic mapping $Q : M \to Y$ satisfying

$$\|f(x) - Q(x)\|_Y \leq \frac{\theta}{2(1-2^{p-2})}\,\|x\|_2^p, \qquad x \in M, \tag{3.18}$$

and

$$Q(ax) = \pi(a)Q(x), \qquad a \in \mathcal{A}, \quad x \in M. \tag{3.19}$$

The dyadic intertwining error is bounded uniformly in $a \in \mathcal{A}$ by

$$4^{-m}\,\|f(a2^m x) - \pi(a)f(2^m x)\|_Y \leq \big(C(\sqrt{n}\,L)^q + D\big)2^{m(q-2)}\,\|x\|_2^q. \tag{3.20}$$

If $\mathcal{A}$ is a multiplicative subsemigroup of R and $\pi : \mathcal{A} \to \mathcal{L}(Y)$ is a representation, then Q is multiplier-equivariant for that represented semigroup.

Proof. Corollary 3.1.1 yields the unique quadratic approximation and (3.18). Corollary 3.4.1 gives (3.19), (3.20), and the represented-semigroup assertion. □

Theorem 3.6 (Multiplier-word relations and range restrictions)**.** *Let $M = R^k$, $R = \mathbb{Z}[X]/(X^n + 1)$, and let the assumptions of Theorem 3.1 hold on the Abelian group underlying M. Let Q be defined by* (3.4)*. Let $g_1, \dots, g_s \in R$, and let $U_1, \dots, U_s \in \mathcal{L}(Y)$. Assume that, for each $i \in \{1, \dots, s\}$, there exists $\rho_i : M \to [0, \infty)$ such that*

$$\|f(g_i x) - U_i f(x)\|_Y \leq \rho_i(x), \qquad x \in M, \tag{3.21}$$

and

$$\lim_{m\to\infty} 4^{-m}\rho_i(2^m x) = 0, \qquad x \in M. \tag{3.22}$$

For a word $w = i_1 \cdots i_t$ in the alphabet $\{1, \dots, s\}$, define

$$g_w = g_{i_1} \cdots g_{i_t} \in R, \qquad U_w = U_{i_1} \cdots U_{i_t} \in \mathcal{L}(Y).$$

The operator word U_w is ordered exactly as the word w is written; no commutativity among the operators U_i is assumed. Put $g_\varnothing = 1$ and $U_\varnothing = I_Y$. Then

$$Q(g_w x) = U_w Q(x), \qquad x \in M, \tag{3.23}$$

for every word w. Consequently, if two words u, v satisfy

$$g_u = \sigma g_v \quad \text{in } R, \qquad \sigma \in \{1, -1\}, \tag{3.24}$$

then

$$(U_u - U_v)Q(x) = 0, \qquad x \in M. \tag{3.25}$$

Equivalently,

$$Q(M) \subseteq \ker(U_u - U_v).$$

By linearity, this inclusion also holds with $Q(M)$ replaced by $W = \operatorname{span}_{\mathbb{F}} Q(M)$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ according to the scalar field of Y. If $U_u - U_v$ is injective for one such relation, then $Q \equiv 0$.

If the generator defects are controlled in the coefficient norm by

$$\|f(g_i x) - U_i f(x)\|_Y \leq C_i\,\|g_i x\|_2^q + D_i\,\|x\|_2^q, \qquad x \in M, \tag{3.26}$$

where $0 < q < 2$ *and* $C_i, D_i \geq 0$, *then* (3.22) *follows from the estimate*

$$4^{-m} \left\| f(g_i 2^m x) - U_i f(2^m x) \right\|_Y \leq \left(C_i (\sqrt{n} \, \|g_i\|_2)^q + D_i \right) 2^{m(q-2)} \, \|x\|_2^q . \tag{3.27}$$

Proof. Theorem 3.3 applied to each multiplier g_i gives

$$Q(g_i x) = U_i Q(x), \qquad x \in M.$$

Induction on the length of $w = i_1 \cdots i_t$ gives

$$Q(g_w x) = Q(g_{i_1}(g_{i_2} \cdots g_{i_t} x)) = U_{i_1} Q(g_{i_2} \cdots g_{i_t} x) = U_w Q(x).$$

This proves (3.23). If $g_u = \sigma g_v$ in R, then

$$U_u Q(x) = Q(g_u x) = Q(\sigma g_v x) = Q(g_v x) = U_v Q(x),$$

because every quadratic map is even and therefore $Q(-y) = Q(y)$. This proves (3.25). The injective case is immediate.

Finally, (3.26), the identity $g_i 2^m x = 2^m (g_i x)$, and Lemma 1 give

$$\begin{aligned} 4^{-m} \left\| f(g_i 2^m x) - U_i f(2^m x) \right\|_Y &\leq 4^{-m} \left(C_i \, \|g_i 2^m x\|_2^q + D_i \, \|2^m x\|_2^q \right) \\ &\leq \left(C_i (\sqrt{n} \, \|g_i\|_2)^q + D_i \right) 2^{m(q-2)} \, \|x\|_2^q , \end{aligned}$$

which tends to zero since $q < 2$. □

Corollary 3.6.1 (Commutator restrictions on the stabilized range)**.** *Assume the hypotheses of Theorem 3.6. Then, for all generators* g_i, g_j,

$$(U_i U_j - U_j U_i) Q(x) = 0, \qquad x \in M. \tag{3.28}$$

Thus the commutators vanish on $W = \operatorname{span}_{\mathbb{F}} Q(M)$*; the operators need not commute on all of* Y.

Proof. Apply Theorem 3.6 to $g_i g_j = g_j g_i$. □

Finite-dimensional stabilized span. Put $d = nk$, $W = \operatorname{span}_{\mathbb{F}} Q(M)$, and $B(x, y) = \frac{1}{4}(Q(x + y) - Q(x - y))$. Symmetry of B follows from evenness of Q. The quadratic identity gives the Jensen relation $B(x + y, z) + B(x - y, z) = 2B(x, z)$. As $B(0, z) = 0$, taking $y = x$ yields $B(2x, z) = 2B(x, z)$. Replacing (x, y) by $(u + v, u - v)$ in the Jensen relation gives $B(u + v, z) = B(u, z) + B(v, z)$; symmetry now proves biadditivity. Also $Q(x) = B(x, x)$. For any $\mathbb{Z}$-basis $e_1, \ldots, e_d$ of M, biadditivity gives $Q(\sum_i a_i e_i) = \sum_{i,j} a_i a_j B(e_i, e_j)$, while $2B(e_i, e_j) = Q(e_i + e_j) - Q(e_i) - Q(e_j)$. Therefore

$$W = \operatorname{span}_{\mathbb{F}} \{ B(e_i, e_j) : 1 \leq i \leq j \leq d \}, \qquad \dim_{\mathbb{F}} W \leq \frac{d(d+1)}{2}.$$

Since $Q(g_i x) = U_i Q(x)$, we also have $U_i W \subseteq W$. The word and commutator identities therefore hold on the finite-dimensional invariant space W. No regularity of Q is used.

Theorem 3.7 (One-generator negacyclic multiplier criterion and range obstruction)**.** *Let $M = R^k$, where $R = \mathbb{Z}[X]/(X^n + 1)$. Let $U \in \mathcal{L}(Y)$. Assume that $f : M \to Y$, $f(0) = 0$, satisfies the quadratic control* (3.16) *for some $0 < p < 2$ and $\theta \geq 0$. Assume also that, for some $0 < q < 2$ and $C, D \geq 0$,*

$$\|f(Xx) - Uf(x)\|_Y \leq C \|Xx\|_2^q + D \|x\|_2^q, \qquad x \in M. \tag{3.29}$$

Then there is a unique quadratic mapping $Q : M \to Y$ satisfying (3.18) *and*

$$Q(Xx) = UQ(x), \qquad x \in M. \tag{3.30}$$

Consequently,

$$Q(X^r x) = U^r Q(x), \qquad r \geq 0, \quad x \in M. \tag{3.31}$$

Since $X^n = -1$ in R, the range of Q satisfies

$$Q(M) \subseteq \ker(U^n - I_Y). \tag{3.32}$$

In particular, $U^n|_W = I_W$ on $W = \operatorname{span}_{\mathbb{F}} Q(M)$. If $\ker(U^n - I_Y) = \{0\}$, then $Q \equiv 0$, and hence

$$\|f(x)\|_Y \leq \frac{\theta}{2(1 - 2^{p-2})} \|x\|_2^p, \qquad x \in M. \tag{3.33}$$

Moreover, since multiplication by X is an isometry in the coefficient norm, the dyadic X-intertwining error satisfies

$$4^{-m} \|f(X2^m x) - Uf(2^m x)\|_Y \leq (C + D)2^{m(q-2)} \|x\|_2^q. \tag{3.34}$$

Proof. Multiplication by X is a signed coordinate permutation in each copy of R. Hence

$$\|Xx\|_2 = \|x\|_2, \qquad x \in M.$$

Corollary 3.1.1 gives the unique quadratic mapping Q and the estimate (3.18). The intertwining hypothesis gives, for every $m \geq 0$,

$$4^{-m} \|f(X2^m x) - Uf(2^m x)\|_Y \leq (C + D)2^{m(q-2)} \|x\|_2^q,$$

because $\|X2^m x\|_2 = \|2^m x\|_2 = 2^m \|x\|_2$. Since $q < 2$, Theorem 3.3, applied to the multiplier X, gives (3.30). This also proves (3.34).

Iterating (3.30) gives (3.31). Since $X^n = -1$ in R, and since every quadratic map satisfies $Q(-x) = Q(x)$, we obtain

$$U^n Q(x) = Q(X^n x) = Q(-x) = Q(x), \qquad x \in M.$$

Thus (3.32) holds. If $\ker(U^n - I_Y) = \{0\}$, then $Q = 0$; (3.18) gives (3.33). □

Remark 3.2 (One-generator assumption)**:** The hypothesis concerns only X. All power identities and (3.32) then follow. Applying the family criterion to $\{1, X, \ldots, X^{2n-1}\}$ would instead require separate defect bounds.

Example 3.1 (A zero-limit obstruction)**.** Let $n = 1$, so $R = \mathbb{Z}[X]/(X + 1)$ and multiplication by X is multiplication by -1. Let $M = R^k \cong \mathbb{Z}^k$, let Y be a real or complex Banach space, and put $U = -I_Y$. Then

$$\ker(U - I_Y) = \ker(-2I_Y) = \{0\}.$$

Assume that $f : M \to Y$, $f(0) = 0$, satisfies the quadratic control (3.16) and, for some $0 < q < 2$ and $C, D \geq 0$,

$$\|f(Xx) - Uf(x)\|_Y = \|f(-x) + f(x)\|_Y \leq (C + D) \|x\|_2^q, \qquad x \in M.$$

Theorem 3.7 gives $Q(M) \subset \ker(-2I_Y) = \{0\}$. Hence $Q \equiv 0$, and therefore

$$\|f(x)\|_Y \leq \frac{\theta}{2(1 - 2^{p-2})} \|x\|_2^p, \qquad x \in M.$$

Thus subquadratic approximate oddness rules out a nonzero quadratic limit in this case.

Example 3.2 (A nonzero negacyclic generator-equivariant quadratic map)**.** Let $n = 2$, $k = 1$, and $R = \mathbb{Z}[X]/(X^2 + 1)$. Write an element of R as $a + bX$, and take $Y = \mathbb{R}^2$ with $U = -I_{\mathbb{R}^2}$. Define

$$Q(a + bX) = (a^2 - b^2, 2ab).$$

This map is quadratic on the underlying additive group $\mathbb{Z}^2$. Since

$$X(a + bX) = -b + aX,$$

a direct computation gives

$$Q(X(a + bX)) = Q(-b + aX) = (b^2 - a^2, -2ab) = -Q(a + bX) = UQ(a + bX).$$

Theorem 3.7 applies to $f = Q$ with zero quadratic defect and zero generator-intertwining defect. It gives

$$Q(X^r x) = (-1)^r Q(x), \qquad r \geq 0.$$

Here $U^2 = I_Y$ and $Q \neq 0$; the range restriction does not force a zero limit.

Example 3.3 (Failure without dyadic decay)**.** Let $G = \mathbb{Z}$, $Y = \mathbb{R}$, and $f(x) = x^2$. Then $\Delta_f = 0$ and $Q = f$. Take $S = \mathrm{id}_G$ and $\Pi(S) = 0$. For every $m \geq 0$,

$$4^{-m} |f(S2^m x) - \Pi(S) f(2^m x)| = x^2.$$

For $x \neq 0$ the defect does not decay, and $Q(Sx) \neq \Pi(S)Q(x)$. The same example occurs for $n = k = 1$, $M = \mathbb{Z}[X]/(X + 1) \cong \mathbb{Z}$, with multiplier $a = 1$ and $\pi(1) = 0$.

4. Bochner-Space Formulation of Expected Control

Apply the deterministic theorem with target $L^1(\Omega; Y)$. For random-normed-space and fixed-point approaches see [20, 18]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $f : M \times \Omega \to Y$ be such that $f(x, \cdot)$ is strongly measurable and Bochner integrable for each $x \in M$, with $f(0, \cdot) = 0$ almost surely. Write

$$\Delta_f(x, y; \omega) = f(x + y, \omega) + f(x - y, \omega) - 2f(x, \omega) - 2f(y, \omega).$$

Bochner integrability and the L^1 norm are used as in [10, 16].

4.1. **Quadratic stability in L^1.**

Theorem 4.1 (L^1-quadratic stability)**.** *Suppose that $\varepsilon : M \times M \to [0,\infty)$ satisfies*

$$\mathbb{E}\left\|\Delta_f(x,y;\cdot)\right\|_Y \le \varepsilon(x,y), \qquad x,y \in M, \tag{4.1}$$

and that

$$\Phi(x) := \sum_{j=0}^{\infty} 4^{-(j+1)}\varepsilon(2^j x, 2^j x) < \infty, \qquad x \in M, \tag{4.2}$$

$$4^{-m}\varepsilon(2^m x, 2^m y) \longrightarrow 0, \qquad x,y \in M. \tag{4.3}$$

Then the limit

$$Q(x) = \lim_{m\to\infty} 4^{-m} f(2^m x,\cdot) \tag{4.4}$$

exists in $L^1(\Omega;Y)$. It defines the unique quadratic mapping $Q : M \to L^1(\Omega;Y)$ satisfying

$$\|f(x,\cdot) - Q(x)\|_{L^1(\Omega;Y)} \le \Phi(x), \qquad x \in M. \tag{4.5}$$

Proof. Set $F(x) = f(x,\cdot) \in L^1(\Omega;Y)$. Then $\|\Delta_F(x,y)\|_{L^1} = \mathbb{E}\left\|\Delta_f(x,y;\cdot)\right\|_Y$, so Theorem 3.1 applies with target $L^1(\Omega;Y)$, including its uniqueness conclusion. □

Corollary 4.1.1 (Two direct L^1 bounds)**.** *In each of the following cases, there is a unique quadratic mapping $Q : M \to L^1(\Omega;Y)$.*

(i) *If $0 < p < 2$, $\theta \ge 0$, and*

$$\mathbb{E}\left\|\Delta_f(x,y;\cdot)\right\|_Y \le \theta(\|x\|_2^p + \|y\|_2^p), \qquad x,y \in M,$$

then

$$\|f(x,\cdot) - Q(x)\|_{L^1(\Omega;Y)} \le \frac{\theta}{2(1-2^{p-2})}\,\|x\|_2^p, \qquad x \in M.$$

(ii) *If $\theta \ge 0$ and $\mathbb{E}\left\|\Delta_f(x,y;\cdot)\right\|_Y \le 2\theta$ for all $x,y \in M$, then*

$$\|f(x,\cdot) - Q(x)\|_{L^1(\Omega;Y)} \le \frac{2\theta}{3}, \qquad x \in M.$$

Proof. Apply Theorem 4.1. The diagonal series sums to $\theta\,\|x\|_2^p\,/\,[2(1-2^{p-2})]$ in (i), and to $2\theta/3$ in (ii). Both controls satisfy the dyadic-vanishing condition. □

4.2. **Inheritance of endomorphism intertwining.** For $S \in \mathrm{End}_{\mathbb{Z}}(M)$, any $\Pi(S) \in \mathcal{L}(Y)$ induces a bounded operator $\Pi_1(S)$ on $L^1(\Omega;Y)$ by

$$(\Pi_1(S)H)(\omega) = \Pi(S)H(\omega), \qquad \|\Pi_1(S)\| \le \|\Pi(S)\|_{\mathrm{op}}.$$

Theorem 4.2 (L^1-inheritance of intertwining)**.** *Let Q be given by Theorem 4.1. Let $\Gamma \subset \mathrm{End}_{\mathbb{Z}}(M)$, with $\Pi(S) \in \mathcal{L}(Y)$ and $\rho_S : M \to [0,\infty)$ assigned to each $S \in \Gamma$. If*

$$\mathbb{E}\,\|f(Sx,\cdot) - \Pi(S)f(x,\cdot)\|_Y \le \rho_S(x), \qquad x \in M, \tag{4.6}$$

and

$$4^{-m}\rho_S(2^m x) \longrightarrow 0, \qquad x \in M, \tag{4.7}$$

for every $S \in \Gamma$, then

$$Q(Sx) = \Pi_1(S)Q(x), \qquad S \in \Gamma,\ x \in M, \tag{4.8}$$

in $L^1(\Omega; Y)$.

Proof. For $F(x) = f(x, \cdot)$, condition (4.6) is exactly $\|F(Sx) - \Pi_1(S)F(x)\|_{L^1} \leq \rho_S(x)$. Apply Theorem 3.2 in $L^1(\Omega; Y)$. □

Corollary 4.2.1 (Immediate L^1 specializations)**.** *Assume the hypotheses of Theorem 4.1. Let $\Gamma \subset \mathrm{End}_{\mathbb{Z}}(M)$, and assign operators $\Pi(S) \in \mathcal{L}(Y)$ and nonnegative controls ρ_S satisfying* (4.6) *for every $S \in \Gamma$. The conclusion* (4.8) *follows whenever, for every $S \in \Gamma$, either $\rho_S(x) \leq C_S \|x\|_2^{q_S}$ with $0 < q_S < 2$ and $C_S \geq 0$, or $\rho_S(x) \leq C_S$ with $C_S \geq 0$ for all $x \in M$. In particular, when $T_a \in \Gamma$, take $\pi(a) = \Pi(T_a)$; then $Q(ax) = \pi_1(a)Q(x)$, where $\pi_1(a)$ is induced by $\pi(a)$. The same single-multiplier conclusion holds under the general dyadic decay condition* (4.7)*, by Theorem 4.2. If Γ is a composition semigroup and Π is a representation, these intertwining identities are compatible with composition.*

Proof. Both bounds imply (4.7), since $4^{-m}\rho_S(2^m x) \leq C_S 2^{m(q_S-2)} \|x\|_2^{q_S}$ in the first case and $\leq C_S 4^{-m}$ in the second. Apply Theorem 4.2. □

Remark 4.1 (Almost-sure representatives)**:** The conclusions above are L^1 identities. Since M is countable, simultaneous almost-sure representatives exist. Set $D_m(x, \omega) = 4^{-m} f(2^m x, \omega)$. For each $x \in M$,

$$\mathbb{E} \sum_{m=0}^{\infty} \|D_{m+1}(x, \cdot) - D_m(x, \cdot)\|_Y \leq \Phi(x) < \infty.$$

Tonelli's theorem and countability of $M \cong \mathbb{Z}^{nk}$ give a single full-measure event on which $Q_\omega(x) := \lim_m D_m(x, \omega)$ exists for every $x \in M$. Since $D_m(x, \cdot) \to Q(x)$ in L^1, uniqueness of the limit in probability identifies $Q_\omega(x)$ as a representative of $Q(x)$, simultaneously in x outside a common null set. The identities $\Delta_Q(x, y) = 0$ in L^1 then show that Q_ω is quadratic simultaneously for all $(x, y) \in M^2$, after discarding another null set. Under the hypotheses of Theorem 4.2, $\Gamma \subset \mathrm{End}_{\mathbb{Z}}(M) \cong \mathrm{Mat}_{nk}(\mathbb{Z})$ is also countable; hence $Q_\omega(Sx) = \Pi(S)Q_\omega(x)$ holds for every $(S, x) \in \Gamma \times M$ on one full-measure event. No bound $\|f(x, \omega) - Q_\omega(x)\|_Y \leq \Phi(x)$ is asserted pointwise or uniformly over dyadic stages.

5. Finite-Grid High-Probability Bounds via Bounded Differences

Fix $d \geq 1$ random coordinates, not necessarily equal to the module rank nk. Let

$$Z = (Z_1, \ldots, Z_d)$$

be a random vector with independent coordinates, where each Z_i takes values in a finite set $\mathcal{S}_i \subset \mathbb{Z}$. Write

$$\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_d.$$

Let

$$f : M \times \mathcal{S} \to Y$$

satisfy

$$f(0,z) = 0, \qquad z \in \mathcal{S}.$$

For $x, y \in M$ and $z \in \mathcal{S}$, define

$$\Delta_f(x,y;z) = f(x+y,z) + f(x-y,z) - 2f(x,z) - 2f(y,z).$$

One realization of Z is used for the entire testing grid. We use McDiarmid's bounded-difference inequality [1, 13, 17, 5].

5.1. **Finite-grid discrepancy envelopes.**

Theorem 5.1 (McDiarmid's inequality)**.** *Let $g : \mathcal{S} \to \mathbb{R}$. Suppose that, for each $i \in \{1,\dots,d\}$, there exists $c_i \geq 0$ such that*

$$|g(z) - g(z')| \leq c_i$$

whenever $z, z' \in \mathcal{S}$ differ only in their i-th coordinate. If $\sum_i c_i^2 > 0$, then, for every $t > 0$,

$$\mathbb{P}\big(g(Z) - \mathbb{E}g(Z) \geq t\big) \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^d c_i^2}\right). \tag{5.1}$$

If all $c_i = 0$, then $g(Z)$ is constant almost surely.

For $x, y \in M$, define

$$g_{x,y}(z) = \Big\|\Delta_f(x,y;z)\Big\|_Y, \qquad z \in \mathcal{S}.$$

Let $\mathcal{G} \subset M \times M$ be a finite testing grid. Assume that there exist constants $c_1,\dots,c_d \geq 0$, independent of $(x,y) \in \mathcal{G}$, such that

$$|g_{x,y}(z) - g_{x,y}(z')| \leq c_i \tag{5.2}$$

whenever $z, z' \in \mathcal{S}$ differ only in coordinate i. Set

$$V = \sum_{i=1}^d c_i^2. \tag{5.3}$$

Corollary 5.1.1 (Finite-grid discrepancy envelope)**.** *Let $\mathcal{G} \subset M \times M$ be nonempty and finite. Assume that*

$$\mathbb{E}\,\Big\|\Delta_f(x,y;Z)\Big\|_Y \leq \varepsilon(x,y), \qquad (x,y) \in \mathcal{G},$$

and that (5.2) *holds uniformly on $\mathcal{G}$. Then, for every $\delta \in (0,1)$, with probability at least $1-\delta$,*

$$\Big\|\Delta_f(x,y;Z)\Big\|_Y \leq \varepsilon_\delta(x,y), \qquad (x,y) \in \mathcal{G}, \tag{5.4}$$

where

$$\varepsilon_\delta(x,y) = \varepsilon(x,y) + \sqrt{\frac{V}{2}\log\left(\frac{|\mathcal{G}|}{\delta}\right)}. \tag{5.5}$$

Proof. If $V = 0$, then every $g_{x,y}$, $(x, y) \in \mathcal{G}$, is constant in each coordinate. Thus $g_{x,y}(Z) = \mathbb{E}g_{x,y}(Z) \le \varepsilon(x, y)$. The claim follows. Assume $V > 0$. For each $(x, y) \in \mathcal{G}$, apply Theorem 5.1 with

$$t = \sqrt{\frac{V}{2} \log\Big(\frac{|\mathcal{G}|}{\delta}\Big)}.$$

Then

$$\mathbb{P}\left(g_{x,y}(Z) > \mathbb{E}g_{x,y}(Z) + t\right) \le \frac{\delta}{|\mathcal{G}|}.$$

Since $\mathbb{E}g_{x,y}(Z) \le \varepsilon(x, y)$, the union bound over $\mathcal{G}$ proves (5.4). □

If $c_i \le c$ for every i, then $V \le dc^2$, and one may enlarge the envelope (5.5) to

$$\varepsilon_\delta^{\mathrm{unif}}(x, y) = \varepsilon(x, y) + c\sqrt{\frac{d}{2} \log\Big(\frac{|\mathcal{G}|}{\delta}\Big)}.$$

Proposition 5.1 (A common-input perturbation model)**.** *Let $f(u, z) = F_0(u) + \eta(u, z)$. Assume that $F_0 : M \to Y$ is quadratic. Let $U \subset M$ be finite. Suppose that, for each coordinate i, numbers $L_i(u) \ge 0$, $u \in U$, satisfy*

$$\left\|\eta(u, z) - \eta(u, z')\right\|_Y \le L_i(u)$$

whenever $z, z' \in \mathcal{S}$ differ only in coordinate i. If $x + y, x - y, x, y \in U$, then

$$g_{x,y}(z) = \left\|\Delta_f(x, y; z)\right\|_Y$$

satisfies

$$|g_{x,y}(z) - g_{x,y}(z')| \le L_i(x + y) + L_i(x - y) + 2L_i(x) + 2L_i(y).$$

Thus Corollary 5.1.1 applies with any c_i dominating this bound on the grid.

Proof. The quadratic part cancels from Δ_f at each tested pair. Hence

$$|g_{x,y}(z) - g_{x,y}(z')| \le \left\|\Delta_\eta(x, y; z) - \Delta_\eta(x, y; z')\right\|_Y.$$

Expanding the right-hand side gives the bound. □

Proposition 5.2 (Common-input multiplier sensitivity)**.** *Let $\mathcal{T} \subset R \times M$ be a finite collection of tested multiplier–input pairs, let $\pi(a) \in \mathcal{L}(Y)$ for every multiplier appearing in $\mathcal{T}$, and let $f(u, z) = F_0(u) + \eta(u, z)$ on $M \times \mathcal{S}$. Choose a finite set $\mathcal{U} \subset M$ containing u and au for every $(a, u) \in \mathcal{T}$, and suppose that $F_0(au) = \pi(a)F_0(u)$ for these tested pairs. For each coordinate i, assume there is a function $L_i : \mathcal{U} \to [0, \infty)$ such that*

$$\left\|\eta(v, z) - \eta(v, z')\right\|_Y \le L_i(v), \qquad v \in \mathcal{U},$$

when z, z' differ only in coordinate i. Then for $h_{a,u}(z) = \|f(au, z) - \pi(a)f(u, z)\|_Y$,

$$|h_{a,u}(z) - h_{a,u}(z')| \le L_i(au) + \|\pi(a)\|_{\mathrm{op}} L_i(u), \qquad (a, u) \in \mathcal{T}.$$

Proof. The F_0 terms cancel on each tested pair. Apply the reverse triangle inequality to $\eta(au, z) - \pi(a)\eta(u, z)$ and use boundedness of $\pi(a)$. □

5.2. **Finite-stage dyadic stability.**

Theorem 5.2 (Finite-grid stability with high probability)**.** *Let $F \subset M$ be nonempty and finite, let $J \geq 0$ be an integer, and fix $\delta \in (0,1)$. Define*

$$\mathcal{G}_{F,J} = \{(2^j x, 2^j x) : x \in F,\ 0 \leq j < J\} \cup \{(2^J x, 2^J y) : x, y \in F\}.$$

Assume that the hypotheses of Corollary 5.1.1 hold on $\mathcal{G}_{F,J}$, with ε_δ defined using this grid.

For $z \in \mathcal{S}$, define the finite-stage dyadic mapping

$$Q_J(\,\cdot\,; z) : M \to Y, \qquad Q_J(x; z) = 4^{-J} f(2^J x, z). \tag{5.6}$$

Then, with probability at least $1 - \delta$, the following two estimates hold simultaneously:

$$\|f(x, Z) - Q_J(x; Z)\|_Y \leq \sum_{j=0}^{J-1} 4^{-(j+1)} \varepsilon_\delta(2^j x, 2^j x), \qquad x \in F, \tag{5.7}$$

and

$$\left\|\Delta_{Q_J(\cdot;Z)}(x, y)\right\|_Y \leq 4^{-J} \varepsilon_\delta(2^J x, 2^J y), \qquad x, y \in F. \tag{5.8}$$

When $J = 0$, the sum in (5.7) *is understood to be zero.*

Proof. By Corollary 5.1.1, there is an event of probability at least $1 - \delta$ on which

$$\left\|\Delta_f(u, v; Z)\right\|_Y \leq \varepsilon_\delta(u, v), \qquad (u, v) \in \mathcal{G}_{F,J}.$$

Fix an outcome in this event. Since $f(0, Z) = 0$,

$$\Delta_f(2^j x, 2^j x; Z) = f(2^{j+1} x, Z) - 4 f(2^j x, Z).$$

The telescoping identity

$$f(x, Z) - 4^{-J} f(2^J x, Z) = \sum_{j=0}^{J-1} 4^{-(j+1)} \big(4 f(2^j x, Z) - f(2^{j+1} x, Z)\big)$$

therefore yields (5.7) after taking norms.

For $x, y \in F$, the definition of Q_J gives

$$\Delta_{Q_J(\cdot;Z)}(x, y) = 4^{-J} \Delta_f(2^J x, 2^J y; Z).$$

This pair belongs to $\mathcal{G}_{F,J}$, proving (5.8). □

Theorem 5.3 (Finite-grid multiplier equivariance with high probability)**.** *Let $F \subset M$ and $\mathcal{A} \subset R$ be nonempty finite sets, let $J \geq 0$ be an integer, and let $\delta \in (0,1)$. For each $a \in \mathcal{A}$, let $\pi(a) \in \mathcal{L}(Y)$. Let Q_J be defined by* (5.6)*. For $a \in \mathcal{A}$ and $x \in F$, define*

$$h_{a,x}^{(J)}(z) = \left\|f(a 2^J x, z) - \pi(a) f(2^J x, z)\right\|_Y, \qquad z \in \mathcal{S}.$$

Assume that there are functions $\rho_a : M \to [0, \infty)$ such that

$$\mathbb{E} h_{a,x}^{(J)}(Z) \leq \rho_a(2^J x), \qquad a \in \mathcal{A}, \quad x \in F,$$

and assume that there are constants $b_1, \dots, b_d \geq 0$, independent of $a \in \mathcal{A}$ and $x \in F$, such that

$$|h_{a,x}^{(J)}(z) - h_{a,x}^{(J)}(z')| \leq b_i$$

whenever $z, z' \in \mathcal{S}$ differ only in the i-th coordinate. Put

$$B = \sum_{i=1}^{d} b_i^2.$$

Then, with probability at least $1 - \delta$, for all $a \in \mathcal{A}$ and all $x \in F$,

$$\left\| Q_J(ax; Z) - \pi(a) Q_J(x; Z) \right\|_Y \leq 4^{-J} \left(\rho_a(2^J x) + \sqrt{\frac{B}{2} \log\left(\frac{|\mathcal{A}||F|}{\delta} \right)} \right). \tag{5.9}$$

If, in addition, for some $0 < q < 2$ and constants $C_a, D_a \geq 0$,

$$\rho_a(2^J x) \leq C_a \left\| a 2^J x \right\|_2^q + D_a \left\| 2^J x \right\|_2^q, \qquad a \in \mathcal{A}, \quad x \in F,$$

then the deterministic part of the right-hand side in (5.9) *is bounded by*

$$\left(C_a (\sqrt{n}\ \|a\|_2)^q + D_a \right) 2^{J(q-2)} \|x\|_2^q. \tag{5.10}$$

Proof. If $B = 0$, each $h_{a,x}^{(J)}$ is constant in every coordinate, and the probabilistic term in (5.9) is zero. The claim follows from the expectation bound. Assume $B > 0$. By McDiarmid's inequality,

$$\mathbb{P}\left(h_{a,x}^{(J)}(Z) > \mathbb{E} h_{a,x}^{(J)}(Z) + t \right) \leq \exp\left(-\frac{2t^2}{B} \right).$$

Take

$$t = \sqrt{\frac{B}{2} \log\left(\frac{|\mathcal{A}||F|}{\delta} \right)}.$$

A union bound over $\mathcal{A} \times F$ gives, with probability at least $1 - \delta$,

$$h_{a,x}^{(J)}(Z) \leq \rho_a(2^J x) + t, \qquad a \in \mathcal{A}, \quad x \in F.$$

On this event,

$$\begin{aligned} \left\| Q_J(ax; Z) - \pi(a) Q_J(x; Z) \right\|_Y &= 4^{-J} \left\| f(2^J ax, Z) - \pi(a) f(2^J x, Z) \right\|_Y \\ &= 4^{-J} h_{a,x}^{(J)}(Z), \end{aligned}$$

because $a(2^J x) = 2^J (ax)$. This proves (5.9). The power-type estimate gives

$$4^{-J} \rho_a(2^J x) \leq 2^{J(q-2)} \left(C_a \|ax\|_2^q + D_a \|x\|_2^q \right),$$

and Lemma 1 gives (5.10). □

Remark 5.1 (Fixed-stage scope)**:** Both results hold for a fixed J on finite grids. Applying Theorems 5.2 and 5.3 with failure probability $\delta/2$ gives simultaneous bounds with probability at least $1 - \delta$. The map $Q_J(\cdot; z)$ need not be quadratic. No uniform-in-J or global pathwise bound follows.

Remark 5.2 (Dependence on dyadic depth)**:** The inputs $2^J x$, $a2^J x$, sensitivity constants b_i, c_i, and union-bound costs may grow with J. Thus the factor 4^{-J} in (5.9) does not imply convergence by itself.

Remark 5.3 (Bounded coordinate ranges)**:** For each $i \in \{1, \dots, d\}$, let $A_i, L_i \geq 0$. Suppose that

$$\mathcal{S}_i \subset [-A_i, A_i] \cap \mathbb{Z}$$

and that, uniformly over the testing grid,

$$|g_{x,y}(z) - g_{x,y}(z')| \leq L_i |z_i - z_i'|$$

whenever $z, z' \in \mathcal{S}$ differ only in coordinate i. Since

$$|z_i - z_i'| \leq 2A_i,$$

one may take the bounded-difference constants

$$c_i = 2L_i A_i.$$

Consequently,

$$V \leq 4 \sum_{i=1}^{d} L_i^2 A_i^2.$$

In the uniform case $A_i = A$ and $L_i = L$, this reduces to

$$V \leq 4dL^2 A^2.$$

These parameters bound only the concentration correction. The expectation $\varepsilon(x, y)$ may depend on the law of Z.

Example 5.1 (A linear common-input finite-grid model)**.** Take a quadratic $F_0 : M \to Y$ and maps $\psi_i : M \to Y$ with $\psi_i(0) = 0$. Define

$$f(u, z) = F_0(u) + \eta(u, z), \qquad \eta(u, z) = \sum_{i=1}^{d} z_i \psi_i(u),$$

where $A_i \geq 0$ and $\mathcal{S}_i \subset [-A_i, A_i] \cap \mathbb{Z}$ for every i. Thus $f(0, z) = 0$, as required above, and

$$\|\eta(u, z) - \eta(u, z')\|_Y \leq 2A_i \|\psi_i(u)\|_Y$$

when only coordinate i is changed. To apply Theorem 5.2, choose a finite $\mathcal{U}$ containing, for every $(v, w) \in \mathcal{G}_{F,J}$, all four inputs $v + w, v - w, v, w$. Proposition 5.1 then applies with $L_i(u) = 2A_i \|\psi_i(u)\|_Y$ for $u \in \mathcal{U}$. For example, uniform coordinate constants on this grid are

$$c_i = 2A_i \max_{(v,w) \in \mathcal{G}_{F,J}} \big(\|\psi_i(v + w)\|_Y + \|\psi_i(v - w)\|_Y + 2 \|\psi_i(v)\|_Y + 2 \|\psi_i(w)\|_Y \big).$$

For Theorem 5.3, also include $2^J F$ and $a2^J F$, $a \in \mathcal{A}$, in $\mathcal{U}$, and assume $F_0(a2^J x) = \pi(a) F_0(2^J x)$ for $a \in \mathcal{A}$, $x \in F$. Apply Proposition 5.2 to $\mathcal{T} = \{(a, 2^J x) : a \in \mathcal{A},\ x \in F\}$; a valid uniform choice is

$$b_i = \max_{a \in \mathcal{A},\, x \in F} 2A_i \Big(\Big\| \psi_i(a2^J x) \Big\|_Y + \|\pi(a)\|_{\mathrm{op}} \Big\| \psi_i(2^J x) \Big\|_Y \Big).$$

Since F_0 is quadratic,

$$\mathbb{E}\left\|\Delta_f(x,y;Z)\right\|_Y \le \sum_{i=1}^{d} \mathbb{E}|Z_i| \ \left\|\Delta_{\psi_i}(x,y)\right\|_Y .$$

On a tested pair (a,u) where $F_0(au)=\pi(a)F_0(u)$, likewise

$$\mathbb{E}\left\|f(au,Z)-\pi(a)f(u,Z)\right\|_Y \le \sum_{i=1}^{d} \mathbb{E}|Z_i| \ \left\|\psi_i(au)-\pi(a)\psi_i(u)\right\|_Y .$$

These bounds provide ε and ρ_a on the finite grids. Independence is used only for concentration.

Corollary 5.3.1 (Coefficient-box finite-grid certificates)**.** *For positive integers N, L, set*

$$F_N=\{x=(x_{j,i})\in R^k : |x_{j,i}|\le N\}, \qquad A_L=\{a=(a_i)\in R : |a_i|\le L\}.$$

Write $H_N=(2N+1)^{nk}$ and $K_L=(2L+1)^n$. At each fixed $J\ge 0$, the finite-grid theorem uses

$$H_N^2 \le |\mathcal{G}_{F_N,J}| \le H_N^2 + JH_N,$$

so its logarithmic union-bound cost is $\log|\mathcal{G}_{F_N,J}| \le \log(H_N^2+JH_N)$. For the multiplier theorem the cost is instead

$$\log(|A_L||F_N|) = n\log(2L+1)+nk\log(2N+1).$$

Proof. The coefficient embedding gives $|F_N|=H_N$ and $|A_L|=K_L$. The second part of $\mathcal{G}_{F_N,J}$ has precisely H_N^2 points, since multiplication by 2^J is injective on M; the first part has at most JH_N points. For $J=0$, $|\mathcal{G}_{F_N,0}|=H_N^2$. The multiplier grid has cardinality K_LH_N. □

6. Conclusion

Dyadically negligible intertwining defects pass to the classical quadratic limit. On R^k, coefficient-norm bounds give multiplier criteria, while ring relations restrict the invariant span $W=\mathrm{span}_{\mathbb{F}}\,Q(M)$. For $X^n=-1$, one generator relation gives $U^n|_W=I_W$, not necessarily $U^n=I_Y$.

Expected defects yield L^1 stability. McDiarmid's inequality controls finite-stage maps on prescribed finite grids, not a global pathwise limit.

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LY VAN NGOC
INDEPENDENT RESEARCHER
QUANG TRI, VIETNAM
Email address: Ngoclyvan@gmail.com

Received 07.06.2026
Revised 13.10.2026
Accepted 15.10.2026